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Research Article

Integrability for Solutions of Anisotropic Obstacle Problems

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This paper deals with anisotropic obstacle problem for the $\mathcal{A}$-harmonic equation $\sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0$. An integrability result is given under suitable assumptions, which show higher integrability of the boundary datum, and the obstacle force solutions u have higher integrability as well.

1. Introduction and Statement of Result

Let Ω be a bounded open subset of $\mathbb{R}^n$. For $p_i > 1$, $i = 1, 2, \ldots, n$, we denote $p_m = \max_{i=1,2,\ldots,n} p_i$ and $\overline{p}$ is the harmonic mean of p_i, that is,

$$\frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}. \tag{1.1}$$

The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$ is defined by

$$W^{1,(p_i)}(\Omega) = \left\{v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \ldots, n\right\}. \tag{1.2}$$

Let us consider solutions $u \in W^{1,(p_i)}(\Omega)$ of the following $\mathcal{A}$-harmonic equation:

$$\sum_{i=1}^{n} D_i(a_i(x, Du(x))) = 0, \tag{1.3}$$

where $D = (D_1, D_2, \dots, D_n)$ is the gradient operator, and the Carathéodory functions $a_i(x, \xi) : \Omega \times R^n \to R$, $i = 1, 2, \dots, n$, satisfy

$$|a_i(x, z)| \leq c_2 (h(x) + |z_i|)^{p_i - 1}, \tag{1.4}$$

for almost every $x \in \Omega$, for every $z \in R^n$, and for any $i = 1, 2, \dots, n$, and there exists $\tilde{\nu} \in (0, +\infty)$ such that

$$\tilde{\nu} \sum_{i=1}^{n} |z_i - \tilde{z}_i|^{p_i} \leq \sum_{i=1}^{n} (a_i(x, z) - a_i(x, \tilde{z}))(z_i - \tilde{z}_i), \tag{1.5}$$

for almost every $x \in \Omega$, for any $z, \tilde{z} \in R^n$. The integrability condition for $h(x) \geq 0$ in (1.4) will be given later.

Let ψ be any function in Ω with values in $R \cup \{\pm\infty\}$ and $\theta \in W^{1,(p_i)}(\Omega)$, and we introduce

$$\mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega) = \left\{ v \in W^{1,(p_i)}(\Omega) : v \geq \psi, \text{ a.e. and } v - \theta \in W_0^{1,(p_i)}(\Omega) \right\}. \tag{1.6}$$

Note that

$$W_0^{1,(p_i)}(\Omega) = \left\{ v \in W_0^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \dots, n \right\}. \tag{1.7}$$

The function ψ is an obstacle and θ determines the boundary values.

Definition 1.1. A solution to the $\mathcal{K}_{\psi,\theta}^{(p_i)}$-obstacle problem is a function $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$ such that

$$\int_{\Omega} \sum_{i=1}^{n} a_i(x, Du(x))(D_i v(x) - D_i u(x))dx \geq 0, \tag{1.8}$$

whenever $v \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega)$.

Higher integrability property is important among the regularity theories of nonlinear elliptic PDEs and systems, see the monograph [1] by Bensoussan and Frehse. Meyers and Elcrat [2] first considered the higher integrability for weak solutions of (1.3) in 1975. Iwaniec and Sbordone [3] obtained a regularity result for very weak solutions of the $\mathcal{A}$-harmonic equation (1.3) by using the celebrated Gehring's Lemma. Global integrability for anisotropic equation is contained in [4]. As far as higher integrability of ∇u is concerned, in problems with nonstandard growth a delicate interplay between the regularity with respect to x and the growth with respect to ξ appears: see [5]. For a global boundedness result of anisotropic variational problems, see [6]. For other related works, see [7]. We refer the readers to the classical books by Ladyženskaya and Ural'ceva [8], Morrey [9], Gilbarg and Trudinger [10] and Giaquinta [11] for some details of isotropic cases.

In the present paper, we consider integrability for solutions of anisotropic obstacle problems of the $\mathcal{A}$-harmonic equation (1.3), which show higher integrability of the boundary datum, and the obstacle force solutions u, have higher integrability as well. The idea of this paper comes from [4], and the result can be considered as a generalization of [4, Theorem 2.1].

Theorem 1.2. *Let* $u \in \mathcal{K}^{(p_i)}_{\psi,\theta}(\Omega)$ *be a solution to the* $\mathcal{K}^{(p_i)}_{\psi,\theta}$ *obstacle problem and* $\theta \in W^{1,(q_i)}(\Omega)$, $q_i \in (p_i, +\infty)$, $i = 1, 2, \ldots, n$, $0 \le h \in L^{q_m}(\Omega)$ *with* $q_m = \max_{i=1,\ldots,n} q_i$, $\psi \in [-\infty, +\infty]$ *is such that* $\theta_* = \max\{\psi, \theta\} \in \theta + W_0^{1,(q_i)}(\Omega)$. *Moreover,* $\overline{p} < n$. *Then*

$$u \in \theta_* + L^t_{weak}(\Omega), \tag{1.9}$$

where

$$t = \frac{\overline{p}^*}{1 - (b\overline{p}^*/\overline{p})(p_m/p_m - 1)} > \overline{p}^*, \tag{1.10}$$

and b is any number verifying

$$0 < b \le \min_{j=1,\ldots,n} \left(1 - \frac{p_j}{q_j}\right)\left(1 - \frac{1}{p_j}\right),$$
$$b < \frac{p_m - 1}{p_m} \frac{\overline{p}}{\overline{p}^*}. \tag{1.11}$$

Remark 1.3. Take the obstacle function ψ to be minus infinity in Theorem 1.2, and the condition (1.4) replaced by

$$|a_i(x, z)| \le c_2 (1 + |z_i|)^{p_i - 1} \tag{1.2$'$}$$

for almost every $x \in \Omega$, for every $z \in R^n$, and for any $i = 1, 2, \ldots, n$, then we arrive at Theorem 2.1 in [4].

2. Proof of the Main Theorem

Proof of Theorem 1.2. Let $u \in \mathcal{K}^{(p_i)}_{\psi,\theta}(\Omega)$ be a solution to the $\mathcal{K}^{(p_i)}_{\psi,\theta}$-obstacle problem. Take $\theta_* = \max\{\psi, \theta\} \in \theta + W_0^{1,(q_i)}(\Omega)$. Let us consider $L \in (0, +\infty)$ and

$$v = \begin{cases} \theta_* - L, & \text{for } u - \theta_* < -L, \\ u, & \text{for } -L \le u - \theta_* \le L, \\ \theta_* + L, & \text{for } u - \theta_* > L. \end{cases} \tag{2.1}$$

Then $v \in \mathcal{K}^{(p_i)}_{\psi,\theta}(\Omega)$. Indeed, for the second and the third cases of the above definition for v, we obviously have $v \ge \psi$, and for the first case, $u - \theta_* < -L$, we have $\theta_* > u + L \ge \psi + L$; this implies $v = \theta_* - L \geq \psi$. Since $u = \theta_* = \theta$ on $\partial\Omega$, then $v = u$ on $\partial\Omega$, this implies $v = \theta$ on $\partial\Omega$. By Definition 1.1, one has

$$\begin{aligned} 0 &\leq \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} a_i(x, Du(x))(D_i v(x) - D_i u(x))dx \\ &= \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} a_i(x, Du(x))(D_i \theta_*(x) - D_i u(x))dx. \end{aligned} \tag{2.2}$$

Monotonicity (1.5) allows us to write

$$\begin{aligned} &\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} |D_i u(x) - D_i \theta_*(x)|^{p_i} dx \\ &\quad\leq \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} (a_i(x, Du(x)) - a_i(x, D\theta_*(x)))(D_i u(x) - D_i \theta_*(x))dx, \end{aligned} \tag{2.3}$$

which together with (2.2) implies

$$\begin{aligned} &\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} |D_i u(x) - D_i \theta_*(x)|^{p_i} dx \\ &\quad\leq -\int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} a_i(x, D\theta_*)(D_i u(x) - D_i \theta_*(x))dx. \end{aligned} \tag{2.4}$$

We now use anisotropic growth (1.4) and the Hölder inequality in (2.4), obtaining that

$$\begin{aligned} &\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} |D_i u - D_i \theta_*|^{p_i} dx \\ &\quad\leq -\sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} a_i(x, D\theta_*)(D_i u - D_i \theta_*)dx \\ &\quad\leq c_2 \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} (h + |D_i \theta_*|)^{p_i - 1} |D_i u - D_i \theta_*| dx \\ &\quad\leq c_2 \sum_{i=1}^{n} \left(\int_{\{|u-\theta_*|>L\}} (h + |D_i \theta_*|)^{p_i} dx \right)^{(p_i-1)/p_i} \left(\int_{\{|u-\theta_*|>L\}} |D_i u - D_i \theta_*|^{p_i} dx \right)^{1/p_i}. \end{aligned} \tag{2.5}$$

Let t_i be such that

$$p_i < t_i \leq q_i, \tag{2.6}$$

for every $i = 1, \ldots, n$; t_i will be chosen later. We use the Hölder inequality as follows:

$$\begin{aligned}
&\left(\int_{\{|u-\theta_*|>L\}} (h + |D_i\theta_*|)^{p_i} dx\right)^{(p_i-1)/p_i} \\
&\quad \le \left(\int_{\{|u-\theta_*|>L\}} (h + |D_i\theta_*|)^{t_i} dx\right)^{(p_i-1)/t_i} (|\{|u-\theta_*| > L\}|)^{(t_i-p_i)(p_i-1)/t_i p_i}.
\end{aligned} \tag{2.7}$$

The following proof is similar to that of [4, Theorem 2.1]; we only list the necessary changes: instead of [4, (3.14)] by

$$\begin{aligned}
&\left(\int_{\{|u-\theta_*|>L\}} (h + |D_i\theta_*|)^{p_i} dx\right)^{(p_i-1)/p_i} \\
&\quad \le \left(\int_{\{|u-\theta_*|>L\}} (h + |D_i\theta_*|)^{t_i} dx\right)^{(p_i-1)/t_i} |\{|u-\theta_*| > L\}|^{b} \\
&\quad \le M|\{|u-\theta_*| > L\}|^{b},
\end{aligned} \tag{2.8}$$

where

$$M = \max_{j=1,\ldots,n}\left(\int_{\Omega} (h + |D_j\theta_*|)^{t_j} dx\right)^{(p_j-1)/t_j} < \infty, \tag{2.9}$$

and instead of [4, (3.19)] we use anisotropic Sobolev Embedding Theorem for $v - u$,

$$\begin{aligned}
&\left(\int_{\Omega} |v-u|^{\overline{p}^*} dx\right)^{1/\overline{p}^*} \\
&\quad \le c_* \left[\prod_{i=1}^{n}\left(\int_{\Omega} |D_i(v-u)|^{p_i} dx\right)^{1/p_i}\right]^{1/n} \\
&\quad \le c_* \left[\prod_{i=1}^{n}\left(\int_{\{|u-\theta_*|>L\}} |D_i u - D_i\theta_*|^{p_i} dx\right)^{1/p_i}\right]^{1/n}.
\end{aligned} \tag{2.10}$$

By $|v-u| = (|u-\theta_*| - L)1_{\{|u-\theta_*|>L\}}$, we obtain

$$\left(\int_{\{|u-\theta_*|>L\}} (|u-\theta_*| - L)^{\overline{p}^*} dx\right)^{1/\overline{p}^*} = \left(\int_{\Omega} |v-u|^{\overline{p}^*} dx\right)^{1/\overline{p}^*}. \tag{2.11}$$

Following the idea of the proof of Theorem 2.1 in [4], we complete the proof of Theorem 1.2. □

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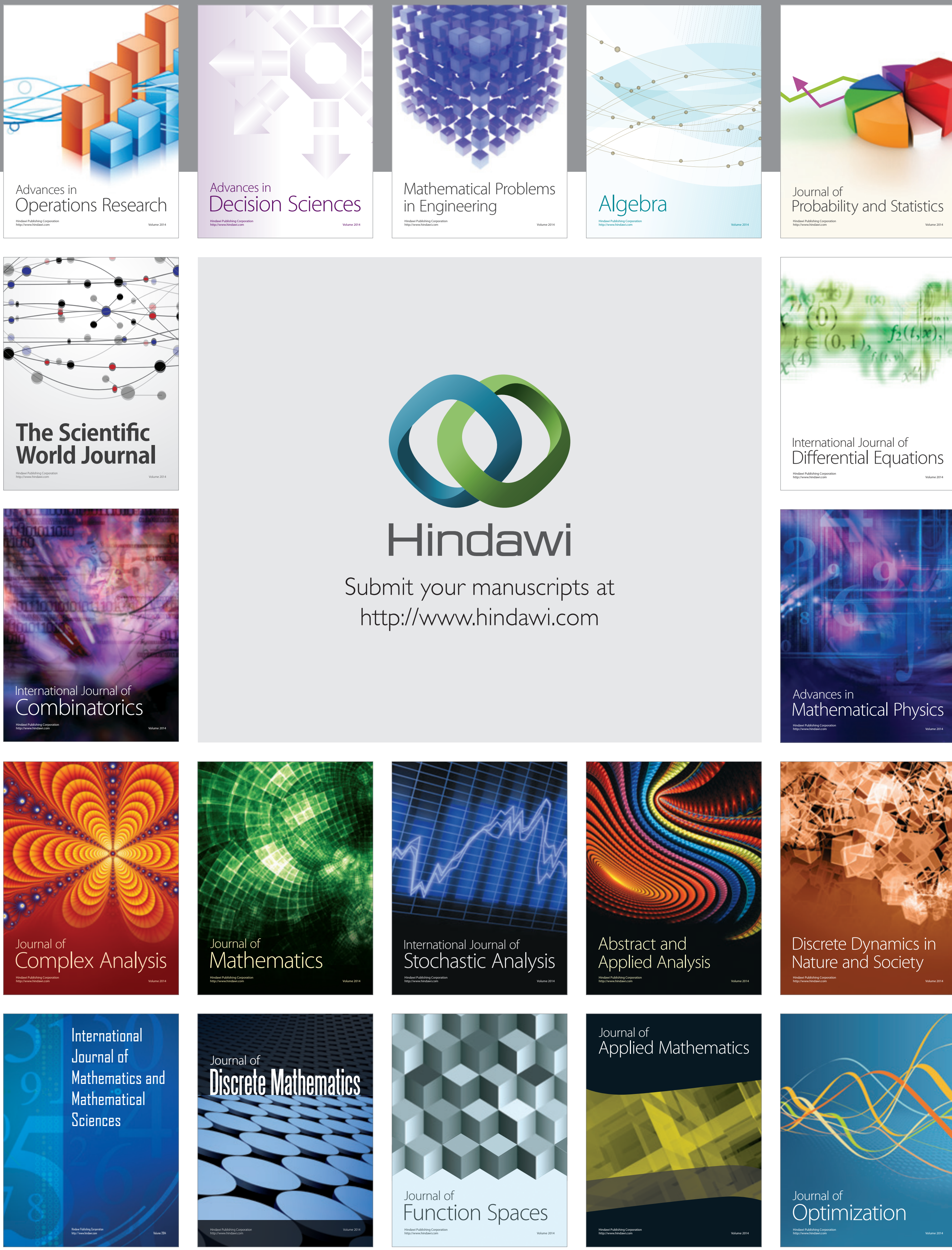
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