Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 595429, 10 pages
doi:10.1155/2012/595429

Research Article

Existence and Multiplicity of Solutions for Semipositone Problems Involving p-Laplacian

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Received 22 March 2012; Accepted 19 April 2012

Academic Editor: Peter Takac

We prove existence and multiplicity of positive solutions for semipositone problems involving p-Laplacian in a bounded smooth domain of $\mathbb{R}^N$ under the cases of sublinear and superlinear nonlinearities term.

1. Introduction

In this paper, we shall study the following semipositone problem involving the p-Laplacian:

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, λ is a positive parameter, and $f : [0,+\infty) \to \mathbb{R}$ is a continuous function satisfying the condition

(F0) $f(0) = -a < 0$.

Such problems are usually referred in the literature as semipositone problems. We refer the reader to [1], where Castro and Shivaji initially called them nonpositone problems, in contrast with the terminology positone problems, coined by Keller and Cohen in [2], when the nonlinearity f was positive and monotone.

Under the case of $p \equiv 2$, a novel variational approach is presented by Costa et al. [3] to the question of existence and multiplicity of positive solutions to problem (1.1), where they consider both the sublinear and superlinear cases. The aim of this paper is to extend their results to the case of p-Laplacian. The main difficulty is in verifying $(\mathrm{PS})_c$-condition because of the operator $-\Delta_p$ is not self-adjoint linear.

We define the discontinuous nonlinearity $g(s)$ by

$$g(s) = \begin{cases} 0, & \text{if } s \leq 0, \\ f(s), & \text{if } s > 0. \end{cases} \tag{1.2}$$

We shall consider the modified problem

$$\begin{gathered} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda g(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

We note that the set of positive solutions of (1.1) and (1.3) do coincide.

Our main results concerning problems (1.1) and (1.3) are the following:

Theorem 1.1. *Assume (F0) and the following assumptions:*

(F1) $\lim_{s\to+\infty}(f(s)/s^{p-1}) = 0$ *(the sublinear case);*

(F2) $F(\delta) > 0$ *for some* $\delta > 0$, *where* $F(u) = \int_0^u f(s)\, ds$.

Then, there exist $0 < \lambda_0 \leq \lambda_*$ *such that* (1.3) *has no nontrivial nonnegative solution for* $0 < \lambda < \lambda_0$, *and has at least two nontrivial nonnegative solutions* u_λ, v_λ *for all* $\lambda > \lambda_*$. *Moreover, when* Ω *is a ball* $B_R = B_R(0)$, *these two solutions are non-increasing, radially symmetric and, if* $N \geq 2$, *at least one of them is positive, hence a solution of* (1.1).

Theorem 1.2. *Assume (F0), (F2) and the following assumptions:*

(F3) $\lim_{s\to+\infty}(f(s)/s^{p-1}) = +\infty$, $\lim_{s\to+\infty}(f(s)/s^{p^*-2}) = 0$ *(the sublinear, subcritical case);*

(F4) $\theta F(s) \leq f(s)s$ *for all* $s \geq K$ *and some* $\theta > p$ *(the AR-condition).*

Then, (1.3) *has at least one nonnegative solution* v_λ *for all* $\lambda > 0$. *If* $\Omega = B_R$ *then* v_λ *is nonincreasing, radially symmetric and one of the two alternatives occurs.*

(i) *There exists* $\lambda^* > 0$ *such that, for all* $0 < \lambda < \lambda^*$, v_λ *is a positive solution of* (1.1) *having negative normal derivative on* ∂B_R.

(ii) *For some sequence* $\mu_n \to 0$, *problem* (1.1) *with* $\lambda \equiv \mu_n$ *has a positive solution* w_{μ_n} *with zero normal derivative on* ∂B_R.

2. Preliminaries

We start by recalling some basic results on variational methods for locally Lipschitz functionals. Let $(X, \|\cdot\|)$ be a real Banach space and X^* is its topological dual. A function $f : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood Ω_u

such that $|f(u_1) - f(u_2)| \le L\|u_1 - u_2\|$ for all $u_1, u_2 \in \Omega_u$, for a constant $L > 0$ depending on Ω_u. The generalized directional derivative of f at the point $u \in X$ in the direction $v \in X$ is

$$f^0(u,v) = \limsup_{w \to u, t \to 0^+} \frac{1}{t}(f(w+tv) - f(w)). \tag{2.1}$$

The generalized gradient of f at $u \in X$ is defined by

$$\partial f(u) = \left\{u^* \in X^* : \langle u^*, \varphi\rangle \le f^0(u;\varphi)\ \forall \varphi \in X\right\}, \tag{2.2}$$

which is a nonempty, convex, and w^*-compact subset of X, where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X. We say that $u \in X$ is a critical point of f if $0 \in \partial f(u)$. For further details, we refer the reader to Chang [4].

We list some fundamental properties of the generalized directional derivative and gradient that will be possibly used throughout the paper.

Proposition 2.1 (see [4, 5]). (1) *Let $j : X \to \mathbb{R}$ be a continuously differentiable function. Then $\partial j(u) = \{j'(u)\}$, $j^0(u;z)$ coincides with $\langle j'(u), z\rangle_X$, and $(f+j)^0(u,z) = f^0(u;z) + \langle j'(u), z\rangle_X$ for all $u, z \in X$.*

(2) *$\partial(\lambda f)(u) = \lambda \partial(u)$ for all $\lambda \in \mathbb{R}$.*

(3) *If f is a convex functional, then $\partial f(u)$ coincides with the usual subdifferential of f in the sense of convex analysis.*

(4) *If f has a local minimum (or a local maximum) at $u_0 \in X$, then $0 \in \partial f(u_0)$.*

(5) *$\|\xi\|_{X^*} \le L$ for all $\xi \in \partial f(u)$.*

(6) *$f^0(u,h) = \max\{\langle \xi, h\rangle : \xi \in \partial f(u)\}$ for all $h \in X$.*

(7) *The function*

$$m(u) = \min_{w \in \partial f(u)} \|w\|_{X^*}, \tag{2.3}$$

exists and is lower semicontinuous; that is, $\lim_{u \to u_0} \inf m(u) \ge m(u_0)$.

In the following we need the nonsmooth version of the Palais-Smale condition.

Definition 2.2. One says φ that nonsmooth satisfies the $(\text{PS})_c$-condition if any sequence $\{u_n\} \subset X$ such that $\varphi(u_n) \to c$ and $m(u_n) \to 0$.

In what follows we write the $(\text{PS})_c$-condition as simply as the PS-condition if it holds for every level $c \in \mathbb{R}$ for the Palais-Smale condition at level c.

We note that property (4) above says that a local minimum (or local maximum) of φ is a critical point of φ. Finally, we point out that many of the results of the classical critical point theory have been extended by Chang [4] to this setting of locally Lipschitz functionals. For example, one has the following celebrated theorem.

Theorem 2.3 (nonsmooth mountain pass theorem; see [4, 5]). *If X is a reflexive Banach space, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth $(PS)_c$-condition, and for some $r > 0$ and $e_1 \in X$ with $\|e_1\| > r$, $\max\{\varphi(0), \varphi(e_1)\} \le \inf\{\varphi(u) : \|u\| = r\}$. Then φ has* *a nontrivial critical* $u \in X$ *such that the critical value* $c = \varphi(u)$ *is characterized by the following minimax principle:*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)), \tag{2.4}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e_1\}$.

We would like to point out that we can obtain the same results for problem (1.3) through approximation of the discontinuous nonlinearity by a sequence of continuous functions. Variational methods were then applied to the corresponding sequence of problems and limits were taken. For the rest of this paper, we write $X = W_0^{1,p}(\Omega)$ with the norm by $\|u\| = (\int_\Omega |\nabla u|^p\ dx)^{1/p}$ and denote by c and c_i the generic positive constants for simplicity.

3. Proof of the Main Results

Now, having listed some basic results on critical point theory for the Lipschitz functionals, let us consider the functional

$$\varphi_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|\ dx - \lambda\int_\Omega G(u)\ dx, \tag{3.1}$$

where $G(u) = \int_0^u g(s)\ ds$ and $g(s)$ were defined in (1.2). Clearly $G : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function and satisfies $G(s) = 0$ for $s \le 0$. In view of [3, Theorems 2.1 and 2.2], the above formula for $\varphi_\lambda(u)$ defines a locally Lipschitz functional on X whose critical points are solutions of the differential inclusion

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \in \partial G(u) \quad \text{a.e. in } \Omega. \tag{3.2}$$

In our present case, it follows that $\partial G(s) = f(s)$ for $s > 0$, $\partial G(s) = 0$ for $s < 0$, and $\partial G(s) = [-a, 0]$ for $s = 0$.

We start with some preliminary lemmas.

Lemma 3.1. *Assume (F0), (F1), and (F2). Then there exists* λ_0 *such that problem* (1.3) *has no nontrivial solution* $0 \le u \in X$ *for* $0 < \lambda < \lambda_0$.

Proof. If $u \ge 0$ is a solution of problem (1.3), then, multiplying the equation by u and integrating over Ω yields

$$\|u\|^p = \int_\Omega |\nabla u|^p\ dx = \lambda\int_\Omega g(u)u\,dx = \lambda\left(\int_{\{x\in\Omega|u(x)\ge\delta_0\}} ug(u)dx + \int_{\{x\in\Omega|u(x)\ge\delta_0\}} ug(u)dx\right), \tag{3.3}$$

hence

$$\|u\|^p \leq \lambda \int_{\{x\in\Omega|u(x)\geq\delta_0\}} ug(u)dx, \tag{3.4}$$

where we have chosen $\delta_0 > 0$ so that $g(s) \leq 0$ for $0 \leq s \leq \delta_0$ (such a δ_0 exists in view of (F0)). Now, since (F1) implies the existence of $c > 0$ such that

$$sg(s) \leq c(1 + s^p), \tag{3.5}$$

for all $s > 0$, we obtain from (3.4) that

$$\|u\|^p \leq \lambda c \int_{\{x\in\Omega|u(x)\geq\delta_0\}} (1 + u^p)\ dx \leq \lambda c\left(1 + \frac{1}{\delta_0^p}\right) \int_{\{x\in\Omega|u(x)\geq\delta_0\}} u^p\ dx \leq \lambda c_1 \int_\Omega u^p\ dx, \tag{3.6}$$

so that

$$\|u\|^p \leq \lambda c_2 \|u\|^p, \tag{3.7}$$

where this last constant $c_2 > 0$ is independent of both u and λ. Therefore we must have

$$\lambda \geq \frac{1}{c_2} := \lambda_0 > 0. \tag{3.8}$$

□

Lemma 3.2. *Assume (F0) and either (F1) or (F3). Then $u = 0$ is a strict local minimum of the functional φ_λ.*

Proof. Since (F1) or (F3) implies the existence of $c_3 > 0$ such that

$$G(s) \leq c_3\left(1 + |s|^{p^*}\right) \quad \forall s \in \mathbb{R}, \tag{3.9}$$

recall also that $G(s) = 0$ for $s \leq 0$. Then, with $\delta_0 > 0$ as in the proof of Lemma 3.1 and noticing that $G(s) \leq 0$ for all $-\infty < s \leq \delta_0$, we can write for an arbitrary $u \in X$,

$$\begin{aligned} \varphi_\lambda(u) &= \frac{1}{p}\|u\|^p - \lambda \int_\Omega G(u)dx \\ &\geq \frac{1}{p}\|u\|^p - \lambda \int_{\{x\in\Omega|u(x)\geq\delta_0\}} G(u)dx \\ &\geq \frac{1}{p}\|u\|^p - \lambda c_3 \int_{\{x\in\Omega|u(x)\geq\delta_0\}} \left(1 + u^{p^*}\right) dx \\ &\geq \frac{1}{p}\|u\|^p - \lambda c_3\left(1 + \frac{1}{\delta_0^{p^*}}\right) \int_{\{x\in\Omega|u(x)\geq\delta_0\}} u^{p^*}\ dx, \end{aligned} \tag{3.10}$$

so that, using the Sobolev embedding theorem in the last inequality and with a constant $c_4 > 0$ independent of u and Ω, we obtain

$$\varphi_\lambda(u) \geq \frac{1}{p}\|u\|^p - \lambda c_4\|u\|^{p^*} = \frac{1}{p}\|u\|^p\left(1 - p\lambda c_4\|u\|^{p^*-p}\right). \tag{3.11}$$

Therefore, for each $0 < \rho < \rho_\lambda := 1/(p\lambda c_4)^{p^*-p}$, it follows that $\varphi_\lambda(u) \geq \alpha_\rho > 0$ if $\|u\| = \rho$. This shows that $u = 0$ is a strict local minimum of φ_λ. □

Lemma 3.3. *Under the same assumptions as in Lemma 3.2, let $\widehat{u} \in X$ be a critical point of φ_λ. Then, $\widehat{u} \in C^{1,\alpha}(\Omega)$ and $\widehat{u}$ is a nonnegative solution of problem* (1.3), *where* $0 < \alpha < 1$.

Proof. We shall follow some of the arguments in [3]. As mentioned earlier, if $\widehat{u}$ is a critical point of φ_λ, then it is shown in [3] that $\widehat{u}$ is a solution of the differential inclusion

$$-\operatorname{div}\left(|\nabla\widehat{u}|^{p-2}\nabla\widehat{u}\right) \in \partial G(\widehat{u}) \quad \text{a.e. in } \Omega. \tag{3.12}$$

Since g is only discontinuous at $s = 0$, the above differential inclusion reduces to an equality, except possibly on the subset $A \subset \Omega$ where $\widehat{u} = 0$. Since f is subcritical, using the $C^{1,\alpha}$ regularity results for quasilinear elliptic equations with p-growth condition (see, for example, [6]), we have $\widehat{u} \in C^{1,\alpha}(\overline{\Omega} \setminus A)$. And, in this latter case, $-\operatorname{div}(|\nabla\widehat{u}|^{p-2}\nabla\widehat{u})$ takes on values in the bounded interval $[-a, 0]$. Therefore, by the Michael selection theorem (see Theorem 1.2.5 of [5]), we see that $-\Delta_p : u \mapsto [-a, 0]$ admits a continuous selection. Using the $C^{1,\alpha}$ regularity results for quasilinear elliptic equations with p-growth condition again, it follows that $\widehat{u} \in C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$.

Next, using a Morrey-Stampacchia theorem [7, Theorem 3.2.2, page 69], we have that $-\operatorname{div}(|\nabla\widehat{u}|^{p-2}\nabla\widehat{u}) = 0$ a.e. in A. Therefore, since we defined $g(0) = 0$, it follows that

$$-\operatorname{div}\left(|\nabla\widehat{u}|^{p-2}\nabla\widehat{u}\right) = g(\widehat{u}) \quad \text{a.e. in } \Omega. \tag{3.13}$$

Replacing the inclusion (3.2) on $\widehat{u}$, we conclude that $\widehat{u} \in C^{1,\alpha}(\Omega)$ is a solution of (1.3). Finally, recalling that $g(s) = 0$ for $s \leq 0$, it is clear that $\widehat{u} \geq 0$. The proof of Lemma 3.3 is complete. □

Lemma 3.4. *Assume either (F1) or (F3), (F4). Then φ_λ satisfies the nonsmooth $(PS)_c$-condition at every $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n\geq 1} \subseteq X$ be a sequence such that $|\varphi_\lambda(u_n)| \leq c$ for all $n \geq 1$ and $m(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. In the superlinear and subcritical case, from (F4), we have

$$\begin{aligned} c \geq \varphi_\lambda(u_n) &= \frac{1}{p}\int_\Omega |\nabla u_n|^p \, dx - \lambda \int_\Omega G(u_n) \, dx \\ &\geq \frac{\|u_n\|^p}{p} - \frac{\lambda}{\theta}\int_\Omega \langle \xi(u_n), u_n \rangle \, dx - c \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p + \int_\Omega \frac{1}{\theta}\left(\|u_n\|^p - \lambda\langle \xi(u_n), u_n \rangle\right) dx - c \\ &\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|^p - \frac{\lambda}{\theta}\|\xi\|_{X^*}\|u_n\| - c, \end{aligned} \tag{3.14}$$

where $\xi(u_n) \in \partial G(u_n)$. Hence $\{u_n\}_{n\geq 1} \subseteq X$ is bounded.

Thus by passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in X as $n \longrightarrow +\infty$. We have

$$\langle J'(u_n), u_n - u \rangle - \int_\Omega \xi(u_n)(u_n - u) \, dx \leq \varepsilon_n \|u_n - u\| \tag{3.15}$$

with $\varepsilon_n \downarrow 0$, where $\xi(u_n) \in \partial\Psi(u_n)$ and $J = (1/p)\int_\Omega |\nabla u_n|^p \, dx$. From (F3) and Chang [4] we know that $\xi(u_n(x)) \in L^{(p^*-1)'}(\Omega)$ $((p^*-1)' = (p^*-1)/(p^*-2))$. Since X is embedded compactly in $L^{p^*-1}(\Omega)$, we have that $u_n \longrightarrow u$ as $n \longrightarrow \infty$ in $L^{p^*-1}(\Omega)$. So using the Hölder inequality, we have

$$\int_\Omega \xi_n(x)(u_n - u) \, dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.16}$$

Therefore, we obtain that $\lim_{n\to\infty}\sup\langle J'(u_n), u_n - u \rangle \leq 0$. But we know that J' is a mapping of type (S_+). Thus we have

$$u_n \longrightarrow u \quad \text{in X}. \tag{3.17}$$

Using the similar method, we can more easily get the nonsmooth $(\mathrm{PS})_c$-condition in the case of sublinear. □

Remark 3.5. Note that we cannot directly apply the results of [4] because the operator $-\Delta_p$ is not self-adjoint linear.

Lemma 3.6. *Under assumptions (F0), (F1), and (F2), let $\Omega = B_R \subset \mathbb{R}^N$ with $N \geq 2$, and let $u \in C^1(\overline{B_R})$ be a radially symmetric, nonincreasing function such that $u \geq 0$ and u is a minimizer of $\varphi_\lambda(u)$ with $\varphi_\lambda(u) = m < 0$. Then, u does not vanish in B_R; that is, $u(x) > 0$ for all $x \in B_R$.*

Proof. Since g is discontinuous at zero, we note that the conclusion does not follow directly from uniqueness of solution for the Cauchy problem with data at $r = R$ (in fact, writing $u = u(r)$, $r = |x|$, we may have $u(R) = u'(R) = 0$ and $u \not\equiv 0$).

Now, since $u \not\equiv 0$ by assumption, $R_0 := \inf\{r \leq R : u(s) = 0 \text{ for } r \leq s \leq R\}$ satisfies $0 < R_0 \leq R$. If $R_0 = R$ there is nothing to prove in view of the fact that u is non-increasing. On the other hand, if $R_0 < R$ then $u'(R_0) = 0$ and $u(r) > 0$ for $r \in [0, R_0)$. It is not hard to prove that this contradicts that u is a minimizer of φ_λ. Indeed, if $R_0 < R$ then

$$\varphi_\lambda(u) = \frac{1}{p}\int_{B_{R_0}} |\nabla u|^p \, dx - \lambda \int_{B_{R_0}} G(u) \, dx = m < 0. \tag{3.18}$$

A simple calculation shows that the rescaled function $v(r) = u(R_0 r/R) \in W_0^{1,p}(B_R) \cap C^1(\overline{B_R})$ satisfies

$$\varphi_\lambda(v) = \delta^{p-N}\left[\frac{1}{p}\int_{B_{R_0}} |\nabla u|^p \, dx - \delta^{-p}\lambda \int_{B_{R_0}} G(u) \, dx\right], \tag{3.19}$$

where $\delta := R_0/R$ is less than 1. Therefore, since we are assuming $1 < p \leq N$, we would reach the contradiction $\varphi_\lambda(v) < m$. □

Remark 3.7. Note that the condition (F2) is necessary because of which guarantee G can have positive values.

Proof of Theorem 1.1. We observe that the functional φ_λ is weakly lower semicontinuous on X. Moreover, the sublinearity assumption (F1) on $g(u)$ implies that φ_λ is coercive. Therefore, the infimum of φ_λ is attained at some u_λ:

$$\inf_{u \in X} \varphi_\lambda(u) = \varphi_\lambda(u_\lambda). \tag{3.20}$$

And, in view of Lemma 3.3, $u_\lambda \in C^{1,\alpha}(\Omega)$ is a nonnegative solution of (1.3). We now claim that u_λ is nonzero for all $\lambda > 0$ large. □

Claim 1. There exists $\Lambda > 0$ such that $\varphi_\lambda(u_\lambda) < 0$ for all $\lambda \geq \Lambda$.

In order to prove the claim it suffices to exhibit an element $w \in X$ such that $\varphi_\lambda(w) < 0$ for all $\lambda > 0$ large. This is quite standard considering that $G(\delta) > 0$ by (F2). Indeed, letting $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ for $\varepsilon > 0$ small, define w so that $w(x) = \delta$ for $x \in \Omega_\varepsilon$ and $0 \leq w(x) \leq \delta$ for $x \in \Omega \setminus \Omega_\varepsilon$. Then

$$\begin{aligned}\varphi_\lambda(w) &= \frac{1}{p}\|w\|^p - \lambda\left(\int_{\Omega_\varepsilon} G(w) \, dx + \int_{\Omega\setminus\Omega_\varepsilon} G(w) \, dx\right) \\ &\leq \frac{1}{p}\|w\|^p - \lambda\left(G(\delta)\operatorname{meas}(\Omega_\varepsilon) - c(1+\delta^p)\operatorname{meas}(\Omega \setminus \Omega_\varepsilon)\right),\end{aligned} \tag{3.21}$$

where we note that the expression in the above parenthesis is positive if we choose $\varepsilon > 0$ sufficiently small. Therefore, there exists $\Lambda > 0$ such that $\varphi_\lambda(w) < 0$ for all $\lambda \geq \Lambda$, which proves the claim.

On the other hand, when $\Omega = B_R$, let u_λ^* denote the Schwarz symmetrization of u_λ, namely, u_λ^* is the unique radially symmetric, nonincreasing, nonnegative function in X which is equi-measurable with u_λ. As is well known,

$$\int_{B_R} G(u_\lambda^*)\, dx = \int_{B_R} G(u_\lambda)\, dx, \tag{3.22}$$

and $\|u_\lambda^*\| \leq \|u_\lambda\|$ (the Faber-Krahn inequality; see [8]), so that $\varphi_\lambda(u_\lambda^*) \leq \varphi_\lambda(u_\lambda)$. Therefore, we necessarily have $\varphi_\lambda(u_\lambda^*) = \varphi_\lambda(u_\lambda)$ and may assume that $u_\lambda = u_\lambda^*$. Moreover, $u_\lambda > 0$ in Ω by Lemma 3.6. Therefore, u_λ is a positive solution of both problems (1.1) and (1.3).

Next, we recall that $u = 0$ is a strict local minimum of φ_λ by Lemma 3.2. Therefore, since φ_λ satisfies the nonsmooth Palais-Smale condition by Lemma 3.4, we can use the minima $u = 0$ and $u = u_\lambda$ of φ_λ to apply the nonsmooth mountain pass theorem and conclude that there exists a second nontrivial critical point v_λ with $\varphi_\lambda(v_\lambda) > 0$. Again, v_λ is a nonnegative solution of problem (1.3) in view of Lemma 3.3. In addition, when $\Omega = B_R$, arguments similar to above passage show that we may assume $v_\lambda = v_\lambda^*$. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. As is well-known, the AR condition (F4) readily implies the existence of an element $e_\lambda \in X$ such that $\varphi_\lambda(e_\lambda) \leq 0$. On the other hand, Lemma 3.2 says that $u = 0$ is a (strict) local minimum of φ_λ and Lemma 3.4 says that φ_λ satisfies nonsmooth $(\mathrm{PS})_c$ for every $c \in \mathbb{R}$. Therefore, an application of the nonsmooth mountain pass theorem stated in section 2 yields the existence of a critical point v_λ such that

$$\varphi_\lambda(v_\lambda) > 0. \tag{3.23}$$

In particular, $v_\lambda \neq 0$, and it follows that v_λ is a nonnegative solution of problem (1.3) by Lemma 3.3. As in the proof of Theorem 1.1, we may assume that $v_\lambda = v_\lambda^*$ in the case of a ball $\Omega = B_R$.

Finally, still in the case of a ball $\Omega = B_R$, we claim that there exists $\lambda^* > 0$ such that, for all $0 < \lambda < \lambda^*$, $v_\lambda = v_\lambda^*$ is a positive solution of problem (1.3) (hence of problem (1.1)) having negative normal derivative on ∂B_R. Indeed, if that is not the case, then, for any given $\lambda > 0$, we can find $0 < \mu = \mu(\lambda) < \lambda$ such that the nonnegative solution $v_\mu = v_\mu^*$ of problem (1.1) with $\lambda = \mu$ obtained a bove satisfies

$$v_\mu(r) > 0 \quad \text{for } r \in [0, R_0), \qquad v_\mu'(R_0) = 0, \qquad v_\mu(r) = 0 \quad \text{for } r \in [R_0, R], \tag{3.24}$$

for some $0 < R_0 \leq R$. Therefore, the rescaled function $w_\mu(r) := v_\mu(R_0 r/R)$ is a positive solution of (1.1) with $\lambda = \mu_0$ (again in the ball B_R), with $\mu_0 := \mu R_0^p / R^p \leq \mu$. This shows that we can always construct a decreasing sequence $\mu_n > 0$ satisfying alternative (ii) of Theorem 1.2, in case alternative (i) does not hold. □

Acknowledgments

The author is very grateful to an anonymous referee for his or her valuable suggestions. Research supported by NSFC (no. 11061030, no. 10971087).

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