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Research Article

Modular Identities and Explicit Evaluations of a Continued Fraction of Ramanujan

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We study a new continued fraction of Ramanujan. We prove its modular identities and give some explicit evaluations.

1. Introduction

Throughout the paper, we assume $|q| < 1$. As usual, for positive integers n and any complex number a, we write

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1}\left(1 - aq^j\right), \qquad (a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n). \tag{1.1}$$

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \tag{1.2}$$

where $|ab| < 1$. After Ramanujan, we define

$$\phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty}, \tag{1.3}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$f(-q) := f\left(-q,-q^2\right) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty, \tag{1.5}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \left(-q;q^2\right)_\infty. \tag{1.6}$$

Ramanujan recorded many q-continued fractions and some of their explicit values in his second notebook [1] and in his lost notebook [2]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [3, p. 11, Entry 11]:

$$\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a-b}{1-q} + \frac{(a-bq)(aq-b)}{1-q^3} + \frac{q(a-bq^2)(aq^2-b)}{1-q^5} + \cdots, \tag{1.7}$$

where either q, a, and b are complex numbers with $|q| < 1$, or q, a, and b are complex numbers with $a = bq^m$ for some integer m. Several elegant q-continued fractions have representations as q-products and some of them can be expressed in terms of Ramanujan's theta-functions. An account of this can be found in in Chapter 32 of Berndt's book [4] (also see [5]). The most famous one, of course, is the Rogers-Ramanujan continued fraction $R(q)$ defined by

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots. \tag{1.8}$$

The continued fraction $R(q)$ has a very beautiful and extensive theory almost all of which was developed by Ramanujan. In particular, his lost notebook [2] contains several results on the Rogers-Ramanujan continued fraction. We refer to the paper by Berndt et al. [6], Kang [7, 8] for proofs of many of these theorems.

In this paper, we examine another continued fraction $T(q)$ of Ramanujan arising from (1.7) and is defined by

$$T(q) := \frac{q}{1-q^2} + \frac{q^4}{1-q^6} + \frac{q^8}{1-q^{10}} + \cdots. \tag{1.9}$$

Note that, replacing q by q^2 and then setting $a = q$ and $b = 0$ in (1.7), we obtain (1.9).

In Section 2, we record some preliminary results. Section 3 is devoted to prove some modular identities for the continued fraction $T(q)$. Finally, in Section 4, we give some explicit evaluations of $T(q)$.

We complete this introduction by defining Ramanujan's modular equation from Berndt's book [3]. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;k^2\right), \tag{1.10}$$

where $0 < k < 1$, ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.11}$$

holds for some positive integer n. Then, a modular equation of degree n is a relation between the moduli k and l which is implied by (1.11). If we set

$$q = \exp\left(-\pi\frac{K'}{K}\right), \qquad q' = \exp\left(-\pi\frac{L'}{L}\right), \tag{1.12}$$

we see that (1.11) is equivalent to the relation $q^n = q'$. Thus, a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n. Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. The multiplier m connecting α and β is defined by

$$m = \frac{K}{L}, \tag{1.13}$$

where $z_r = \phi^2(q^r)$.

2. Preliminary Results

In this section, we record some results that will be used in the subsequent sections.

Lemma 2.1 (see [3, p. 124, Entry 12(i) and (ii)]). *One has*

$$f(q) = \sqrt{z_1}2^{-1/6}\{\alpha(1-\alpha)\}^{1/24}q^{-1/24}, \qquad f(-q) = \sqrt{z_1}2^{-1/6}(1-\alpha)^{1/6}\alpha^{1/24}q^{-1/24}. \tag{2.1}$$

Lemma 2.2 (see [3, p. 214, Entry 24(iii)]). *If β has degree 2 over α, then*

$$\begin{aligned} m\sqrt{\alpha-1} + \sqrt{\beta} &= 1, \\ m^2\sqrt{\alpha-1} + \beta &= 1. \end{aligned} \tag{2.2}$$

Lemma 2.3 (see [3, p. 230, Entry 5(ii)]). *If β has degree 3 over α, then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \tag{2.3}$$

Lemma 2.4 (see [3, p. 215, (24.22)]). *If β has degree 4 over α, then*

$$\sqrt{\beta} = \left(\frac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^2. \tag{2.4}$$

Lemma 2.5 (see [3, p. 280-281, Entry 13(v) and (vi)]). *If β has degree 5 over α, then*

$$\begin{aligned} m &= \frac{1+\left((1-\beta)^5/(1-\alpha)\right)^{1/8}}{1+\left\{(1-\alpha)^3(1-\beta)\right\}^{1/8}}, \\ \frac{5}{m} &= \frac{1-\left((1-\alpha)^5/(1-\beta)\right)^{1/8}}{1-\left\{(1-\alpha)(1-\beta)^3\right\}^{1/8}}. \end{aligned} \tag{2.5}$$

Lemma 2.6 (see [3, p. 314, Entry 19(i)]). *If β has degree 7 over α, then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \tag{2.6}$$

3. Modular Identitites for $T(q)$

In this section, we use Ramanujan's modular equations to prove certain modular identities for $T(q)$.

Theorem 3.1. *One has*

$$T(q) = \frac{f(q)-f(-q)}{f(q)+f(-q)}. \tag{3.1}$$

Proof. Replacing q by q^2 and the setting $a = q$ and $b = 0$ in (1.7) and simplifying, we obtain

$$\frac{(-q;q^2)_\infty - (q;q^2)_\infty}{(-q;q^2)_\infty + (q;q^2)_\infty} = \frac{q}{1-q^2} + \frac{q^4}{1-q^6} + \frac{q^8}{1-q^{10}} + \cdots. \tag{3.2}$$

Employing (1.6) and (1.9) in (3.2) and simplifying, we complete the proof. □

Corollary 3.2. *One has*

$$\frac{1+T(q)}{1-T(q)} = \frac{f(q)}{f(-q)}. \tag{3.3}$$

Proof. Dividing numerator and denominator on right-hand side of the identity in Theorem 3.1 by $f(-q)$ and simplifying, we complete the proof. □

Theorem 3.3. *One has*

$$(i)\ \alpha = 1 - \left(\frac{1-T(q)}{1+T(q)}\right)^8, \qquad (ii)\ \beta = 1 - \left(\frac{1-T(q^n)}{1+T(q^n)}\right)^8, \tag{3.4}$$

where β has degree n over α.

Proof. We employ Lemma 2.1 in Corollary 3.2 to complete the proof. □

Theorem 3.4. *Let $u = T(q)$ and $v = T(-q)$. Then,*

$$u + v = 0. \tag{3.5}$$

Proof. Replacing q by $-q$ in Corollary 3.2, we obtain

$$\frac{1+T(-q)}{1-T(-q)} = \frac{f(-q)}{f(q)}. \tag{3.6}$$

Now, eliminating $f(q)/f(-q)$ between (3.6) and Corollary 3.2 and simplifying, we complete the proof. □

Theorem 3.5. *Let $u = T(q)$ and $v = T(q^2)$. Then,*

$$u^2 - v - 2u^2v - u^4v + 6u^2v^2 - v^3 - 2u^2v^3 - u^4v^3 + u^2v^4 = 0. \tag{3.7}$$

Proof. Eliminating m in (2.2) and then simplifying, we deduce that

$$\left(1+\beta+(\beta-1)\sqrt{1-\alpha}\right)^2 - 4\beta = 0. \tag{3.8}$$

From Theorem 3.3(i), we have

$$\sqrt{1-\alpha} = \left(\frac{1-T(q)}{1+T(q)}\right)^4. \tag{3.9}$$

Now, employing Theorem 3.3(ii) with $n = 2$ and (3.9) in (3.8) and factorizing using *Mathematica*, we obtain

$$(-1+v)^8\left(u^2 - v - 2u^2v - u^4v + 6u^2v^2 - v^3 - 2u^2v^3 - u^4v^3 + u^2v^4\right) \times \left(1 + 2u^2 + u^4 - 16u^2v + 6v^2 + 12u^2v^2 + 6u^4v^2 - 16u^2v^3 + v^4 + 2u^2v^4 + u^4v^4\right) = 0. \tag{3.10}$$

It can be seen that the first and the last factors in (3.10) do not vanish for $|q| \to 0$. So, by identity theorem, we have

$$u^2 - v - 2u^2v - u^4v + 6u^2v^2 - v^3 - 2u^2v^3 - u^4v^3 + u^2v^4 = 0. \tag{3.11}$$

□

Theorem 3.6. *Let $u = T(q)$ and $v = T(q^3)$. Then,*

$$u^3 - v - 3u^2v + 3uv^2 + 3u^3v^2 - 3u^2v^3 - u^4v^3 + uv^4 = 0. \tag{3.12}$$

Proof. From Lemma 2.3, we obtain

$$\alpha\beta - \left(1 - (1-\alpha)^{1/4}(1-\beta)^{1/4}\right)^4 = 0. \tag{3.13}$$

From Theorem 3.3, we deduce that

$$\alpha = 1 - \left(\frac{1-u}{1+u}\right)^8, \qquad \beta = 1 - \left(\frac{1-v}{1+v}\right)^8,$$
$$(1-\alpha)^{1/4} = \left(\frac{1-u}{1+u}\right)^2, \qquad (1-\beta)^{1/4} = \left(\frac{1-v}{1+v}\right)^2, \tag{3.14}$$

where β has degree 3 over α.

Employing (3.14) in (3.13) and factorizing using *Mathematica*, we arrive at

$$\left(-u^3 + v + 3u^2v - 3uv^2 - 3u^3v^2 + 3u^2v^3 + u^4v^3 - uv^4\right)$$
$$\times\left(-u + 3u^2v + u^4v - 3uv^2 - 3u^3v^2 + v^3 + 3u^2v^3 - u^3v^4\right) = 0. \tag{3.15}$$

It can be seen that the second factor of (3.15) does not vanish for $|q| \to 0$, so by identity theorem, we have

$$u^3 - v - 3u^2v + 3uv^2 + 3u^3v^2 - 3u^2v^3 - u^4v^3 + uv^4 = 0. \tag{3.16}$$

□

Theorem 3.7. *Let $u = T(q)$ and $v = T(q^4)$. Then,*

$$u^4 - v - 4u^2v + 2u^4v - 4u^6v - u^8v + 28u^4v^2 - 7v^3 - 28u^2v^3 + 14u^4v^3 - 28u^6v^3$$
$$- 7u^8v^3 + 70u^4v^4 - 7v^5 - 28u^2v^5 + 14u^4v^5 - 28u^6v^5 - 7u^8v^5 + 28u^4v^6 - v^7 - 4u^2v^7$$
$$+ 2u^4v^7 - 4u^6v^7 - u^8v^7 + u^4v^8 = 0. \tag{3.17}$$

Proof. Squaring the modular equation in Lemma 2.4 and simplifying, we obtain

$$\beta - \left(\frac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^4 = 0. \tag{3.18}$$

From Theorem 3.3(i), we have

$$(1-\alpha)^{1/4} = \left(\frac{1-T(q)}{1+T(q)}\right)^2. \tag{3.19}$$

Now, employing Theorem 3.3(ii) with $n = 4$ and (3.19) in (3.18) and simplifying, we complete the proof. □

Theorem 3.8. *Let $u = T(q)$ and $v = T(q^5)$. Then,*

$$u^5 - v - 5u^2v + 10u^3v^2 + 5u^5v^2 - 10u^2v^3 - 10u^4v^3 + 5uv^4 + 10u^3v^4 - 5u^4v^5 - u^6v^5 + uv^6 = 0. \tag{3.20}$$

Proof. From Theorem 3.3, we obtain

$$c := (1-\alpha)^{1/8} = \left(\frac{1-u}{1+u}\right), \qquad d := (1-\beta)^{1/8} = \left(\frac{1-v}{1+v}\right), \tag{3.21}$$

where β has degree 5 over α.

Employing (3.21) in (2.5), we find that

$$m = \frac{c+d^5}{c(1+c^3d)}, \tag{3.22}$$

$$\frac{5}{m} = \frac{d-c^5}{d(1-cd^3)}, \tag{3.23}$$

respectively.

Eliminating m between (3.22) and (3.23) and simplifying, we deduce that

$$5cd\left(1+c^3d\right)\left(1-cd^3\right) - \left(c+d^5\right)\left(d-c^5\right) = 0. \tag{3.24}$$

Substituting for c and d from (3.21) in (3.24) and simplifying, we arrive at

$$u^5 - v - 5u^2v + 10u^3v^2 + 5u^5v^2 - 10u^2v^3 - 10u^4v^3 + 5uv^4 + 10u^3v^4 - 5u^4v^5 - u^6v^5 + uv^6 = 0. \tag{3.25}$$

□

Theorem 3.9. *Let $u = T(q)$ and $v = T(q^7)$. Then,*

$$\begin{aligned} &u^8 - uv - 7u^3v - 7u^5v + 7u^7v + 28u^6v^2 - 7uv^3 - 49u^3v^3 + 7u^5v^3 - 7u^7v^3 + 70u^4v^4 \\ &\quad - 7uv^5 + 7u^3v^5 - 49u^5v^5 - 7u^7v^5 + 28u^2v^6 + 7uv^7 - 7u^3v^7 - 7u^5v^7 - u^7v^7 + v^8 = 0. \end{aligned} \tag{3.26}$$

Proof. From Lemma 2.6, we obtain

$$\alpha\beta - \left(1 - (1-\alpha)^{1/4}(1-\beta)^{1/4}\right)^8 = 0. \tag{3.27}$$

Again, from Theorem 3.3, we deduce that

$$\begin{aligned} \alpha &= 1 - \left(\frac{1-u}{1+u}\right)^8, \qquad \beta = 1 - \left(\frac{1-v}{1+v}\right)^8, \\ (1-\alpha)^{1/8} &= \left(\frac{1-u}{1+u}\right), \qquad (1-\beta)^{1/8} = \left(\frac{1-v}{1+v}\right), \end{aligned} \tag{3.28}$$

where β has degree 7 over α.

Employing (3.28) in (3.27) and simplifying using *Mathematica*, we arrive at

$$\begin{aligned} &u^8 - uv - 7u^3v - 7u^5v + 7u^7v + 28u^6v^2 - 7uv^3 - 49u^3v^3 + 7u^5v^3 - 7u^7v^3 + 70u^4v^4 \\ &\quad - 7uv^5 + 7u^3v^5 - 49u^5v^5 - 7u^7v^5 + 28u^2v^6 + 7uv^7 - 7u^3v^7 - 7u^5v^7 - u^7v^7 + v^8 = 0. \end{aligned} \tag{3.29}$$

□

4. Explicit Evaluations of $T(q)$

In this section, we establish some general theorems for the explicit evaluations of the continued fraction $T(q)$ and give examples.

For $q := e^{-\pi\sqrt{n}}$, Ramanujan's two class invariants G_n and g_n are defined by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q), \qquad g_n = 2^{-1/4}q^{-1/24}\chi(-q). \tag{4.1}$$

The class invariants G_n and g_n are connected by the relation [4, p. 187, Entry 2.1]:

$$g_{4n} = 2^{1/4}g_nG_n. \tag{4.2}$$

The singular modulus α_n is defined by $\alpha_n := \alpha(e^{-\pi\sqrt{n}})$, where n is a positive integer and unique positive number between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha_n)}{{}_2F_1(1/2, 1/2; 1; \alpha_n)}. \tag{4.3}$$

Class invariants and singular moduli are intimately related by the equalities [4, p. 185, (1.6)]:

$$G_n = (4\alpha_n(1-\alpha_n))^{-1/24}, \qquad g_n = \left(4\alpha_n(1-\alpha_n)^{-2}\right)^{-1/24}. \tag{4.4}$$

An account of Ramanujan's class invariants and singular moduli can be found in Chapter 34 of Berndt's book [4].

Theorem 4.1. *One has*

$$T\left(e^{-\pi\sqrt{n}}\right) = \frac{1-(1-\alpha_n)^{1/8}}{1+(1-\alpha_n)^{1/8}}. \tag{4.5}$$

Proof. We set $q := e^{-\pi\sqrt{n}}$ in Theorem 3.3(i) and use the definition of singular moduli α_n and simplifying, we complete the proof. □

In the scattered places of his first notebook [1], Ramanujan calculated over 30 singular moduli α_n. See Chapter 34 of Berndt's book [4] for details. Thus, one can use Theorem 4.1 to find the values of $T(e^{-\pi\sqrt{n}})$ if the corresponding values of α_n are known. For example, from [4, p. 281, Theorem 9.2], we note that

$$\alpha_2 = \left(\sqrt{2}-1\right)^2. \tag{4.6}$$

Employing (4.6) in Theorem 4.1, we calculate

$$T\left(e^{-\pi\sqrt{2}}\right) = \frac{1-\left(-2+2\sqrt{2}\right)^{1/8}}{1+\left(-2+2\sqrt{2}\right)^{1/8}}. \tag{4.7}$$

Many other values of $T(e^{-\pi\sqrt{n}})$ can be computed by using the known values of α_n.

Theorem 4.2. *One has*

$$T\left(e^{-\pi\sqrt{n}}\right) = \left(\frac{g_{4n}-2^{1/4}g_n^2}{g_{4n}-2^{1/4}g_n^2}\right). \tag{4.8}$$

Proof. Dividing numerator and denominator of right-hand side of Theorem 3.1 and employing (1.6), we obtain

$$T(q) = \frac{\chi(q)-\chi(-q)}{\chi(q)+\chi(-q)}. \tag{4.9}$$

Setting $q := e^{-\pi\sqrt{n}}$, employing the definitions of G_n and g_n from (4.1) in (4.9) and simplifying, we obtain

$$T\left(e^{-\pi\sqrt{n}}\right) = \frac{G_n - g_n}{G_n + g_n}. \tag{4.10}$$

Substituting for G_n from (4.2) in (4.10) and simplifying, we complete the proof. □

Theorem 4.2 implies that if we know the values of g_n and g_{4n} for any positive number n, then corresponding values of $T(e^{-\pi\sqrt{n}})$ can easily be calculated. Saikia [9] evaluated several values of g_n and g_{4n} for positive number n. For example, noting from [9, Theorem 3.5], we have

$$g_3 = 2^{-1/6}\left(2+\sqrt{3}\right)^{1/8}, \qquad g_{12} = 2^{1/6}\left(2+\sqrt{3}\right)^{1/8}. \tag{4.11}$$

Employing (4.11) in Theorem 4.2, we obtain

$$T\left(e^{-\pi\sqrt{3}}\right) = \frac{2 - 2^{3/4}\left(2+\sqrt{3}\right)^{1/8}}{2 + 2^{3/4}\left(2+\sqrt{3}\right)^{1/8}}. \tag{4.12}$$

Many other values of $T(e^{-\pi\sqrt{n}})$ can be determined by using the values of g_n and g_{4n} evaluated in [9].

References

[1] S. Ramanujan, *Notebooks*, vol. 1,2, Tata Institute of Fundamental Research, Bombay, India, 1957.

[2] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, India, 1988.

[3] B. C. Berndt, *Ramanujan's Notebooks*, part 3, Springer, New York, NY, USA, 1991.

[4] B. C. Berndt, *Ramanujan's Notebooks*, Part 4, Springer, New York, NY, USA, 1998.

[5] B. C. Berndt, "Flowers which we cannot yet see growing in Ramanujan's garden of hypergeometric series, elliptic functions, and q's," in *Special Functions 2000: Current Perspective and Future Directions*, J. Bustoz, M. E. H. Ismail, and S. K. Suslov, Eds., vol. 30, pp. 61–85, Kluwer Academic, Dordrecht, The Netherlands, 2001.

[6] B. C. Berndt, S. S. Huang, J. Sohn, and S. H. Son, "Some theorems on the Rogers-Ramanujan continued fraction in Ramanujan's lost notebook," *Transactions of the American Mathematical Society*, vol. 352, no. 5, pp. 2157–2177, 2000.

[7] S.-Y. Kang, "Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook," *The Ramanujan Journal*, vol. 3, no. 1, pp. 91–111, 1999.

[8] S.-Y. Kang, "Ramanujan's formulas for the explicit evaluation of the Rogers-Ramanujan continued fraction and theta-functions," *Acta Arithmetica*, vol. 90, no. 1, pp. 49–68, 1999.

[9] N. Saikia, "Ramanujan's modular equations and Weber-Ramanujan's class invariants G_n and g_n," *Bulletin of Mathematical Sciences*, vol. 2, no. 1, pp. 205–223, 2012.