Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 975760, 11 pages
doi:10.1155/2012/975760

Research Article

Quasilinearization Technique for Φ-Laplacian Type Equations

Inara Yermachenko[1] and Felix Sadyrbaev[1, 2]

[1] *Faculty of Natural Sciences and Mathematics, Daugavpils University, Daugavpils, LV-5400, Latvia*
[2] *Institute of Mathematics and Computer Science, University of Latvia, Riga, LV-1459, Latvia*

Correspondence should be addressed to Inara Yermachenko, inara.jermacenko@du.lv

Received 31 March 2012; Revised 8 August 2012; Accepted 4 September 2012

Academic Editor: Paolo Ricci

An equation $d/dt(\Phi(t,x')) + f(t,x) = 0$ is considered together with the boundary conditions $\Phi(a,x'(a)) = 0$, $x(b) = 0$. This problem under appropriate conditions can be reduced to quasilinear problem for two-dimensional differential system. The conditions for existence of multiple solutions to the original problem are obtained by multiply applying the quasilinearization technique.

1. Introduction

Consider the Φ-Laplacian type equation

$$\frac{d}{dt}\Phi(t,x') + f(t,x) = 0, \quad t \in I := [a,b], \tag{1.1}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R})$ is Lipschitz function with respect to x, $\Phi \in C(I \times \mathbb{R}, \mathbb{R})$ is Lipschitz and monotone function with respect to x', together with the boundary conditions

$$\Phi(a, x'(a)) = 0, \qquad x(b) = 0. \tag{1.2}$$

This equation (even in a greater generality) was intensively studied in the last time ([1–3] and references therein). If $\Phi(t,x') = x'$ then it reduces to

$$x'' + f(t,x) = 0. \tag{1.3}$$

The equation (1.1) can also be interpreted as the Euler equation for the functional

$$J(x) = \int_0^1 (\Psi(t, x') - F(t, x))dt, \tag{1.4}$$

where $\Phi(t, x') = (\partial\Psi(t, x')/\partial x')$ and $f(t, x) = (\partial F(t, x)/\partial x)$.

Our aim is to obtain the multiplicity results. For this we denote $y = \Phi(t, x')$ and rewrite (1.1) as a two-dimensional differential system of the form

$$\begin{aligned} x' &= \Phi^{-1}(t, y), \\ y' &= -f(t, x) \end{aligned} \tag{1.5}$$

and apply the quasilinearization process described in [4–7]. Namely, we reduce the system (1.5) to a quasilinear one of the form

$$\begin{aligned} x' - ky &= F_k(t, y), \\ y' + kx &= H_k(t, x), \end{aligned} \tag{1.6}$$

so that both systems (1.5) and (1.6) are equivalent in some domain $\Omega_k = \{(t, x, y) : a \leq t \leq b, |x| \leq N_x, |y| \leq N_y\}$ and moreover the extracted linear part $(LX)(t) := \binom{x'-ky}{y'+kx}$ is nonresonant with respect to the boundary conditions

$$y(a) = 0, \qquad x(b) = 0. \tag{1.7}$$

If any solution of the quasilinear problem (1.6), (1.7) satisfies the inequalities $|x(t)| \leq N_x$, $|y(t)| \leq N_y$ for all $t \in [a, b]$, then we say that the original problem for Φ-Laplacian type equation (1.1), (1.2) allows for quasilinearization.

If a solution $(x(t), y(t))$ of the problem (1.6), (1.7) is located in Ω_k, then this $(x(t), y(t))$ also solves the problem (1.5), (1.7) and therefore the respective $x(t)$ solves the original problem (1.1), (1.2). Notice that the type of a solution $x(t)$ to the problem (1.1), (1.2) is induced by oscillatory type of a solution $(x(t), y(t))$ to the quasilinear problem (1.6), (1.7), which, in turn, is defined by oscillatory properties of the extracted nonresonant linear part $(LX)(t)$ (see below).

If the original nonlinear problem allows for quasilinearization with respect to the linear parts with different types of nonresonance, then this problem is expected to have multiple solutions.

The paper is organized as follows. In Section 2 definitions are given. In Section 3 the main result is proved concerning the solvability of a quasilinear boundary value problem. Section 4 contains application of the main result and the quasilinearization technique for studying a nonlinear system; the numerical results are provided and a corresponding example was analyzed.

2. Definitions

Consider the quasilinear system (1.6), where functions F_k, H_k are continuous, bounded (i.e., there exists a positive constant K such that $|F_k| < K$ and $|H_k| < K$ for all values of arguments) and satisfy the Lipschitz conditions in y and x, respectively. Consider also the relevant homogeneous system

$$\begin{aligned} x' - ky &= 0, \\ y' + kx &= 0. \end{aligned} \tag{2.1}$$

Definition 2.1. A linear part $(LX)(t) := \binom{x'-ky}{y'+kx}$ is nonresonant with respect to the boundary conditions (1.7) if the homogeneous problem (2.1), (1.7) has only the trivial solution.

In order to classify the linear parts for different values of k let us introduce polar coordinates as

$$x(t) = r(t)\sin\varphi(t), \qquad y(t) = r(t)\cos\varphi(t). \tag{2.2}$$

Then the angular function $\varphi(t)$ for (2.1) satisfies $\varphi'(t) = k$. Suppose that $k > 0$, then $\varphi(t)$ is monotonically increasing and the boundary conditions (1.7) take the form $\varphi(a) = \pi/2$, $\varphi(b) = \pi n$, $n \in \mathbb{N}$.

Definition 2.2. One says that a linear part $(LX)(t)$ in (2.1) is i-nonresonant with respect to the boundary conditions (1.7) if the angular function $\varphi(t)$, defined by the initial condition $\varphi(a) = \pi/2$, takes exactly i times $(i = 0, 1, \ldots)$ values of the form πn in the interval (a, b) and $\varphi(b) \neq \pi n$, $n \in \mathbb{N}$.

The linear part $(LX)(t)$ in (2.1) is nonresonant with respect to the boundary conditions (1.7) if the coefficient $k > 0$ satisfies $\cos k(b-a) \neq 0$, this means that k belongs to a certain interval of

$$\left(0, \frac{\pi}{2(b-a)}\right), \left(\frac{\pi}{2(b-a)}, \frac{3\pi}{2(b-a)}\right), \ldots, \left(\frac{(2i-1)\pi}{2(b-a)}, \frac{(2i+1)\pi}{2(b-a)}\right), \ldots, \tag{2.3}$$

$i = 1, 2, 3, \ldots$. So the linear part $(LX)(t)$ is 0-nonresonant with respect to the boundary conditions (1.7) if $0 < k < (\pi/2(b-a))$ and it is i-nonresonant with respect to the boundary conditions mentioned above if

$$k \in \left(\frac{(2i-1)\pi}{2(b-a)}, \frac{(2i+1)\pi}{2(b-a)}\right), \quad i \in \mathbb{N}. \tag{2.4}$$

Let $(\xi(t), \eta(t))$ be a solution of the quasilinear problem (1.6), (1.7).

Definition 2.3. One says that $(x(t;\delta), y(t;\delta))$ is a neighboring solution of a solution $(\xi(t), \eta(t))$, if $(x(t;\delta), y(t;\delta))$ solves the same system (1.6), satisfies the condition $y(a;\delta) = 0$ and there exists $\varepsilon > 0$ such that for all $\delta \in (0, \varepsilon]$ $x(a;\delta) = \xi(a) + \delta$.

In order to classify solutions of the quasilinear problem under consideration introduce local polar coordinates for the difference between neighboring solution $(x(t;\delta), y(t;\delta))$ and investigated solution $(\xi(t), \eta(t))$ as

$$x(t;\delta) - \xi(t) = \rho(t)\sin\Theta(t;\delta), \qquad y(t;\delta) - \eta(t) = \rho(t)\cos\Theta(t;\delta), \tag{2.5}$$

where $\Theta(a;\delta) = \pi/2$ and $\rho(a) = \delta$.

Definition 2.4. One says that $(\xi(t), \eta(t))$ is an i type solution of the problem (1.6), (1.7), if there exists $\varepsilon > 0$ such that for any $\delta \in (0, \varepsilon]$ the angular function $\Theta(t;\delta)$, defined by the initial condition $\Theta(a;\delta) = \pi/2$, takes exactly i values of the form πn in the interval (a, b) and $\Theta(b;\delta) \neq \pi n,\ n \in \mathbb{N}$.

Remark 2.5. If $(\xi(t), \eta(t))$ is an i type solution of (1.6), (1.7), then the angular function $\Theta(t;\delta)$ in (2.5) satisfies the inequalities

$$\begin{gathered} \frac{\pi}{2} < \Theta(b;\delta) < \pi \quad \text{if } i = 0, \\ i\pi < \Theta(b;\delta) < (i+1)\pi \quad \text{if } i \neq 0. \end{gathered} \tag{2.6}$$

3. Results for Quasilinear Systems

Consider the quasilinear system (1.6), where the linear part $(LX)(t) := \begin{pmatrix} x'-ky \\ y'+kx \end{pmatrix}$ is nonresonant with respect to the boundary conditions (1.7) and functions F_k, H_k are continuous, bounded and satisfy the Lipschitz conditions with respect to y and x, respectively. By a solution we mean a two-dimensional vector function (x, y) with continuously differentiable components an element of the space $C_2^1([a,b] \times \mathbb{R}^2, \mathbb{R})$.

Lemma 3.1. *A set $\mathbb{S}$ of solutions to the problem* (1.6), (1.7) *is nonempty and compact in* $C_2^1([a,b] \times \mathbb{R}^2, \mathbb{R})$.

Proof. The problem (1.6), (1.7) has a solution if the right sides F_k and H_k are bounded. This can be proved by direct application of Schauder fixed point theorem and follows from the well-known results ([8, 9], for instance). The Existence Theorem of [9][Ch. 2, § 2] when adapted for the problem (1.6), (1.7) says that this problem is solvable if the homogeneous one (2.1), (1.7) has only the trivial solution. This is the case since the nonresonance condition $\cos k(b-a) \neq 0$ fulfils.

Compactness follows from the integral representation of a solution of the problem (1.6), (1.7) via the Green's matrix (4.14) and standard evaluations in order to show that the Arzela-Ascoli criterium is satisfied. □

Lemma 3.2. *There exists a maximal solution $X_{\max} = (x^*, y^*)$ of quasilinear problem* (1.6), (1.7) *with the property that $y^*(a) = 0$ and $x^*(a) = \max\{x(a) : (x, y) \in \mathbb{S}, y(a) = 0\}$. Similarly there exists a minimal solution $X_{\min} = (x_*, y_*)$ of* (1.6), (1.7) *with a property $y_*(a) = 0$ and $x_*(a) = \min\{x(a) : (x, y) \in \mathbb{S}, y(a) = 0\}$.*

Proof. A set $\mathbb{S}_1 = \{(x(a), y(a)) \in R^2 : (x, y) \in \mathbb{S}\}$ is the image of a continuous map $M : C_2^1([a,b]) \to R^2$ defined by $M(x,y) = (x(a), y(a))$. Since $\mathbb{S}$ is compact $\mathbb{S}_1$ is compact also. Moreover $\mathbb{S}_1$ is compact in a straight line $y = 0$. Thus $\mathbb{S}_1$ is bounded and closed and therefore there exist the maximal and the minimal elements. The case of $X_{\max} = X_{\min}$ corresponds to a unique solution of the BVP (1.6), (1.7). □

Lemma 3.3. *Suppose that the linear part $(LX)(t)$ in* (1.6) *is i-nonresonant with respect to the boundary conditions* (1.7). *Let $(\xi(t), \eta(t))$ be any element of $\mathbb{S}$. Then the angular function $\Theta(t;\delta)$ introduced by* (2.5) *for large enough δ takes exactly i times values of the form πn, $n \in \mathbb{N}$ in the interval (a,b) and $\Theta(b;\delta) \neq \pi n$.*

Proof. Consider the neighboring solution $(x(t;\delta), y(t;\delta))$ (see Definition 2.3). Notice that both $(\xi(t), \eta(t))$ and $(x(t;\delta), y(t;\delta))$ are solutions of (1.6) and $\Theta(a;\delta) = \pi/2$. The normalized functions $u = (1/\delta)(x(t;\delta) - \xi(t))$ and $v = (1/\delta)(y(t;\delta) - \eta(t))$ satisfy the system

$$u' - kv = \frac{1}{\delta}\left[F_k(t, y(t;\delta)) - F_k(t, \eta(t))\right],$$
$$v' + ku = \frac{1}{\delta}\left[H_k(t, x(t;\delta)) - H_k(t, \xi(t))\right]. \tag{3.1}$$

The right sides in (3.1) tend to zero uniformly in $t \in [a,b]$ as $\delta \to +\infty$ since F_k and H_k are bounded functions. The functions $u(t)$, $v(t)$ tend to solutions $x(t)$, $y(t)$ of the homogeneous equation (2.1), which satisfy the initial conditions $\varphi(a) = \pi/2$, $x(a) = 1$, where $\varphi(t)$ is the angular function for $(x(t), y(t))$. Therefore $\Theta(t;\delta) \to \varphi(t)$ as $\delta \to +\infty$, uniformly in $t \in [a,b]$. As a consequence, $\Theta(b;\delta)$ takes exactly i times values of the form πn together with $\varphi(b)$. □

The main theorem follows.

Theorem 3.4. *If a linear part $(LX)(t)$ in the quasilinear system* (1.6) *is i-nonresonant with respect to the boundary conditions* (1.7), *then the quasilinear problem* (1.6), (1.7) *has an i type solution.*

Proof. Consider a solution $X_{\max} = (\xi^*, \eta^*)$, mentioned in Lemma 3.2 and neighboring solutions $(x(t;\delta), y(t;\delta))$ (see Definition 2.3). We claim that $X_{\max}$ is an i type solution to the problem. Suppose that this is not true. According to (2.6) there are two possibilities.

Case 1. For any $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that $\Theta(b;\delta) = \pi n$ (for some natural value of n). Therefore $(x(t;\delta), y(t;\delta))$ solves the BVP (1.6), (1.7) as well. Since $x(a;\delta) - \xi^*(a) = \delta > 0$ by virtue of (2.5), that is, $x(a;\delta) > \xi^*(a)$, a solution $X_{\max} = (\xi^*, \eta^*)$ is not maximal in the sense of Lemma 3.2 This case is ruled out.

Case 2. $\Theta(b;\delta) \notin [i\pi, (i+1)\,\pi]$. Then there exists small positive δ_1 such that $j\pi \le \Theta(b;\delta_1) \le (j+1)\,\pi$, where $j \neq i$. By Lemma 3.3 exists δ_2 such that for all $\delta \ge \delta_2 \Theta(b;\delta)$ satisfies

$$i\pi < \Theta(b;\delta) < (i+1)\pi. \tag{3.2}$$

Since $\Theta(b;\delta)$ is continuous then there exists $\delta_* \in [\delta_1, \delta_2]$ such that $\Theta(b;\delta_*) = \pi n$. It follows again that $(x(t;\delta_*), y(t;\delta_*))$ is a solution of the BVP (1.6), (1.7). Therefore $x(a;\delta_*) - \xi^*(a) = \delta_* > 0$ and $X_{\max}$ is not a maximal solution. The obtained contradiction completes the proof. □

4. Application

Consider the differential equation

$$\frac{d}{dt}\Phi(t,x') + f(t,x) = 0, \tag{4.1}$$

where $\Phi(t,x') = r(t)|x'|^{1/p}\operatorname{sgn} x'$, $f(t,x) = q(t)|x|^p \operatorname{sgn} x$, $t \in I := [a,b]$, $p > 1$, $r,q \in C(I;(0,+\infty))$ together with the boundary conditions

$$\Phi(a,x'(a)) = 0, \qquad x(b) = 0, \tag{4.2}$$

which in polar coordinates take the form $\varphi(a) = \pi/2, \varphi(b) = \pi n,\ n \in \mathbb{N}$.

It is worth mentioning that the problem of minimizing the functional

$$J(x) = \int_a^b \left[pr(t)\left|x'\right|^{((1+p)/p)} - q(t)|x|^{p+1}\right]dt, \tag{4.3}$$

with respect to the class of curves joining an arbitrary point of the line $t = a$ with a given point $(b,0)$ leads just to the boundary value problem (4.1), (4.2) [10][Ch. 1, Sec. 6].

Denote $\Phi(t,x') = y$, then obtain a two-dimensional differential system

$$\begin{aligned} x' &= (r(t))^{-p}\left|y\right|^p \operatorname{sgn} y, \\ y' &= -q(t)|x|^p \operatorname{sgn} x, \end{aligned} \tag{4.4}$$

together with the boundary conditions

$$y(a) = 0, \qquad x(b) = 0. \tag{4.5}$$

The obtained system (4.4) is equivalent to a system

$$\begin{aligned} x' - ky &= (r(t))^{-p}\left|y\right|^p \operatorname{sgn} y - ky, \\ y' + kx &= kx - q(t)|x|^p \operatorname{sgn} x, \end{aligned} \tag{4.6}$$

where the coefficient $k > 0$ satisfies $\cos k(b-a) \neq 0$. This means if coefficient $k \in (((2i-1)\pi/2(b-a)),((2i+1)\pi/2(b-a)))$, $i \in \mathbb{N}$ then extracted linear part $(LX)(t)$ in (4.6) is i-nonresonant with respect to the boundary conditions (4.5).

Denote $U_k(t,y) := (r(t))^{-p}|y|^p \operatorname{sgn} y - ky$.

Function $U_k(t,y)$ is odd in y for fixed $t = t^*$. We calculate the value of this function at the point of local extremum y_0. Set

$$m_y(t^*) = \left|U_k(t^*,y_0)\right| = \left(\frac{k}{p}\right)^{p/(p-1)}(p-1)(r(t^*))^{p/(p-1)}. \tag{4.7}$$

Choose $n_y(t^*)$ such that $|y| \le n_y(t^*) \Rightarrow |U_k(t^*, y)| \le m_y(t^*)$. Computation gives that

$$n_y(t^*) = k^{1/(p-1)}(r(t^*))^{p/(p-1)}\gamma, \tag{4.8}$$

where a constant γ is a root of the equation $\gamma^p = \gamma + (p-1)p^{(p/(1-p))}$.

Similarly we transform the function $V_k(t,x) := kx - q(t)|x|^p \operatorname{sgn} x$ and instead of the functions $U_k(t,y)$, $V_k(t,x)$ consider

$$\begin{aligned} \widehat{U_k(t,y)} &:= U_k(t, \Delta(-N_y, y, N_y)), \\ \widehat{V_k(t,x)} &:= V_k(t, \Delta(-N_x, x, N_x)), \end{aligned} \tag{4.9}$$

where the truncation function Δ is given by

$$\Delta(x,y,z) = \begin{cases} x, & y < x \\ y, & x \le y \le z \\ z, & y > z, \end{cases} \tag{4.10}$$

$N_y = \min\{n_y(t) : t \in [a,b]\}$ and $N_x = \min\{n_x(t) : t \in [a,b]\}$, besides

$$\begin{aligned} \sup\left|\widehat{U_k(t,y)}\right| &= M_y = \max\{m_y(t) : t \in [a,b]\}, \\ \sup\left|\widehat{V_k(t,x)}\right| &= M_x = \max\{m_x(t) : t \in [a,b]\}. \end{aligned} \tag{4.11}$$

The nonlinear system (4.6) and the quasilinear one,

$$\begin{aligned} x' - ky &= \widehat{U_k(t,y)}, \\ y' + kx &= \widehat{V_k(t,x)}, \end{aligned} \tag{4.12}$$

are equivalent in a domain

$$\Omega_k = \{(t,x,y) : a \le t \le b, |x(t)| \le N_x, |y(t)| \le N_y\}. \tag{4.13}$$

The modified quasilinear problem (4.12), (4.5) is solvable if k belongs to one from the intervals mentioned above. The respective solution $(x_k(t), y_k(t))$ can be written in the integral form

$$\begin{aligned} x_k(t) &= \int_a^b \left(G_k^{11}(t,s)\widehat{U_k(s,y(s))} + G_k^{12}(t,s)\widehat{V_k(s,x(s))} \right) ds, \\ y_k(t) &= \int_a^b \left(G_k^{21}(t,s)\widehat{U_k(s,y(s))} + G_k^{22}(t,s)\widehat{V_k(s,x(s))} \right) ds, \end{aligned} \tag{4.14}$$

where $G_k^{ij}(t,s)$ $(i,j=1,2)$ are the elements of the Green's matrix to the respective homogeneous problem

$$\begin{aligned} x' - ky &= 0, \\ y' + kx &= 0, \end{aligned} \quad y(a)=0,\ x(b)=0. \tag{4.15}$$

Then

$$\begin{aligned} |x_k(t)| &\le (b-a)\bigl(\Gamma_{11}(k)\cdot M_y + \Gamma_{12}(k)\cdot M_x\bigr), \\ |y_k(t)| &\le (b-a)\bigl(\Gamma_{21}(k)\cdot M_y + \Gamma_{22}(k)\cdot M_x\bigr), \end{aligned} \tag{4.16}$$

where $\Gamma_{ij}(k)(i,j=1,2)$ are the best estimates (which are known precisely) of the respective elements $G_k^{ij}(t,s)$ of the Green's matrix.

If the inequalities

$$\begin{aligned} (b-a)\bigl(\Gamma_{11}(k)\cdot M_y + \Gamma_{12}(k)\cdot M_x\bigr) &< N_x, \\ (b-a)\bigl(\Gamma_{21}(k)\cdot M_y + \Gamma_{22}(k)\cdot M_x\bigr) &< N_y \end{aligned} \tag{4.17}$$

hold then the nonlinear problem (4.6), (4.5) (or, equivalently, the original problem (4.1), (4.2)) allows for quasilinearization and therefore has a solution of definite type.

Since the Green's matrix of the homogeneous linear problem (4.15) is given by

$$\mathbb{G}_k(t,s) = \begin{cases} \dfrac{1}{\triangle}\begin{pmatrix} -\sin(k(s-a))\sin(k(b-t)) & -\cos(k(s-a))\sin(k(b-t)) \\ \sin(k(s-a))\cos(k(b-t)) & \cos(k(s-a))\cos(k(b-t)) \end{pmatrix} & \\ \hfill \text{if } a \le s \le t \le b, & \\ \dfrac{1}{\triangle}\begin{pmatrix} -\cos(k(t-a))\cos(k(b-s)) & -\cos(k(t-a))\sin(k(b-s)) \\ \sin(k(t-a))\cos(k(b-s)) & \sin(k(t-a))\sin(k(b-s)) \end{pmatrix} & \\ \hfill \text{if } a \le t < s \le b, & \end{cases} \tag{4.18}$$

where $\triangle = \cos k(b-a)$, therefore

$$\left|G_k^{ij}(t,s)\right| \le \frac{1}{|\cos k(b-a)|} =: \Gamma_k, \quad (i,j=1,2). \tag{4.19}$$

Suppose that $0 < r_1 \le r(t) \le r_2$ and $0 < q_1 \le q(t) \le q_2$ for all $t \in [a,b]$.

Taking into consideration the expressions for M_y, M_x, N_y, N_x, γ, and the estimate Γ_k we obtain that both inequalities in (4.17) hold if the following inequality is fulfilled

$$\frac{k(b-a)}{|\cos k(b-a)|}\cdot p^{p/(1-p)}\cdot(p-1)\cdot\left(r_2^{p/(p-1)} + q_1^{1/(1-p)}\right) < A\cdot\gamma, \tag{4.20}$$

where $A = \min\{r_1^{p/(p-1)}, q_2^{1/(1-p)}\}$.

Thus a fulfilment of the inequality (4.20) is a sufficient condition for existence of a solution of definite oscillatory type to the problem (4.1), (4.2).

Depending on the functions $r(t)$ and $q(t)$ and parameter p there are 4 different possible cases. Denote:

$$\begin{aligned}
\mu &= \frac{1}{r_2^p \cdot q_2}, \quad \text{if } r_1^{-p} < q_2,\ r_2^{-p} < q_1 \\
\mu &= r_1^p \cdot q_1, \quad \text{if } r_1^{-p} > q_2,\ r_2^{-p} > q_1 \\
\mu &= \left(\frac{r_1}{r_2}\right)^p, \quad \text{if } r_1^{-p} > q_2,\ r_2^{-p} < q_1 \\
\mu &= \frac{q_1}{q_2}, \quad \text{if } r_1^{-p} < q_2,\ r_2^{-p} > q_1,
\end{aligned} \tag{4.21}$$

then inequality (4.20) is fulfilled if the following inequality holds

$$\frac{2k(b-a)}{|\cos\, k(b-a)|} \cdot p^{p/(1-p)} \cdot (p-1) \cdot \mu^{1/(1-p)} < \gamma. \tag{4.22}$$

The following theorem is valid.

Theorem 4.1. *Suppose that functions $r(t)$ and $q(t)$ in the Φ-Laplacian type equation* (4.1) *are such that $0 < r_1 \le r(t) \le r_2$ and $0 < q_1 \le q(t) \le q_2$ for all $t \in [a,b]$. If there exists some number $k \in (((2i-1)\pi/2(b-a)), ((2i+1)\pi/2(b-a)))$, $i \in \mathbb{N}$, which satisfies the inequality*

$$\frac{2k(b-a)}{|\cos k(b-a)|} \cdot p^{p/(1-p)} \cdot (p-1) \cdot \mu^{1/(1-p)} < \gamma, \tag{4.23}$$

where γ is a root of the equation $\gamma^p = \gamma + (p-1) \cdot p^{p/(1-p)}$ and μ is number of the form (4.21), *then there exists an i type solution of the nonlinear problem* (4.1), (4.2).

Corollary 4.2. *If there exist numbers $k_j \in (((2j-1)\pi/2(b-a)), ((2j+1)\pi/2(b-a)))$, $j = 1, 2, \ldots, n$, which satisfy the inequality* (4.23), *then there exist at least n solutions of different types to the problem* (4.1), (4.2).

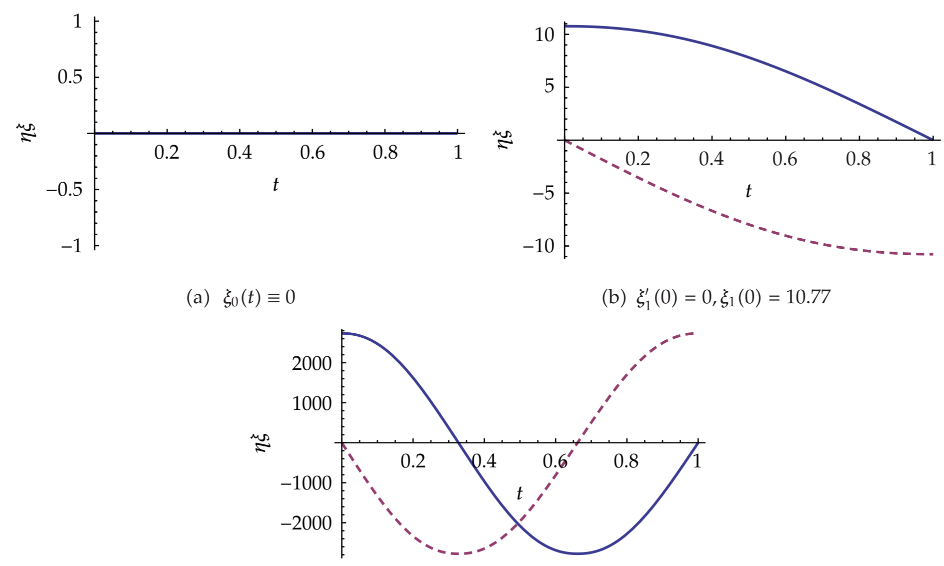
Denote: τ_i is a root of the equation $\tau = -\cot\tau$, which belongs to the interval $(((2i-1)\pi/2), \pi i)$, $i \in \mathbb{N}$. If the inequality

$$2\tau_i \cdot p^{p/(1-p)} \cdot (p-1) \cdot \mu^{1/(1-p)} < \gamma \tag{4.24}$$

holds then (4.23) is fulfilled also. The results of calculations are provided in Table 1. For certain values of p and μ this table shows which numbers k_{i-1} of the form $k_{i-1} = (\tau_i/(b-a))$, $i \in \mathbb{N}$ satisfy the inequality (4.23). The subscript of number k in Table 1 indicates that nonlinear problem under consideration has a solution of definite type, for instance, k_0, k_1 show that there exist 0 type and 1 type solutions.

Table 1: Results of calculations for the problem (4.1), (4.2).

p	γ	μ	k_i
3/2	1.2509	$\mu \geq 0.8390$	$k_0;\ k_1$
4/3	1.2703	$\mu \geq 0.7903$	$k_0;\ k_1$
5/4	1.2813	$\mu \geq 0.7852$	$k_0;\ k_1$
		$\mu \geq 0.9437$	$k_0;\ k_1;\ k_2$
6/5	1.2884	$\mu \geq 0.7907$	$k_0;\ k_1$
		$\mu \geq 0.9161$	$k_0;\ k_1;\ k_2$
		$\mu \geq 0.9949$	$k_0;\ k_1;\ k_2;\ k_3$
7/6	1.2933	$\mu \geq 0.7991$	$k_0;\ k_1$
		$\mu \geq 0.9034$	$k_0;\ k_1;\ k_2$
		$\mu \geq 0.9702$	$k_0;\ k_1;\ k_2;\ k_3$



(a) $\xi_0(t) \equiv 0$

(b) $\xi_1'(0) = 0, \xi_1(0) = 10.77$

(c) $\xi_2'(0) = 0$, $\xi_2(0) = 2733.569$

Figure 1: Different type solutions of the problem (4.25).

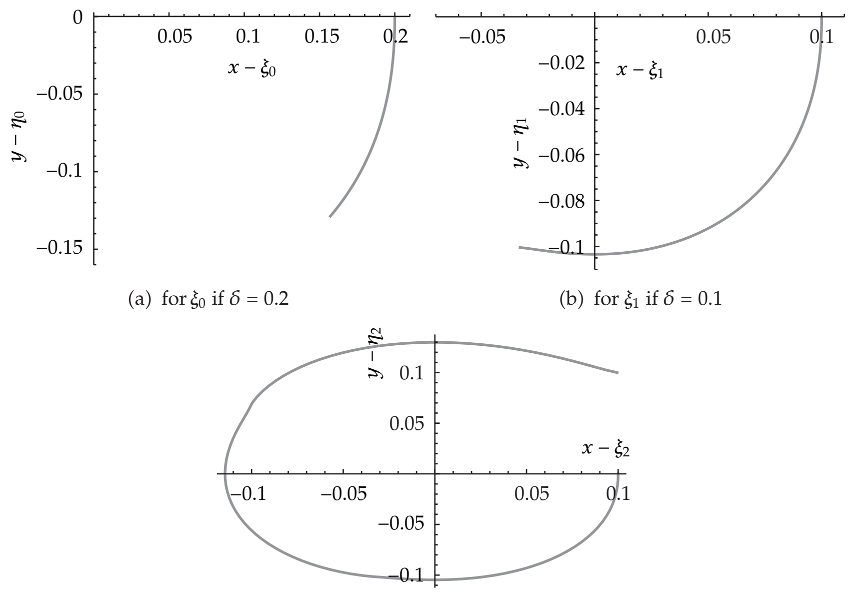
4.1. Example

Consider the problem

$$\frac{d}{dt}\left(\left|x'\right|^{5/6} \operatorname{sgn} x'\right) + 0.04(\cos \pi t + 25)|x|^{6/5} \operatorname{sgn} x = 0, \quad x'(0) = 0,\ x(1) = 0, \tag{4.25}$$

which is a special case of the problem (4.1), (4.2) with $p = 6/5$, $r(t) \equiv 1$, and $q(t) = 0.04(\cos \pi t + 25)$.

For all $t \in [0,1]$ $0.96 \leq q(t) \leq 1.04$, since $q_2 > 1 = r_1^{-p}$ and $q_1 < 1 = r_2^{-p}$, then $\mu = q_1/q_2$, $\mu \approx 0.923$.



(a) for ξ_0 if $\delta = 0.2$

(b) for ξ_1 if $\delta = 0.1$

(c) for ξ_2 if $\delta = 0.1$

Figure 2: Phase portraits of the differences between solution ξ_i $(i = 0, 1, 2)$ of (4.25) and respective neighboring solution in the interval $t \in [0, 1]$.

In accordance with calculations (see Table 1) there exist at least three different solutions of the problem (4.25) of 0 type, 1 type and 2 type, respectively. We have computed them (see Figures 1 and 2).

References

[1] A. Cabada and R. L. Pouso, “Extremal solutions of strongly nonlinear discontinuous second-order equations with nonlinear functional boundary conditions,” *Nonlinear Analysis*, vol. 42, no. 8, pp. 1377–1396, 2000.

[2] M. Cherpion, C. de Coster, and P. Habets, “Monotone iterative methods for boundary value problems,” *Differential and Integral Equations*, vol. 12, no. 3, pp. 309–338, 1999.

[3] C. Bereanu and J. Mawhin, “Existence and multiplicity results for some nonlinear problems with singular φ-Laplacian,” *Journal of Differential Equations*, vol. 243, no. 2, pp. 536–557, 2007.

[4] I. Yermachenko and F. Sadyrbaev, “Types of solutions and multiplicity results for two-point nonlinear boundary value problems,” *Nonlinear Analysis*, vol. 63, no. 5–7, pp. e1725–e1735, 2005.

[5] I. Yermachenko and F. Sadyrbaev, “Quasilinearization and multiple solutions of the Emden-Fowler type equation,” *Mathematical Modelling and Analysis*, vol. 10, no. 1, pp. 41–50, 2005.

[6] I. Yermachenko and F. Sadyrbaev, “Multiplicity results for two-point nonlinear boundary value problems,” *Studies of the University of Žilina, Mathematical Series*, vol. 20, no. 1, pp. 63–72, 2006.

[7] F. Sadyrbaev and I. Yermachenko, “Multiple solutions of two-point nonlinear boundary value problems,” *Nonlinear Analysis*, vol. 71, no. 12, pp. e176–e185, 2009.

[8] R. Conti, “Equazioni differenziali ordinarie quasilineari con condizioni lineari,” *Annali di Matematica Pura ed Applicata*, vol. 57, pp. 49–61, 1962.

[9] N. I. Vasilyev and A. Y. Klokov, *Foundations of the Theory of Boundary Value Problems for Ordinary Differential Equations*, Zinatne, Riga, Latvia, 1978.

[10] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1963.