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Research Article

Some Nonunique Common Fixed Point Theorems in Symmetric Spaces through $\mathrm{CLR}_{(S,T)}$ Property

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We introduce a new class of mappings satisfying the "common limit range property" in symmetric spaces and utilize the same to establish common fixed point theorems for such mappings in symmetric spaces. Our results generalize and improve some recent results contained in the literature of metric fixed point theory. Some illustrative examples to highlight the realized improvements are also furnished.

1. Introduction

In 1986, Jungck [1] generalized the idea of weakly commuting pair of mappings due to Sessa [2] by introducing the notion of compatible pair of mappings and also showed that compatible pair of mappings commute on the set of coincidence points of the involved mappings. Recall that a point $x \in X$ is called a coincidence point of the pair of self-mappings (f, g) defined on X if $fx = gx(= w)$ while the point w is then called a point of coincidence for the pair (f, g). In the recent past and even now, the concept of compatible mappings is frequently used to prove results on the existence of common fixed points. The study of common fixed points of noncompatible pairs is also equally natural and fascinating. Pant [3] initiated the study of noncompatible pairs employing the idea of pointwise R-weakly commuting pairs. Pant [4] proved an interesting fixed point theorem for maps satisfying Lipschitz type conditions. In recent years, the result of Pant [4] was generalized and improved by Sastry and Murthy [5] (see also [6]) by introducing the idea of tangential maps (or the property (E.A)) and g-continuity. In continuation of this, Imdad and Soliman [7] and Soliman et al. [8] extended the results of Sastry and Murthy [5] as well as Pant [4] to symmetric space utilizing the idea of weakly compatible pair together with common property (E.A) (a notion due to Liu et al. [9]). For more references on the recent development of common fixed point theory in symmetric spaces, we refer readers to [10–14]. Most recently, Gopal et al. [15] improved these results by utilizing the idea of absorbing pair which is essentially due to Gopal et al. [16].

In this paper, we introduce a new notion called the common limit range property and show that this new notion buys a typically required condition up to a pair of mappings along with the notion of absorbing property in proving common fixed point theorems for Lipschitz type mappings in symmetric spaces. Consequently, the relevant recent fixed point theorems due to Soliman et al. [8] and Gopal et al. [15] are generalized and improved.

2. Preliminaries

A symmetric d on a nonempty set X is a function $d : X\times X \to [0,\infty)$ which satisfies $d(x, y) = d(y, x)$ and $d(x, y) = 0 \Leftrightarrow x = y$ (for all $x, y \in X$). If d is a symmetric on a set X, then for $x \in X$ and $\epsilon > 0$, we write $B(x,\epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is given by the sets U (along with empty set) in which for each $x \in U$, one can find some $\epsilon > 0$ such that $B(x,\epsilon) \subset U$. A set $S \subset X$ is a neighbourhood of $x \in X$ if and only if there is a U containing x such that $x \in$

$U \subset S$. A symmetric d is said to be a semimetric if for each $x \in X$ and for each $\epsilon > 0$, $B(x,\epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. Thus a symmetric (resp. a semimetric) space X is a topological space whose topology $\tau(d)$ on X is induced by a symmetric (resp. a semimetric) d. Notice that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$. The distinction between a symmetric and a semimetric is apparent as one can easily construct a semimetric d such that $B(x,\epsilon)$ need not be a neighbourhood of x in $\tau(d)$.

Since symmetric spaces are not necessarily Hausdorff and the symmetric d is not generally continuous, in the course of proving fixed point theorems, some additional axioms are required. The following axioms are relevant to this note which are available in the papers of Aliouche [17], Galvin and Shore [18], Hicks and Rhoades [19], and Wilson [20].

(W_3) [20] Given $\{x_n\}$, x and y in X with $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply $x = y$.

(W_4) [20] Given $\{x_n\}, \{y_n\}$ and an x in X with $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply $d(y_n, x) \to 0$.

(HE) [17] Given $\{x_n\}, \{y_n\}$ and an x in X with $d(x_n, x) \to 0$ and $d(y_n, x) \to 0$ imply $d(x_n, y_n) \to 0$.

($1C$) [18] A symmetric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

(CC) [18] A symmetric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where x_n, y_n are sequences in X and $x, y \in X$.

Clearly, (CC) implies ($1C$) but not conversely. Also (W_4) implies (W_3) and ($1C$) implies (W_3) but converse implications are not true. All other possible implications amongst (W_3), ($1C$), and (HE) are not true in general. A nice illustration via demonstrative examples is given by Cho et al. [21]. However, (CC) implies all the remaining four conditions namely: (W_3), (W_4), (HE), and ($1C$).

Recall that a sequence $\{x_n\}$ in a semimetric space (X, d) is said to be d-Cauchy if it satisfies the usual metric condition. Here, one needs to notice that in a semimetric space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if semimetric is suitably restricted (see Wilson [20]). In [22], Burke furnished an illustrative example to show that a convergent sequence in semimetric spaces need not admit a Cauchy subsequence. He was able to formulate an equivalent condition under which every convergent sequence in semimetric space admits a Cauchy subsequence. There are several concept of completeness in semimetric space for example, S-completeness, d-Cauchy completeness, strong and weak completeness (see Wilson [20]). We omit the details of these notions which are not relevant to this paper.

Let (f, g) be a pair of self-mappings defined on a nonempty set X equipped with a symmetric (semimetric) d. Then for the pair (f, g), we recall some relevant concepts as follows.

A pair (f, g) of self-mappings is said to be

(i) compatible (cf. [1]) if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X,

(ii) noncompatible (cf. [4, 23]) if there exists at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but $\lim_{n\to\infty}(fgx_n, gfx_n)$ is either nonzero or nonexistent,

(iii) tangential (or satisfying the property (E.A)) (cf. [5, 24]) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Let Y be an arbitrary set and X be a nonempty set equipped with symmetric (semimetric) d. Then the pairs (f, S) and (g, T) of mappings from Y into X are said to have,

(iv) (cf. [9]) the common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t \quad \text{for some } t \in X, \tag{1}$$

while the pair (g, T) is said to have

(v) the common limit range property with respect to the map g (denoted by (CLRg) (cf. [25–29]) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_n = gu$ for some $u \in X$,

(vi) let Y be an arbitrary set and X be a nonempty set equipped with symmetric (semimetric) d. Then f is said to be g-continuous (cf. [5]) if $gx_n \to gx \Rightarrow fx_n \to fx$ whenever $\{x_n\}$ is a sequence in Y and $x \in Y$,

(vii) a pair (f, g) of self-mappings defined on a set X is said to be weakly compatible (or partially commuting or coincidentally commuting (cf. [5, 30])) if the pair commutes on the set of coincidence points that is, $fx = gx$ (for $x \in X$) implies that $fgx = gfx$,

(viii) let f and $g (f \neq g)$ be two self-mappings defined on a symmetric (or semimetric) space (X, d), then f is called g-absorbing if there exists some real number $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$ for all x in X. Analogously, g will be called f-absorbing (cf. [16]) if there exists some real number $R > 0$ such that $d(fx, fgx) \leq Rd(fx, gx)$ for all x in X. The pair of self maps (f, g) will be called absorbing if it is both g-absorbing as well as f-absorbing,

(ix) let f and g $(f \neq g)$ be two self-mappings defined on a symmetric (or semimetric) space (X, d), then f is called pointwise g-absorbing if for given x in X, there exists some $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$,

On similar lines, we can define pointwise f-absorbing map. If we take $g = I$, the identity map on X, then f is trivially I-absorbing. Similarly, I is f-absorbing in respect of any f. It has been shown in [16] that a pair of compatible or R-weakly commuting pair need not be g-absorbing or

f-absorbing. Also absorbing pairs are neither a subclass of compatible pairs nor a subclass of noncompatible pairs as the absorbing pairs need not commute at their coincidence points. For other properties and related results for absorbing pair of maps, one can consult [16].

For the sake of completeness, we state below some theorems contained in Soliman et al. [8] and Gopal et al. [15].

Theorem 1 (see cf. [8]). *Let Y be an arbitrary nonempty set while X be another nonempty set equipped with a symmetric (semimetric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE). Let $f, g, S, T : Y \to X$ be four mappings which satisfy the following conditions:*

(i) *f is S-continuous and g is T-continuous,*

(ii) *the pairs (f, S) and (g, T) share the common property (E.A),*

(iii) *SX and TX are d-closed ($\tau(d)$-closed) subset of X (resp., $fX \subset TX$ and $gX \subset SX$).*

Then there exist $u, w \in X$ such that $fu = Su = Tw = gw$. Moreover, if $Y = X$ along with

(iv) *the pairs (f, S) and (g, T) are weakly compatible and*

(v) *$(fx, gfx) \neq \max\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\}$, whenever the right hand side is nonzero.*

Then, f, g, S, and T have a common fixed point in X.

Theorem 2 (see cf. [15]). *Let Y be an arbitrary nonempty set while X be another nonempty set equipped with a symmetric (semimetric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE). Let $f, g, S, T : Y \to X$ be four mappings which satisfy the following conditions:*

(i) *f is S-continuous and g is T-continuous,*

(ii) *the pairs (f, S) and (g, T) share the common property (E.A),*

(iii) *TY is a d-closed ($\tau(d)$-closed) subset of X and $gY \subset SY$ (resp., SY is a d-closed ($\tau(d)$-closed) subset of X and $fY \subset TY$).*

Then, there exist $u, w \in Y$ such that $fu = Su = Tw = gw$.

Moreover, if $Y = X$, then f, g, S and T have a common fixed point provided the pairs (f, S) and (g, T) are pointwise absorbing.

Theorem 3 (see cf. [15]). *Let Y be an arbitrary set while (X, d) be a symmetric (semimetric) space equipped with a symmetric (semimetric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE). Let $f, g, S, T : Y \to X$ be four mappings which satisfy the following conditions:*

(i) *the pair (g, T) satisfies the property (E.A) (resp., (f, S) satisfies the property (E.A)),*

(ii) *TY is a d-closed ($\tau(d)$-closed) subset of X and $gY \subset SY$ (resp., SY is a d-closed ($\tau(d)$-closed) subset of X and $fY \subset TY$) and*

(iii) *$d(fx, gy) \leq km(x, y)$ for any $x, y \in X$ where $k \geq 0$ and $m(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(gy, Ty)\}, \min\{d(fx, Ty), d(gy, Sx)\}\}$.*

Then, there exist $u, w \in Y$ such that $fu = Su = Tw = gw$.

Moreover, if $Y = X$, then f, g, S and T have a common fixed point provided the pairs (f, S) and (g, T) are pointwise absorbing.

In this paper, we provide a unified approach to certain theorems in symmetric (semimetric) spaces using a blend of common limit range property along with absorbing pair property and obtain generalizations of various results due to Gopal et al. [15], Soliman et al. [8], Pant [31], Sastry and Murthy [5], Imdad et al. [7], Cho et al. [21], and some others.

3. Main Results

We start to section with the following definition.

Definition 4. Let f, g, S, and T be four self-mappings defined on a symmetric space (X, d). Then the pairs (f, S) and (g, T) are said to have the common limit range property (with respect to S and T), often denoted by $\mathrm{CLR}_{(S,T)}$, if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t \tag{2}$$

with $t = Su = Tw$, for some $t, u, w \in X$.

If $f = g$ and $S = T$, then the above definition implies (CLRg) property due to Sintunavarat and Kumam [28]. Also notice that the preceeding definition implies the common property (E.A) but the converse implication is not true in general. The following example substantiates this fact.

Example 5. Consider $X = [2, 20]$ equipped with the symmetric defined by $d(x, y) = (x-y)^2$ for all $x, y \in X$ which satisfies (W_3) and (HE). Define self mappings f, g, S and T on X as

$$\begin{gathered}
fx = \begin{cases} 2 & \text{if } x = 2, \\ 7 & \text{if } 2 < x \leq 5, \\ \dfrac{2x+5}{3} & \text{if } x > 5, \end{cases} \\
Sx = \begin{cases} 3 & \text{if } x = 2, \\ 2 & \text{if } 2 < x \leq 5, \\ \dfrac{x+5}{2} & \text{if } x > 5, \end{cases} \\
gx = \begin{cases} 4 & \text{if } x = 2, \\ \dfrac{4x+7}{3} & \text{if } 2 < x \leq 5, \\ 3 & \text{if } x > 5, \end{cases} \\
Tx = \begin{cases} 6 & \text{if } x = 2, \\ \dfrac{3x+4}{2} & \text{if } 2 < x \leq 5, \\ 4 & \text{if } x > 5. \end{cases}
\end{gathered} \tag{3}$$

For sequences $x_n = 5 + (1/n)$ and $y_n = 2 + (1/n)$, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = 5\,(= t), \quad (4)$$

which shows that the pairs (f, S) and (g, T) share the common property (E.A). However, there does not exist points u and w in X for which $t = Su = Tw$.

In view of the preceeding example, the following proposition is predictable.

Proposition 6. *If the pairs (f, S) and (g, T) share the common property (E.A) and $S(X)$ as well as $T(X)$ are closed subsets of X, then the pairs also share the $CLR_{(S,T)}$ property.*

We now prove our first result employing S-continuity of f and T-continuity of g instead of utilizing some Lipschitz or contractive type condition.

Theorem 7. *Let Y be an arbitrary nonempty set while X be a nonempty set equipped with a symmetric (semimetric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE). If $f, g, S, T : Y \to X$ are four mappings which satisfy the following conditions:*

(i) *f is S-continuous and g is T-continuous,*

(ii) *the pairs (f, S) and (g, T) satisfy the $CLR_{(S,T)}$ property,*

then, (f, S) and (g, T) have a coincidence point. Moreover, if $Y = X$, then $f, g, S,$ and T have a common fixed point provided the pairs (f, S) and (g, T) are pointwise absorbing.

Proof. Since the pairs (f, S) and (g, T) satisfy the $CLR_{(S,T)}$ property, therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t \quad (5)$$

with $t = Su = Tw$, for some $t, u, w \in X$.

On using S-continuity of f along with the condition (W_3), we get $fu = Su$ which shows that u is a coincidence point of the mappings f and S. Similarly, using the T-continuity of g along with the condition (W_3), we obtain $gw = Tw$ which shows that w is a coincidence point of g and T. Owing to $CLR_{(S,T)}$ property, we have $fu = Su = gw = Tw = t$.

As the pairs (f, S) and (g, T) are pointwise absorbing, we can write

$$\begin{aligned} &fu = fSu, \quad Su = Sfu, \quad gw = gTw, \quad Tw = Tgw \\ &\Longrightarrow fu = Sfu, \quad fu = ffu, \quad gw = Tgw, \\ &\quad gw = ggw, \end{aligned} \quad (6)$$

which show that $fu (fu = gw)$ is a common fixed point of f, g, S and T. This concludes the proof. □

With a view to demonstrate the utility of Theorem 7 over Theorem 1 and Theorem 2, we adopt the following example.

Example 8. Consider $X = Y = (-1, 1] \cup \{2, 3, 4\}$ equipped with the symmetric defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$ which satisfies (W_3) and (HE). Define self mappings f, g, S, and T on X as

$$fx = \begin{cases} \frac{3}{5} & \text{if } -1 < x \le \frac{-1}{2}, \\ \frac{x}{4} & \text{if } \frac{-1}{2} < x < \frac{1}{2}, \\ \frac{3}{5} & \text{if } \frac{1}{2} \le x < 1, \\ 3 & \text{if } x = 1, 4, \\ 2 & \text{if } x = 2, 3, \end{cases}$$

$$gx = \begin{cases} \frac{3}{5} & \text{if } -1 < x \le \frac{-1}{2}, \\ \frac{-x}{4} & \text{if } \frac{-1}{2} < x < \frac{1}{2}, \\ \frac{3}{5} & \text{if } \frac{1}{2} \le x < 1, \\ 3 & \text{if } x = 1, 4, \\ 2 & \text{if } x = 2, 3, \end{cases} \quad (7)$$

$$Sx = \begin{cases} \frac{3}{4} & \text{if } -1 < x \le \frac{-1}{2} \\ \frac{x}{2} & \text{if } \frac{-1}{2} < x < \frac{1}{2}, \\ \frac{-3}{4} & \text{if } \frac{1}{2} \le x < 1, \\ 2 & \text{if } x = 1, 2, 3, 4, \end{cases}$$

$$Tx = \begin{cases} \frac{-3}{4} & \text{if } -1 < x \le \frac{-1}{2}, \\ \frac{-x}{2} & \text{if } \frac{-1}{2} < x < \frac{1}{2}, \\ \frac{3}{4} & \text{if } \frac{1}{2} \le x < 1, \\ 2 & \text{if } x = 1, 2, 3, 4. \end{cases} \quad (8)$$

Consider sequences $\{x_n\} = \{1/(n+2)\}$ and $\{y_n\} = \{-1/(n+2)\}$ in X. Clearly,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = 0 \quad (9)$$

with $0 = S(0) = T(0)$, and

$$\begin{aligned} &\lim_{n\to\infty} Sx_n = 0 = S(0) \Longrightarrow \lim_{n\to\infty} fx_n = 0 = f(0), \\ &\lim_{n\to\infty} Ty_n = 0 = T(0) \Longrightarrow \lim_{n\to\infty} gy_n = 0 = g(0), \end{aligned} \quad (10)$$

which show that the pairs (f, S) and (g, T) share the $CLR_{(S,T)}$ property while the map f is S-continuous and the map g is T-continuous. Further $f(X) = (-1/8, 1/8) \cup \{3/5, 2, 3\} \not\subseteq T(X) = (-1/4, 1/4) \cup \{-3/4, 3/4, 2\}$ and $g(X) = (-1/8, 1/8) \cup \{3/5, 2, 3\} \not\subseteq S(X) = (-1/4, 1/4) \cup \{-3/4, 3/4, 2\}$ and evidently none of the involved subspaces are closed. Also, by a routine calculation, one can easily verify that the pairs (f, S) and (g, T) are pointwise absorbing. Thus, the involved pairs of maps (f, S) and (g, T) satisfy all the conditions of Theorem 7 and have two common fixed points namely: $x = 0$ and $x = 2$.

Notice that at $x = 1$, the involved maps do not satisfy the condition

$$d(fx, gfx) \neq \max\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\}, \tag{11}$$

whenever the right hand side is nonzero. Moreover, it can also be verified that at points $x = 1$ and $y = 2$, the involved maps do not satisfy the Lipschitz type condition employed in [4]. Thus, this example substantiates the fact that Theorem 7 is genuine extension of Theorems 1 and 2.

By restricting f, g, S, and T suitably, one can derive corollaries involving two as well as three mappings. Here, it may be pointed out that any result involving three maps is itself a new result. For the sake of brevity, we opt to mention just one such corollary by restricting Theorem 7 to three mappings f, S, and T which is still new and presents yet another sharpened form of a relevant theorem contained in [15] besides admitting a nonself setting upto coincidence points.

Corollary 9. *Let Y be an arbitrary set while (X, d) be a symmetric (semimetric) space equipped with a symmetric (semimetric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE). If $f, S, T : Y \to X$ are three mappings which satisfy the following conditions:*

(i) *f is S-continuous and f is T-continuous,*

(ii) *the pairs (f, S) and (f, T) satisfy the $CLR_{(S,T)}$ property,*

then, there exist $u, w \in Y$ such that $fu = Su = Tw$. Moreover, if $Y = X$, then f, S and T have a common fixed point provided the pairs (f, S) and (f, T) are pointwise absorbing.

The following example illustrates the preceding corollary involving a pair of two self-mappings.

Example 10. Consider $X = Y = [2, 23)$ equipped with the symmetric defined by $d(x, y) = e^{|x-y|} - 1$, for all $x, y \in X$ which satisfies (W_3) and (HE). Define self mappings $f, S : X \to X$ as

$$fx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (5,7) \cup (7,10) \cup (10,11) \cup (11,12) \cup (12,13) \cup (13,21) \cup (21,23), \\ \dfrac{x+5}{2} & \text{if } 2 < x \leq 5, \\ 7 & \text{if } x = 7, \\ 12 & \text{if } x = 10, \\ 11 & \text{if } x = 11, 13, \\ 11.5 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \end{cases}$$

$$Sx = \begin{cases} 2 & \text{if } x \in \{2\} \cup [7,10) \cup (10,11) \cup (11,12) \cup (12,13) \cup (13,21) \cup (21,22) \cup (22,23), \\ 6 & \text{if } 2 < x \leq 5, \\ \dfrac{x+1}{3} & \text{if } x \in (5,7), \\ 11 & \text{if } x = 10, 11, 13, 22, \\ 11.6 & \text{if } x = 12, \\ 10 & \text{if } x = 21. \end{cases} \tag{12}$$

By routine calculations, one can easily verify that the maps in the pair (f, S) satisfies all the conditions of Corollary 9 and have two common fixed points, namely: 2 and 11. Also, the present example does not satisfy the Lipschitz type condition utilized in [4]. To view this claim, consider $x = 13$ and $y = 22$, then we have $e^9 - 1 \leq k\, 0 = 0$, which is a contradiction. Also, observe that at $x = 21$, the involved maps do not satisfy the condition:

$$d(fx, ffx) \neq \max\{d(Sx, Sfx), d(ffx, Sfx), d(fx, Sfx), d(fx, Sx), d(ffx, Sx)\}, \tag{13}$$

whenever the right hand side is nonzero. Here, it is worth noting that none of the earlier relevant theorems for example, Imdad and Soliman [7], Soliman et al. [8] and Gopal et al. [15] can be used in the context of this example as Corollary 9 does not require conditions on containment and closedness amongst the ranges of the involved mappings.

Our next theorem is essentially inspired by Theorem 3 due to Gopal et al. [15].

Theorem 11. *Let Y be an arbitrary set while (X, d) be a symmetric (semimetric) space equipped with a symmetric (semimetric) d which enjoys (W_3) (or Hausdorffness of $\tau(d)$) and (HE). If $f, g, S, T : Y \to X$ are four mappings which satisfy the following conditions:*

(i) *the pairs (f, S) and (g, T) satisfy the $CLR_{(S,T)}$ property,*

(ii) *$d(fx, gy) \leq km(x, y)$, for any $x, y \in X$, where $k \geq 0$ and $m(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(gy, Ty)\}, \min\{d(fx, Ty), d(gy, Sx)\}\}$,*

then, there exist $u, w \in Y$ such that $fu = Su = Tw = gw$. Moreover, if $Y = X$, then f, g, S, and T have a common fixed point provided the pairs (f, S) and (g, T) are pointwise absorbing.

Proof. Since the pairs (f, S) and (g, T) share the $CLR_{(S,T)}$ property, therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = t \tag{14}$$

with $t = Su = Tw$, for some $t, u, w \in X$.

On using condition (ii), we have

$$d(fu, gy_n) \leq k \max\{d(Su, Ty_n), \min\{d(fu, Su), d(gy_n, Ty_n)\}, \min\{d(fu, Ty_n), d(gy_n, Su)\}\} \quad (15)$$

which on letting $n \to \infty$, gives rise $\lim_{n\to\infty} d(fu, gy_n) = 0$. Now appealing to (W_3), we get $fu = Su$ so that $fu = Su = Tw$.

Next, we show that $Tw = gw$. To accomplish this, using (ii), we have

$$\begin{aligned} d(fu, gw) &\leq k \max\{d(Su, Tw), \min\{d(fu, Su), d(gw, Tw)\}, \min\{d(fu, Tw), d(gw, Su)\}\} \\ &= k \max\{d(Tw, Tw), \min\{d(fu, fu), d(gw, Tw)\}, \min\{d(fu, fu), d(gw, gw)\}\} \\ &= 0 \end{aligned} \quad (16)$$

so that $fu = gw$ and hence in all $fu = Su = gw = Tw$ which shows that both the pairs have a point of coincidence.

On using pointwise absorbing property of the pairs (f, S) and (g, T), we have

$$\begin{gathered} fu = fSu, \quad Su = Sfu, \quad gw = gTw, \quad Tw = Tgw, \\ \Longrightarrow fu = Sfu, \quad fu = ffu, \quad gw = Tgw, \\ gw = ggw, \end{gathered} \quad (17)$$

which show that fu $(fu = gw)$ is a common fixed point of f, g, S, and T. □

The following example demonstrates Theorem 11.

Example 12. Consider $X = Y = [0, 20)$ equipped with the symmetric $d(x, y) = (x - y)^2$ for all $x, y \in X$ which satisfies (W_3) and (HE). Set $f = g$ and $S = T$. Define $f, S : X \to X$ as follows:

$$fx = \begin{cases} 2 & \text{if } 0 \leq x \leq 2,\ x \geq \dfrac{11}{2}, \\ 6 & \text{if } 2 < x \leq 5, \\ \dfrac{x+3}{4} & \text{if } 5 < x < \dfrac{11}{2}, \\ 10 & \text{if } x = 10, \end{cases}$$

$$Sx = \begin{cases} 2 & \text{if } 0 \leq x \leq 2,\ x \geq \dfrac{11}{2}, \\ 4 & \text{if } 2 < x \leq 5, \\ \dfrac{x+1}{3} & \text{if } 5 < x < \dfrac{11}{2}, \\ 10 & \text{if } x = 10. \end{cases} \quad (18)$$

Then, by a routine calculation, it can be easily verified that f and S satisfy condition (ii) (of Theorem 11) for $k = 4.271$. Also, the mappings f and S satisfies the $\mathrm{CLR}_{(S,T)}$ property with the sequence $x_n = 5 + 1/n$. The verification of the pointwise absorbing property of the pair (f, S) is straight forward. Thus f and S satisfy all the conditions of Theorem 11 and have two common fixed points, namely: $x = 2$ and $x = 10$.

Observe that $f(X) = [2, 17/8) \cup \{6, 10\} \not\subseteq S(X) = [2, 13/6) \cup \{4, 10\}$ and none of $f(X)$ and $S(X)$ is closed. Further, it is also worth noting that for all x with $2 < x \leq 5$ and with $f = g$ and $S = T$, the involved pair (f, S) does not satisfy the condition

$$d(fx, gfx) \neq \max\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\}, \quad (19)$$

whenever the right hand side is nonzero. Thus, this example also establishes the utility of Theorem 11 over corresponding results proved in Soliman et al. [8] and Gopal et al. [15].

Remark 13. Choosing $k = 1$ in Theorem 11, we can derive a slightly sharpened form of a theorem due to Cho et al. [21] as conditions on the ranges of involved mappings are completely relaxed.

By restricting f, g, S, and T suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 11 to three mappings which is yet another sharpened and unified form of a theorem due to Gopal et al. [15] in symmetric spaces and also remains relevant to some results in Pant [4] and Pant [31].

Corollary 14. *Suppose that (in the setting of Theorem 11) d satisfies (W_3) and (HE). If $f, S, T : Y \to X$ are three mappings which satisfy the following conditions:*

(i) *the pairs (f, S) and (f, T) satisfy the $\mathrm{CLR}_{(S,T)}$ property,*

(ii) *$d(fx, fy) \leq k m_2(x, y)$, for any $x, y \in X$, where $k \geq 0$ and $m_2(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(fy, Ty)\}, \min\{d(fx, Ty), d(fy, Sx)\}\}$,*

then, there exist $u, w \in Y$ such that $fu = Su = Tw$. Moreover, if $Y = X$, then f, S, and T have a common fixed point provided the pair (f, S) is pointwise S-absorbing while the pair (f, T) is pointwise T-absorbing.

Corollary 15. *Let (X, d) be symmetric (semimetric) space wherein d satisfies (W_3) (Hausdoffness of $\tau(d)$) and (HE). If $f, g, S, T : X \to X$ are four self mappings of X which satisfy the following conditions:*

(i) *the pairs (f, S) and (g, T) satisfy the $\mathrm{CLR}_{(S,T)}$ property,*

(ii) *$d(fx, gy) < m(x, y)$, where $m(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(gy, Ty)\}, \min\{d(fx, Ty), d(gy, Sx)\}\}$*

then there exist $u, w \in X$ such that $fu = Su = Tw = gw$.

Moreover, if $Y = X$, then f, g, S, and T have a unique common fixed point provided the pair (f, S) is pointwise S-absorbing whereas the pair (g, T) is pointwise T-absorbing.

Proof. Proof follows from Theorem 11 by setting $k = 1$. □

Our next theorem is essentially inspired by a Lipschitzian condition utilized by Cho et al. [21] as well as Gopal et al. [15].

Theorem 16. *Theorem 11 remains true if (W_3) is replaced by $(1C)$ while condition (ii) (of Theorem 11) is replaced by the following condition (ii′) besides retaining rest of the hypotheses:*

(ii′) *$d(fx, gy) \leq km_1(x, y)$, for any $x, y \in X$, where $k \geq 0$ together with $k\alpha < 1$, and wherein $m_1(x, y) = \max\{d(Sx, Ty), \alpha[d(fx, Sx) + d(gy, Ty)], \alpha[d(fx,Ty) + d(gy,Sx)]\}$.*

Proof. The proof can be completed on the lines of proof of Theorem 11, hence details are not included. □

By restricting f, g, S, and T suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 16 to three mappings which is yet another sharpened form of a theorem contained in [15] which also remains relevant to some results in Pant [4] and Pant [31].

Corollary 17. *Suppose that (in the setting of Theorem 16) d satisfies (IC) and (HE). If $f, S, T : Y \to X$ are three mappings which satisfy the following conditions:*

(i) *the pairs (f, S) and (f, T) satisfy the $CLR_{(S,T)}$ property,*

(ii) *$d(fx, fy) \leq km_3(x, y)$, for any $x, y \in X$, where $k \geq 0$ together with $k\alpha < 1$, and $m_3(x, y) = \max\{d(Sx, Ty), \alpha[d(fx, Sx)+ d(fy, Ty)], \alpha[d(fx, Ty) + d(fy, Sx)]\}$*

then, there exist $u, w \in Y$ such that $fu = Su = Tw$. Moreover, if $Y = X$, then f, S, and T have a common fixed point provided the pair (f, S) is pointwise S-absorbing while the pair (f, T) is pointwise T-absorbing.

Corollary 18. *Let (X, d) be symmetric (semimetric) space wherein d satisfies (IC) and (HE). If f, g, S, and T are four self mappings of X which satisfy the following conditions:*

(i) *the mappings satisfy the $CLR_{(S,T)}$ property,*

(ii) *$d(fx, gy) < m_1(x, y)$, where $m_1(x, y) = \max\{d(Sx, Ty), \alpha[d(fx, Sx) + d(gy, Ty)], \alpha[d(fx, Ty) + d(gy, Sx)]\}$ with $0 < \alpha < 1$,*

then, there exist $u, w \in X$ such that $fu = Su = Tw = gw$. Moreover, if $Y = X$, then f, g, S, and T have a unique common fixed point provided the pair (f, S) is pointwise S-absorbing while the pair (f, T) is pointwise T-absorbing.

Proof. The proof can be completed on the lines of proof of Theorem 11. □

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