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A NOTE ON CONNECTEDNESS IN CARTESIAN CLOSED CATEGORIES

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ABSTRACT. Primarily working in the category of limit spaces and continuous maps we suggest a new concept of connectivity with application in all categories where function space objects satisfy natural exponential laws. In a separate Appendix we motivate the development of a homotopy theory for spaces of real-valued continuous maps endowed with the structure of continuous convergence.

KEY WORDS AND PHRASES: Cartesian closedness, exponential laws, continuous convergence, cT-connectedness, Galois correspondence, homotopy classes.
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1. INTRODUCTION.

Definitions are as in Preuss [1–2], or are given below. Being a cartesian closed topological category, the category of *limit spaces* has natural function spaces $C_c(X,Y)$, where $C(X,Y)$ denotes the continuous functions between limit spaces X, Y, and c is *the limit structure of continuous convergence*. The present note relies on the three exponential laws

$$C_c(X \times Y, Z) \cong C_c(X, C_c(Y,Z));$$

$$C_c(X, \prod_{i\in I} Y_i) \cong \prod_{i\in I} C_c(X, Y_i);$$

$$C_c(\sum_{i\in I} X_i, Y) \cong \prod_{i\in I} C_c(X_i, Y).$$

Since much of the reasoning to follow is category theoretical in nature, the reader can (in most cases) easily restate the results in any other cartesian closed topological category. A good example is compactly generated topological spaces, where the c-structure in the exponential laws corresponds to the "compactly generated modification" (a coreflector) of the compact-open topology.

The notation $\mathcal{F} \to^q x$ (or $\mathcal{F} \to x$ in X) means the filter $\mathcal{F}$ on X q-converges to x. The exact definition of limit structure is given by the three axioms (i) for all $x \in X$, $\dot{x} \to^q x$; (ii) for all $x \in X$, if $\mathcal{F} \to^q x$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \to^q x$; (iii) for all $x \in X$, if $\mathcal{F}, \mathcal{G} \to^q x$, then $\mathcal{F} \cap \mathcal{G} \to^q x$. If q is a limit structure on X, the pair (X,q) is called a (limit) space. When there is no ambiguity, the space (X,q) is shortly denoted by X.

For Φ a filter on $C(X,Y)$ and $\varphi \in C(X,Y)$ we define $\Phi \to^c \varphi \Leftrightarrow w(\Phi \times \mathcal{F}) \to \varphi(x)$ in Y for every $x \in X$ and $\mathcal{F} \to x$, where w denotes the evaluation map $C(X,Y) \times X \to Y$ defined by $w(f,x) = f(x)$. If X, Y are topological spaces and X locally compact, then c is the compact-open topology.

Given a fixed limit space (test space) T, a space X is called *T-connected*, if all continuous maps from X to T are constant (cf. Preuss [1]). If T is any non-singleton space with discrete structure, the T-connected spaces are precisely the connected spaces.

2. A NEW CONCEPT OF CONNECTIVITY.

In the definition below, $C(X,T)$ and T are related not only in terms of cardinality, but also in terms of structure.

Let T be a fixed limit space.

DEFINITION 1. The limit space X is *cT-connected*, if $T \cong C_c(X,T)$. A subset A of X is called cT-connected, if it is cT-connected as a space, that is if $C_c(A,T) \cong T$.

The above homeomorphism can be established other ways than the canonical one.

PROPOSITION 2. For finite T, the cT-connected spaces and the T-connected spaces coincide. For any T, all T-connected spaces are also cT-connected.

The proof is straightforward. Since the space $\mathbf{2}$ is cN-connected, there are cT-connected spaces which are not T-connected. Moreover, let A and B be cN-connected (disjoint) sets in a limit space X. The chain of embeddings $N \hookrightarrow C_c(A \cup B, N) \hookrightarrow C_c(A,N) \times C_c(B,N) \hookrightarrow N \times N \hookrightarrow N$ clearly implies $N \cong C_c(A \cup B, N)$. Since singleton sets are cT-connected (for any T), it follows that any finite set in X is cN-connected.

Examples from classical topology of cT-connectedness abound. We observe that N is cT-connected for $T \equiv$ the Cantor set, since $C_c(N,T) \cong T^N \cong T$.

PROPOSITION 3. The property of being cT-connected is preserved by finite products, however, in general not by arbitrary products. If a space is cT_i-connected for every $i \in I$, then it is $c\prod_{i\in I} T_i$-connected. The sum $\sum_{\iota\in I} X_i$ of cT-connected spaces is cT-connected if and only if $T^I \cong T$.

PROOF. If X, Y are assumed cT- connected, the first exponential law gives $C_c(X \times Y, T) \cong C_c(X, C_c(Y,T)) \cong T$. Since the Cantor space is not cN-connected, cT-connectivity is not preserved by general products. If X is cT_i-connected for every $i \in I$, the second exponential law yields $C_c(X, \prod_{i\in I} T_i) \cong \prod_{i\in I} C_c(X,T_i) \cong \prod_{i\in I} T_i$. The last assertion in the above proposition follows from the third exponential law.

PROPOSITION 4. Let X and Y be limit spaces. The following are equivalent: (1) X is cT-connected; (2) X is $cC_c(Y,T)$-connected for every Y; (3) $C_c(Y \times X, T) \cong C_c(Y,T)$ for every Y.

PROOF. Let Y be an arbitrary limit space. From (1) and the first exponential law it follows $C_c(Y \times X, T) \cong C_c(Y, C_c(X,T)) \cong C_c(Y,T)$. On the other hand, the first exponential law yields $C_c(Y \times X, T) \cong C_c(X, C_c(Y,T))$, and (2) follows. Clearly, (2) and the first exponential law together imply $C_c(Y \times X, T) \cong C_c(X, C_c(Y,T)) \cong C_c(Y,T)$. It is seen that (3) implies (1) by taking Y to be a singleton space.

REMARK 5. Apparently, cT-connectivity yields a meaningful theory in all categories satisfying the exponential laws. (In these categories, the obvious definition of cT- connectivity is Definition 1 above employing the natural function spaces and the natural concept of isomorphism.) An important example is the category of limit vector spaces and linear continuous maps. In this category the reals are cT-connected with respect to any limit vector space T, since apparently $L_c(R,T) \cong T$.

PROPOSITION 6. If T contains no homeomorphic copy of itself, any quotient of a cT-connected space is again cT-connected.

PROOF. Let s be a quotient map from a cT-connected space X onto a space Y. Let F be the set of continuous maps from X to T which factorize over s. Then $\varphi : F_c \to C_c(Y,T)$ defined by $\varphi(f) = f'$ (given f factorizes as $f' \circ s$) is a homeomorphism, and hence $T \hookrightarrow C_c(Y,T) \hookrightarrow C_c(X,T) \cong T$. The assumption on T apparently gives $C_c(Y,T) \cong T$.

In the definition of classical connectivity, any finite discrete space can serve as test space T (Proposition 2). Since such spaces contain no homeomorphic copy of themselves, Proposition 6 covers classical connectivity.

EXAMPLE 7 (Completions). Cauchy spaces S, T, ... and Cauchy maps yield a cartesian closed topological category (cf. Bentley et al. [3]), whose canonical function spaces are denoted by $C_c(S,T)$. In this example, all spaces are assumed Hausdorff. The couple $(j,\hat{S})$ is a completion of S if j is a dense Cauchy embedding of S into the complete Cauchy space $\hat{S}$. The completion is called *strict* if for every Cauchy filter $\mathcal{G}$ of $\hat{S}$ there is a Cauchy filter $\mathcal{F}$ of S with $\mathcal{G} \supseteq \overline{j(\mathcal{F})}^{\hat{S}}$. It has the *extension property* if for every complete Cauchy space T every Cauchy map $\varphi : S \to T$ extends (uniquely) to a Cauchy map $\hat{\varphi} : \hat{S} \to T$ $(\varphi = \hat{\varphi} \circ j)$. Completions which meet these criteria are provided by for instance the topological completion functor and by Wyler's completion functor. *Let T be a complete, regular Cauchy space. If $(j,\hat{S})$ is a strict completion of S with the regular extension property, then $C_c(S,T)$ is Cauchy isomorphic to $C_c(\hat{S},T)$.* Thus, here S is cT-connected if and only if $\hat{S}$ is cT-connected.

PROOF. Let p be the Cauchy extension $C_c(S,T) \to C_c(\hat{S},T)$. Obviously, p^{-1} is a Cauchy bijection, so it remains to prove p is a Cauchy map. Therefore, let $\mathcal{G}$ be a Cauchy filter of $\hat{S}$ and Φ a Cauchy filter of $C_c(S,T)$. It follows $w(p(\Phi) \times \mathcal{G})$ is a Cauchy filter of T, since $w(p(\Phi) \times \mathcal{G})$ $\supseteq w(p(\Phi) \times \overline{j(\mathcal{F})}^{\hat{S}}) \supseteq \overline{w(p(\Phi) \times j(\mathcal{F}))} = \overline{\Phi(\mathcal{F})}$, where $\mathcal{F}$ is a Cauchy filter of S.

The following definitions relate the idea of cT-connectivity to classes of spaces. Let Λ be a class of limit spaces and define classes $d\Lambda$ and $c\Lambda$ through

$$Y \in d\Lambda \iff C_c(X,Y) \cong Y \ \forall X \in \Lambda$$

$$X \in c\Lambda \iff C_c(X,Y) \cong Y \ \forall Y \in \Lambda.$$

Then, for classes $\Lambda, \Lambda_1, \Lambda_2$ it holds

$$\Lambda \subseteq cd\Lambda, \ \ \Lambda \subseteq dc\Lambda$$

$$\Lambda_1 \subseteq \Lambda_2 \ \Rightarrow \ d\Lambda_1 \supseteq d\Lambda_2, \ \ c\Lambda_1 \supseteq c\Lambda_2,$$

which means operators c and d define a Galois correspondence. Thus, the composed operators cd and dc are closure operators, and the class $cd\Lambda$ ($dc\Lambda$) may be regarded a natural connectedness property (disconnectedness property) defined by the class Λ.

It is well-known that established connectivity concepts yield Galois correspondences, cf. Preuss [1–2].

REMARK 8. The spaces in the family $c\Theta$ (where Θ is a given family of limit spaces) are called *$c\Theta$-connected*. Sometimes, it may be practically convenient to assume Θ is closed under formation of arbitrary products (productive), or closed under formation of subspaces (hereditary). Productivity is suggested by Proposition 3, and hereditarity is fulfilled in most natural connectivity concepts (that is, the family of test spaces is hereditary). If Θ is hereditary and contains at least one non-trivial Hausdorff space, then a space is classically connected if it is $c\Theta$-connected.

We shall close the paper with a notion on path connectedness in spaces $C_c(S,T)$.

3. APPENDIX. HOMOTOPY THEORY AND $C_c(S,T)$.

It is worth pointing out that the category of limit spaces provides a natural setting for homotopy theory. Let S and T be limit spaces. A limit structure on $C(S,T)$ is called *conjoining* if for every limit space X, the function $\alpha : X \times S \to T$ is continuous whenever its associate $\hat{\alpha} : X \to C(S,T)$ is continuous (as usual, $\alpha(x,s)=\hat{\alpha}(x)(s)$). On the other hand, a limit structure on $C(S,T)$ is called *splitting* if for every limit space X, the continuity of $\alpha : X \times S \to T$ implies the continuity of $\hat{\alpha} : X \to C(S,T)$.

With no assumptions on spaces S and T, the c-structure used throughout the previous section is known to be the unique finest splitting and coarsest conjoining structure on $C(S,T)$.

FACT 1. In $C_c(S,T)$, path components and homotopy classes coincide.

PROOF. Assume a path $\hat{\varphi}$ joining f and g, or more precisely, a continuous map $\hat{\varphi} : I \to C_c(S,T)$ ($\hat{\varphi}(0) = f$, $\hat{\varphi}(1) = g$). Since c is conjoining, then $\varphi : I \times S \to T$ is continuous ($\varphi(0,s) = f(s)$, $\varphi(1,s) = g(s)$). Secondly, assume there is a continuous map $\varphi : I \times S \to T$ ($\varphi(0,s) = f(s)$, $\varphi(1,s) = g(s)$). Since c is splitting, the associate $\hat{\varphi} : I \to C_c(S,T)$ is continuous ($\hat{\varphi}(0)(s) = f(s)$, $\hat{\varphi}(1)(s) = g(s)$).

REMARK. Let S, T be topological spaces. It is well-known the compact-open topology on C(S,T) is always splitting, but not necessarily conjoining. However, Fact 1 holds if S is assumed to be a topological k-space (cf. Dugundji [4]). If S is a locally compact topological space, the compact-open topology equals the c-structure, and Fact 1 follows from the proof above.

FACT 2. Let S, T and U be limit spaces. If two maps $S \to T$ are homotopic, then their induced maps $C_c(T,U) \to C_c(S,U)$ are homotopic. Similarly, if two maps $T \to U$ are homotopic, then their induced maps $C_c(S,T) \to C_c(S,U)$ are homotopic.

The standard proof for topological spaces and compact-open topology (cf. Dugundji [4]) needs the extra assumption that T be locally compact or S a k-space. However, Fact 2 is easily proved, when it is observed the c-structure is both splitting and conjoining, and that the composition map $C_c(S,T) \times C_c(T,U) \to C_c(S,U)$ is continuous without additional assumptions on T.

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