Internat. J. Math. & Math. Sci.
VOL. 20 NO. 4 (1997) 825-827

SUBRINGS OF I-RINGS AND S-RINGS

MAMADOU SANGHARE

Département de Mathématiques et Informatiques
Faculté des Sciences et Techniques
UCAD
DAKAR (SENEGAL)
e-mail sanghare@ucad.refer.sn

(Received May 6, 1993 and in revised form February 13, 1997)

ABSTRACT. Let R be a non-commutative associative ring with unity $1 \neq 0$, a left R-module is said to satisfy property (I) (resp. (S)) if every injective (resp. surjective) endomorphism of M is an automorphism of M. It is well known that every Artinian (resp. Noetherian) module satisfies property (I) (resp. (S)) and that the converse is not true. A ring R is called a left I-ring (resp. S-ring) if every left R-module with property (I) (resp (S)) is Artinian (resp. Noetherian). It is known that a subring B of a left I-ring (resp. S-ring) R is not in general a left I-ring (resp. S-ring) even if R is a finitely generated B-module, for example the ring $M_3(K)$ of 3×3 matrices over a field K is a left I-ring (resp S-ring), whereas its subring

$$B = \left\{ \begin{bmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & 0 & \alpha \end{bmatrix} / \alpha, \beta, \gamma \in K \right\}$$

which is a commutative ring with a non-principal Jacobson radical

$$J = K.\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + K.\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is not an I-ring (resp. S-ring) (see [4], theorem 8). We recall that commutative I-rings (resp S-rings) are characterized as those whose modules are a direct sum of cyclic modules, these rings are exactly commutative, Artinian, principal ideal rings (see [1]). Some classes of non-commutative I-rings and S-rings have been studied in [2] and [3]. A ring R is of finite representation type if it is left and right Artinian and has (up to isomorphism) only a finite number of finitely generated indecomposable left modules. In the case of commutative rings or finite-dimensional algebras over an algebraically closed field, the classes of left I-rings, left S-rings and rings of finite representation type are identical (see [1] and [4]) A ring R is said to be a ring with polynomial identity (P. I-ring) if there exists a polynomial $f(X_1, X_2, ..., X_n)$, $n \geq 2$, in the non-commuting indeterminates $X_1, X_2, ..., X_n$ over the center Z of R such that one of the monomials of f of highest total degree has coefficient 1, and $f(a_1, a_2, ..., a_n) = 0$ for all $a_1, a_2, ..., a_n$ in R. Throughout this paper all rings considered are associative rings with unity, and by a module M over a ring R we always understand a unitary left R-module. We use M_R to emphasize that M is a unitary right R-module.

KEY WORDS AND PHRASES: Left I-ring, left S-ring, ring with polynomial identity, ring of finite representation type.

1991 AMS SUBJECT CLASSIFICATION CODES: 16D70, 16P10, 16L60.

1. THE MAIN RESULT

THEOREM. Let R be a left I-ring (resp. S-ring), and B be a sub-ring of R contained in the center Z of R Suppose that R is a finitely generated flat B-module Then B is an I-ring (resp S-ring)

To prove this theorem we need some results.

It is easy to see that

LEMMA 1. Every homomorphic image of a left I-ring (resp S-ring) is a left I-ring (resp S-ring)

LEMMA 2. Let P_1 and P_2 be two prime ideals of a ring R. If P_1 is not contained in P_2 then $Hom_R(R/P_1, R/P_2) = \{0\}$

PROOF. Let $f : R/P_1 \to R/P_2$ be an R-homomorphism, and set $f(1 + P_1) = t + P_2$, where $t \in R$. Let $x \in P_1 \backslash P_2$, and let r be any element in R. We have $P_2 = f(xr + P_1) = xrt + P_2$ Thus $xRt \in P_2$. Since P_2 is prime, we have $t \in P_2$, and hence $f = 0$.

LEMMA 3. Let R be a prime ring with polynomial identity. If R is a left I-ring (resp. S-ring), then R is simple Artinian.

PROOF. Let R' be the total ring of fractions of R [5]. It is known that R' is simple Artinian [5], so the R-module R' satisfies (I) (resp. (S)). Since R is a left I-ring (resp. S-ring), then R' is an Artinian (resp. Noetherian) R-module and hence $R' = R$.

LEMMA 4. Let R be a semi-prime ring with polynomial identity. If R is a left I-ring (resp S-ring), then R is semi-simple Artinian.

PROOF. Let $(P_\ell)_{\ell \in L}$ be a family pairwise distinct minimal prime ideals of R such that

$$\bigcap_{\ell \in L} P_\ell = \{0\}.$$

By Lemma 1 the quotient rings $R/P_\ell (\ell \in L)$ are left I-rings (resp. S-rings) with polynomial identity Then it follows from Lemma 3 that the rings $R/P_\ell (\ell \in L)$ are simple Artinian, so the left R-modules $R/P_\ell (\ell \in L)$ satisfy (I) (resp (S)). Following Lemma 1, $Hom_R(R/P_\ell, R/P_{\ell'}) = \{0\}$ for $\ell \neq \ell'$, so the left R-module $M = \oplus_{\ell \in L} R/P_\ell$ satisfies (I) (resp. (S)). Since R is a left I-ring (resp. S-ring), then M is Artinian. But R regarded as left R-module is isomorphic to a submodule of the semi-simple Artinian left R-module M, hence R is semi-simple Artinian.

PROPOSITION 5. Let R be a ring with polynomial identity. If R is a left S-ring (resp. I-ring), then R is left Artinian.

PROOF. Suppose that R is a left S-ring (resp. I-ring) then the quotient ring $R/rad(R)$, where $rad(R)$ is the prime radical of R, is a left S-ring (resp. I-ring), so, following Lemma 4, the ring $R/rad(R)$ is semi-simple Artinian This fact implies that R is semi-perfect and hence $rad(R) = J(R)$, where $J(R)$ is the Jacobson radical of R. Let e be a primitive idempotent of R. Since the endomorphism ring of the R-module Re is isomorphic to the local ring eRe with a nil Jacobson radical $eJ(R)e$, then the R-module Re satisfies property (I) (resp (S)). It follows that the R-module Re is Noetherian (resp. Artinian). Since R regarded as R-module is a direct sum of finitely many left R-modules of the form Re, where e is a primitive idempotent of R, then R is Noetherian. Let P now be a prime ideal of R. Since the prime ring R/P is simple in virtue of Lemma 3, then R is left Artinian.

PROOF OF THE MAIN THEOREM. Since R is a finitely generated Z-module, then R is a ring with polynomial identity (see [6]). So by Proposition 5 R is a left Artinian ring. Thus by [7] the ring B is Artinian. Let $e_1, ..., e_n$ be primitive idempotents of B such that $B = \oplus_{i=1}^n e_i Bei$ For every i, $1 \leq i \leq n$, $B_i = e_i B e_i$ is a local Artinian ring. To show that B is a left I-ring (resp. S-ring) it is enough to show that for every i, $1 \leq i \leq n$, B_i is a left I-ring (resp. S-ring). We have $A = \oplus_{i=1}^n A_i$, where $A_i = e_i A e_i$, $1 \leq i \leq n$. By hypothesis the left B-module $\oplus_{i=1}^n A_i = A$ is flat and finitely generated, so the B_i-module

$$A_i = e_i A e_i \cong e_i A e_i \otimes_B B = A \otimes_b e_i B e_i = A \otimes_B B_i$$

is also flat and finitely generated Since B_i is an Artinian local ring then the B_i-module A_i is faithfully flat (see [8] proposition 1, p. 44)

Suppose now that B_i is not an I-ring (resp. S-ring) for some i, $1 \leq i \leq n$ Then by Proposition 2 of [2], there exists a B_i-module M of infinite length such that, for every integer $n \geq 1$, the B_i-module M^n satisfies both properties (I) and (S) Following [8] (corollary 2, p. 107), the B_i-module A_i is a free module. Let $M' = M \otimes_{B_i} A_i$. Since the B_i-module M is of infinite length and A_i is a faithfully flat B_i-module, then M' is an A_i-module of infinite length. On the other hand, since A_i is a free B_i-module, there exists an integer $s \geq 1$ such that $A_i = B_i^s$. We have then the B_i-module isomorphism

$$M' = M \otimes_{B_i} A_i = M \otimes_{B_i} B_i^s \cong M^s.$$

Hence the B_i-module $M' \cong M^s$ satisfies both properties (I) and (S) and therefore M', regarded as A_i-module, satisfies properties (I) and (S) This fact implies that the homomorphic image A_i of the left I-ring (resp. S-ring) A is not a left I-ring (resp. S-ring), in contradiction with Lemma 1.

COROLLARY. Let R be a left I-ring (resp. S-ring). If R is a finitely generated flat module over its center Z, then Z is an I-ring (resp. S-ring).

The following example shows that the converse of the theorem above is not true· Let K be a field The commutative ring $A = K[X,Y]/(X^2, XY, Y^2)$ is not an I-ring (resp. S-ring) because its Jacobson radical $J = K\overline{X} + K\overline{Y}$ is not principal (see [1], theorem 8). On the other hand K is an I-ring (resp S-ring) and A is a finite-dimensional K-vector space

ACKNOWLEDGEMENT. The author would like to thank the referee for his valuable suggestions and numerous very useful remarks about the text.

REFERENCES

[1] KAIDI, A.M. and SANGHARE, M., Une caractérisation des anneaux artiniens à idéaux principaux, *Lec. Notes in Math.*, Vol. 1328, Springer-Verlag, Berlin (1988), 245-254.

[2] SANGHARE, M., Sur quelques classes d'anneaux liées au lemme de Fitting, *Rend. Sem. Math. Padova*, **87** (1992), 29-37.

[3] SANGHARE, M., On S-duo-rings, *Comm. in Algebra* **20** (8) (1992), 2183-2189

[4] SANGHARE, M., Characterizations of algebras whose modules with Fitting's property are of finite length, *Ext. Mat.* 7 (2) (1992), 1-2.

[5] POSNER, E.C., Prime rings satisfying a polynomial identity, *Proc. Amer. Math. Soc.* **11** (1960), 180-183.

[6] RENAULT, G., *Algèbre non-commutative*, Gauthier-Villars, Paris (1975).

[7] EISENBUD, E., Subrings of Artinian and Noetherian rings, *Math. Ann.* **185** (1970), 247-249

[8] BOURBAKI, N., *Algèbre commutative chap. I*, Hermann, Paris (1961).