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EXISTENCE OF GLOBAL SOLUTION FOR A DIFFERENTIAL SYSTEM WITH INITIAL DATA IN L^p

PETER BATES, FENGXIN CHEN, and PING WANG

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ABSTRACT. In this paper, we study the system governing flows in the magnetic field within the earth. The system is similar to the magnetohydrodynamic (MHD) equations. By establishing a new priori estimates and following Calderón's procedure for the Navier Stokes equations [1], we obtained, for initial data in space L^p, the global in time existence and uniqueness of weak solution of the system subject to appropriate conditions.

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1. Introduction. We consider in this work the following differential system arising from geophysics (cf. Hide [7]), which governs the flow of an electrically-conducting fluid in the presence of a magnetic field, when referred to a frame which rotates with angular velocity Ω relative to an inertial frame

$$\begin{gathered}\frac{\partial v}{\partial t}+(v\cdot\nabla)v=\nu\Delta v-\frac{1}{\rho}\nabla p-2\Omega\times v-\frac{1}{\rho\mu}(\nabla\times b)\times b+f(x),\\ \frac{\partial b}{\partial t}=\lambda\Delta b-\nabla\times(v\times b)-\frac{1}{\mu}\nabla q+g(x),\\ \operatorname{div}v=0,\qquad \operatorname{div}b=0,\end{gathered}\tag{1.1}$$

where v is the Eulerian flow velocity, ρ is the density, b is the magnetic field, p is the pressure, ν, μ are, respectively, constants of kinematical viscosity, magnetic permeability, $\lambda=\eta/\mu$ with electrical resistivity η, and $f(x)$, $g(x)$ are volume forces.

The initial conditions are as follows:

$$v(x,0)=v_0,\qquad b(x,0)=b_0\quad\text{for } x\in R^n.\tag{1.2}$$

The existence of solutions of system (1.1) and (1.2) in L^2 has been proved in [9]. Some regularity properties and large time behaviors of the solutions for a similar system, the MHD equations, are obtained in Sermange [10] and Temam [12]. More recently, we obtained in [2] the local in time existence and uniqueness of weak solutions of the system in L^p with $p>n$.

Motivated by Calderón's work on the Navier Stokes equations [1], we consider in this paper the initial value problem for the above system in the infinite cylinder $S=(0,\infty)\times R^n$ with initial data v_0, $b_0\in L^p$ with $p\le n$.

This article is arranged in the following order: in Section 2, we introduce some notations and definitions. Applying Calderón's partition lemma, we introduce in Section 3

Leray's approximating system for our problem. In Section 4, we state and briefly prove some lemmas similar to those for Navier-Stokes equations. Finally, in Section 5, by establishing a priori estimates for our system and adapting Calderón's technique, we prove the global in time existence and uniqueness of weak solution of (1.1) and (1.2) for initial data in L^p.

2. Notations and definition of weak solution. In this section, we introduce some notations and the definition of a weak solution of the differential system (1.1) and (1.2).

Denote by $L^{p,q}(S_T)$ the standard functional space consisting of Lebesgue measurable vector functions $u=(u_1,u_2,\dots,u_n)$ with the following property:

$$\|u\|_{p,q}=\sum_{j=1}^{n}\left[\int_0^T\left(\int_{R^n}|u_j(x,t)|^p dx\right)^{q/p}\right]^{1/q}<\infty, \tag{2.1}$$

where $S_T=(0,T)\times R^n$. Let $u^*=\sup_t|u|$ and define $\|u^*\|_p(T)=(\int(\sup_{0<t<T}|u|)^p dx)^{1/p}$. Let $\mathcal{L}^{p,q}(S_T)=L^{p,q}(S_T)\times L^{p,q}(S_T)$ with the standard product norm $\|(v,b)\|_{p,q}=\|v\|_{p,q}+\|b\|_{p,q}$ and $Ł^p(R^n)=L^p(R^n)\times\cdots\times L^p(R^n)$ with the norm $\|g\|_p=\sum_{i=1}^n\|g_i\|_p$ for $g\in Ł^p(R^n)$.

Let $\mathcal{S}(R^n)$ denote the space of rapidly decreasing functions on R^n, $\mathcal{S}'(R^n)$ the space of tempered distributions, and $\mathfrak{D}_T$ the space of functions $\phi(x,t)=(\phi_1(x,t),\dots,\phi_n(x,t))$ with the properties: $\phi_i\in\mathcal{S}(R^{n+1})$, $\phi_i(x,t)=0$ for $t\geq T$; $\operatorname{div}\phi=\sum_{i=1}^n D_{x_i}\times\phi(x,t)=0$ for all t.

Definition 2.1. A function $u=(v,b)$ is a weak solution of (1.1) and (1.2) with initial divergence free data $(v_0,b_0)\in Ł^p(R^n)\times Ł^p(R^n)$ if the following conditions hold

(1) $u(x,t)\in\mathcal{L}^{p,q}(S_T)$ for some p,q with $p,q\geq 2$;

(2) for $\phi,\psi\in\mathfrak{D}_T$,

$$\begin{aligned}
&\int_0^T\int_{R^n}\langle v,(\nu\Delta+D_t)\phi\rangle dx\,dt+\int_0^T\int_{R^n}\langle v,(\nabla\phi)v\rangle dx\,dt\\
&\quad+\int_0^T\int_{R^n}\langle v,2\Omega,\times\phi\rangle dx\,dt-\frac{1}{\rho\mu}\int_0^T\int_{R^n}\langle b,(\nabla\phi)b\rangle dx\,dt\\
&\quad=-\int_{R^n}\langle v_0,\phi(x,0)\rangle dx+\int_0^T\int_{R^n}\langle f(x,t),\phi\rangle dx\,dt;\\
&\int_0^T\int_{R^n}\langle b,(\lambda\Delta+D_t)\psi\rangle dx\,dt+\int_0^T\int_{R^n}\langle v,(\nabla\psi)b\rangle dx\,dt\\
&\quad-\int_0^T\int_{R^n}\langle b,(\nabla\psi)v\rangle dx\,dt\\
&\quad=-\int_{R^n}\langle b_0,\psi(x,0)\rangle dx+\int_0^T\int_{R^n}\langle g(x,t),\psi\rangle dx\,dt;
\end{aligned}\tag{2.2}$$

(3) for almost every $t\in[0,T]$, $\operatorname{div}v(x,t)=\operatorname{div}b(x,t)=0$ in the distributional sense.

Following Fabes et al. [4], we can find a divergence free matrix fundamental solution $E_{i,j}$ for the n-dimensional heat equation. We define matrices $(E^k_{i,j})$, $k=1,2$ as follows:

$$E_{i,j}^k = \delta_{i,j}\Gamma_k(x,t) - R_i R_j \Gamma_k(x,t), \tag{2.3}$$

where

$$\Gamma_1 = \frac{e^{-|x|^2/4\nu t}}{(4\pi\nu t)^{n/2}}, \qquad \Gamma_2 = \frac{e^{-|x|^2/4\lambda t}}{(4\pi\lambda t)^{n/2}}, \tag{2.4}$$

R_j is the jth Riesz transform, namely, R_j is a singular integral operator on $L^p(R^n)$, $1 < p < \infty$, defined as

$$R_j(f) = \text{P.V.C}_j \int_{R^n} (x_j - y_j)|x-y|^{-n-1} f(y)\,dy. \tag{2.5}$$

Now, we define an integral operator $A(v,w)$ for $v = (v_1,\dots,v_n)$, $w = (w_1,\dots,w_n)$. Denote

$$B_k(v,w)(x,t) = \int_0^t \int_{R^n} \langle v(y,s), \nabla E^k(x-y,t-s)\rangle w(y,s)\,dy\,ds; \quad \text{for } k = 1,2. \tag{2.6}$$

$$D(v)(x,t) = \int_0^t \int_{R^n} \langle v(y,s), 2\Omega \times E^1(x-y,t-s)\rangle\,dy\,ds. \tag{2.7}$$

For $u_1 = (v_1, b_1)$, $u_2 = (v_2, b_2)$, let

$$A(u_1,u_2) = \begin{pmatrix} B_1(v_1,v_2) - \dfrac{1}{\rho\mu} B_1(b_1,b_2) \\ \dfrac{1}{2}\Big[B_2(v_1,b_1) - B_2(b_1,v_1) + B_2(v_2,b_2) - B_2(b_2,v_2)\Big] \end{pmatrix}. \tag{2.8}$$

3. Approximating system.

The following result was obtained in [2].

Theorem 3.1. *Let v_0, $b_0 \in L^r$, $1 \le r < \infty$, be divergence free weakly. $u(x,t) = (v(x,t), b(x,t)) \in \mathcal{L}^{p,q}(S_T)$, $p,q \ge 2$, $p < \infty$, is a weak solution of (1.1) and (1.2) with initial value (v_0,b_0) if and only if u is a solution of the following integral equation:*

$$u + A(u,u) + D(u) = u^0 + f^0, \tag{3.1}$$

where

$$u^0 = \begin{pmatrix} \displaystyle\int_{R^n} \Gamma_1(x-y,t)v_0(y)\,dy \\ \displaystyle\int_{R^n} \Gamma_2(x-y,t)b_0(y)\,dy \end{pmatrix},$$

$$f^0 = \begin{pmatrix} \displaystyle\int_0^t \int_{R^n} E^1(x-y,t-s)f(y,s)\,dy\,ds \\ \displaystyle\int_0^t \int_{R^n} E^2(x-y,t-s)g(y,s)\,dy\,ds \end{pmatrix}. \tag{3.2}$$

We need the following lemmas obtained by Calderón [1].

Lemma 3.1. *Let $f \in L^p(R^n)$, $2 < p < n$, be a given vector function such that* $\operatorname{div} f = 0$ *in the distributional sense. Then, for each $s > 0$, f can be expressed as $g + h$, where*

$$\begin{aligned} \|g\|_n &\le c s^{1-(p/n)} \|f\|_p^{p/n}, \quad \operatorname{div} g = 0, \\ \|h\|_2 &\le c s^{1-(p/2)} \|f\|_p^{p/2}, \quad \operatorname{div} h = 0, \end{aligned} \tag{3.3}$$

where the constant c depends only on n and p.

Lemma 3.2. *Let $T(u,v) = B(u,v) + l(u) + F$ ($B(u,v)$ is bilinear and $l(u)$ is linear) satisfy*

$$\|T(u,v)\| \le c_1\|u\|\|v\| + c_2\|u\| + \|F\| \tag{3.4}$$

with the same norm in a Banach space. Then the quadratic operator $T(u,v)$ maps the ball $\{\|u\| \le s_1\}$ into itself if s_1 is the smallest root of

$$c_1 s^2 + (c_2 - 1)s + \|F\| = 0, \tag{3.5}$$

provided that c_1, c_2, and $\|F\|$ satisfy

$$(1-c_2)^2 > 4c_1\|F\|, \quad c_1 > 0,\ 0 \le c_2 < 1. \tag{3.6}$$

If $2s_1c_1 + c_2 < 1$, $T(u,v)$ is a contraction mapping in the ball of radius s_1. In particular, $T(u,v)$ is a contraction mapping in the ball of radius s_1 if

$$2c_1\|F\|\big((1-c_2)^2 - 4c_1\|F\|\big)^{-1/2} + c_2 < 1. \tag{3.7}$$

Consider the following system in v_1, v_2, b_1, b_2, p_1, p_2, q_1, and q_2

$$\begin{gathered}
L_1 v_1 + (\nabla v_1)v_1 - \frac{1}{\rho\mu}(\nabla b_1)b_1 + \nabla p_1 = 0,\\
L_1 v_2 + (\nabla v_2)v_2 + (\nabla v_2)v_1 + (\nabla v_1)v_2 - \frac{1}{\rho\mu}(\nabla b_2)b_2 + (\nabla b_2)b_1 + (\nabla b_1)b_2 + \nabla p_2 = 0,\\
L_2 b_1 + (\nabla b_1)v_1 - (\nabla v_1)b_1 + \nabla q_1 = 0,\\
L_2 b_2 + (\nabla b_2)v_2 + (\nabla b_2)v_1 + (\nabla b_1)v_2 - (\nabla v_2)b_2 - (\nabla v_2)b_1 - (\nabla v_1)b_2 + \nabla q_2 = 0,\\
\operatorname{div} v_i = 0, \qquad \operatorname{div} b_i = 0, \quad i = 1,2,\\
v_i(x,0) = h_i(x), \qquad b_i(x,0) = k_i(x), \quad i = 1,2,
\end{gathered} \tag{3.8}$$

where $L_1 = \partial/\partial t - \nu\Delta$, $L_2 = \partial/\partial t - \lambda\Delta$. We have the following definition.

Definition 3.1. The vector $((v_1,v_2),(u_1,u_2))$ is said to be a weak solution of (3.8) if $((v_1+v_2),(b_1+b_2))$ is a weak solution of (1.1) and (1.2) with initial data (h_1+h_2, k_1+k_2).

It then follows from Theorem 3.1 that

Theorem 3.2. *The vector functions $((v_1,v_2),(b_1,b_2)) \in \mathcal{L}^{p,q}(S_T)^4$, $2 \le p$, $q \le \infty$, are weak solutions of (3.8) if and only if they are solutions of the following integral equations:*

$$\begin{gathered}
v_1 + B_1(v_1,v_1) - \frac{1}{\rho\mu}B_1(b_1,b_1) = v_1^0,\\
v_2 + B_1(v_2,v_2) + B_1(v_1,v_2) + B_1(v_2,v_1) - \frac{1}{\rho\mu}\big[B_1(b_2,b_2) + B_1(b_1,b_2) + B_1(b_2,b_1)\big] = v_2^0,\\
b_1 + B_2(v_1,b_1) - B_2(b_1,v_1) = b_1^0,\\
b_2 + B_2(v_2,b_2) + B_2(v_1,b_2) + B_2(v_2,b_1) - \big[B_2(b_2,v_2) + B_2(b_1,v_2) + B_2(b_2,v_1)\big] = b_2^0,
\end{gathered} \tag{3.9}$$

where

$$v_i^0 = \Gamma_1 * h_i, \qquad b_i^0 = \Gamma_2 * k_i, \quad i = 1,2. \tag{3.10}$$

Now, let us introduce Leray's approximating system. Let $\alpha(x)$ be a C^∞ nonnegative, compact supported function on R^n with integral equal to 1, $\alpha_\varepsilon(x) = \varepsilon^{-n}\alpha(\varepsilon^{-1}x)$. Denote the modifying function of $u(x,t)$ by $u^\#(x,t)$, i.e., $u^\# = \alpha_\varepsilon * u$. For each ε, consider the following approximating system

$$L_1 v_1 + (\nabla u)v_1^\# - \frac{1}{\rho\mu}(\nabla b_1)b_1^\# + \nabla p_1 = 0 \tag{3.11}$$

$$\begin{aligned} L_1 v_2 + \nabla(v_1+v_2)v_2^\# + (\nabla v_2)v_1^\# - \frac{1}{\rho\mu}&((\nabla b_2)b_2^\# \\ &+ (\nabla b_2)b_1^\# + (\nabla b_1)b_2^\#) + \nabla p_2 = 0, \end{aligned} \tag{3.12}$$

$$L_2 b_1 + (\nabla b_1)v_1^\# - (\nabla v_1)b_1^\# + \nabla q_1 = 0, \tag{3.13}$$

$$\begin{aligned} L_2 b_2 + (\nabla b_2)v_2^\# + (\nabla b_2)v_1^\# + (\nabla b_1)v_2^\# - (\nabla v_2)b_2^\# \\ - (\nabla v_2)b_1^\# - (\nabla v_1)b_2^\# + \nabla q_2 = 0, \end{aligned} \tag{3.14}$$

$$\operatorname{div} v_i = 0, \qquad \operatorname{div} b_i = 0, \quad i = 1,2, \tag{3.15}$$

$$v_i(x,0) = v_i'^\#(x), \qquad b_i(x,0) = b_i'^\#(x), \quad i = 1,2, \tag{3.16}$$

where $v_i', b_i', i = 1,2$, are partitions of initial data v_0, b_0, respectively, in the sense of Lemma 3.1, i.e., $v_0 = v_1' + v_2'$, $b_0 = b_1' + b_2'$. From Lemma 3.1, we have

$$\begin{aligned} \|v_1'^\#\|_n &\le \|v_1'\|_n \le c s^{1-(p/n)}\|v_0\|_n, \\ \|v_2'^\#\|_2 &\le \|v_2'\|_2 \le c s^{1-(p/2)}\|v_0\|_2, \\ \|b_1'^\#\|_n &\le \|b_1'\|_n \le c s^{1-(p/n)}\|b_0\|_n, \\ \|b_2'^\#\|_2 &\le \|b_2'\|_2 \le c s^{1-(p/2)}\|b_0\|_2. \end{aligned} \tag{3.17}$$

4. Some lemmas. In this section, we present some lemmas without providing much of the details of their proofs for the arguments involved are similar to those used in [1].

First, we consider (3.11), (3.13), and (3.15) with corresponding data $v_1(x,0) = v_1'^\#$, $b_1(x,0) = b_1'^\#$. The problem is equivalent to the following integral equations (cf. [2])

$$\begin{aligned} v_1 + B_1(v_1, v_1^\#) - \frac{1}{\rho\mu}B_1(b_1, b_2^\#) &= v_1^{0\#}, \\ b_1 + B_2(v_1, b_1^\#) - B_2(b_1, v_1^\#) &= b_1^{0\#}, \end{aligned} \tag{4.1}$$

where $v_1^{0\#}$, $v_2^{0\#}$ are defined by (3.10) with h_i, k_i being replaced by $v_1'^\#$, $b_1'^\#$, respectively. Denote $u_1 = (v_1, b_1)$, the solution of (4.1). Define, for $s > 0$, and a function $w, w^s = w$ if $|w| < s$, $w = 0$ otherwise. We have the following lemma:

Lemma 4.1. *The system (4.1), including the limit case, i.e., when* $u_1 = u_1^{\#}$, $(v_1^0, b_1^0) = (v_1^{0\#}, b_1^{0\#})$, *admits a unique solution* $u_1 = (v, b)$, *for all* t, *satisfying*

$$||u_1^*||_n(\infty) \le c s^{1-(p/n)} ||u_1^0||_p^{p/n}, \tag{4.2}$$

provided that $s^{1-(p/n)}\|u_1^0\|_p^{p/n} < \varepsilon_0$, *where* $u_1^0 = (v_1^0, b_1^0)$, *and*

$$||u_1^*||_p(\infty) \le c \max\left(s^{1-(p/n)}||u_1^0||_p^{p/n}, ||(u_1^0)^s||_p\right), \tag{4.3}$$

provided that $\max(s^{1-(p/n)}\|u_1^0\|_p^{p/n}, \|(u_1^0)^s\|_p) < \varepsilon_0$, *where* $(u_1^0)^s = ((v_1^0)^s, (b_1^0)^s)$, ε_0 *is a fixed and a small constant and* c *depends only on* ε_0.

Proof. The proof is a direct extension of that of [1, Lem. III.1]. □

Lemma 4.2. *Let* $u_1' = (v_1', b_1')$ *be chosen such that* $\|(v_1', b_1')\|_p$ *is so small that the existence of solution* u_1 *is assured by Lemma 4.1 and such that, for all* t,

$$||u_1^*||_n < a_0 < c_0^{-1}, \tag{4.4}$$

where c_0 *is an independent constant. Suppose that* $u_2 = (v_2, b_2)$ *is a solution of (3.12), (3.14), (3.15), and (3.16) and suppose that* ∇v_2, ∇b_2, $(\partial/\partial t)v_2$, $(\partial/\partial t)b_2 \in L^2(S_T)$. *Then* u_2 *satisfies the following estimate:*

$$|u_2(t)|_2^2 + 2(1 - c_0 a_0)\int_0^t |\|\nabla u_2\||_2^2 dt \le |u_2(0)|_2^2, \tag{4.5}$$

where

$$\begin{aligned} |u_2|_2^2 &= \left(\|v_2\|_2^2 + \frac{1}{\rho\mu}\|b_2\|_2^2\right), \\ |\|\nabla u_2\||_2^2 &= \left(\nu\|\nabla v_2\|_2^2 + \frac{1}{\rho\mu}\|\nabla b_2\|_2^2\right). \end{aligned} \tag{4.6}$$

Proof. Multiplying (3.12) and (3.14) by v_2, b_2, respectively and integrating over R^n, we get

$$\begin{aligned} &\frac{1}{2}\frac{d\|v_2\|^2}{dt} + \nu\|\nabla v_2\|_2^2 + ((\nabla v_1)v_2^{\#}, v_2) \\ &\qquad - \frac{1}{\rho\mu}\left[((\nabla b_1)b_2^{\#}, v_2) + ((\nabla b_2)b_1^{\#}, v_2) + ((\nabla b_2)b_2^{\#}, v_2)\right] = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\frac{1}{2}\frac{d\|b_2\|^2}{dt} + \lambda\|\nabla b_2\|_2^2 + ((\nabla b_1)v_2^{\#}, b_2) \\ &\qquad - \left[((\nabla v_1)b_2^{\#}, b_2) + ((\nabla v_2)b_2^{\#}, b_2) - ((\nabla v_2)b_1^{\#}, b_2)\right] = 0. \end{aligned} \tag{4.8}$$

Note that, for functions a, b, c, and exponents r, n, 2 such that $(1/r) + (1/n) + (1/2) = 1$, we have

$$|((\nabla a)b, c)| \le \|\nabla a\|_2\|b\|_r\|c\|_n. \tag{4.9}$$

Multiplying (4.8) by $(1/\rho\mu)$ and adding the resulting equation to (4.7), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,|(v_2,b_2)|^2 &+ |\|(\nabla v_2,\nabla b_2)\||_2^2 \\ &\le c_1(\|v_1\|_n+\|b_1\|_n)(\|\nabla v_2\|_2^2+\|\nabla b_2\|^2) \\ &\le c_2(\|v_1\|_n+\|b_1\|_n)|\|(v,b)\||_2^2.\end{aligned} \tag{4.10}$$

It is then standard to obtain (4.5). □

Now, let us consider the existence of a weak solution of (3.12), (3.14), (3.15), and (3.16). It is easy to see that the system is equivalent to the following

$$\begin{aligned} v_2+\mathcal{B}_1(u_1,u_2) &= v_2^{0\#}, \\ b_2+\mathcal{B}_2(u_1,u_2) &= b_2^{0\#}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned}\mathcal{B}_1(u_1,u_2) &= B_1(v_2,v_2^{\#})+B_1(v_1,v_2^{\#})+B_1(v_2,v_1^{\#}) \\ &\quad -\frac{1}{\rho\mu}B_1(b_2,b_2^{\#})+B_1(b_1,b_2^{\#})+B_1(b_2,b_1^{\#}), \\ \mathcal{B}_2(u_1,u_2) &= B_2(v_2,b_2^{\#})+B_2(v_1,b_2^{\#})+B_2(v_2,b_1^{\#}) \\ &\quad -B_2(b_2,v_2^{\#})+B_2(b_1,v_2^{\#})+B_2(b_2,v_1^{\#}).\end{aligned} \tag{4.12}$$

Lemma 4.3. *If T is suitably small, then there exists a solution u_2 of (4.11) such that*

$$||u_2^*||_2(T) < \infty. \tag{4.13}$$

Proof. Applying the standard estimate on E^i, the definition of B_i (cf. (2.6)) and the Hardy-Littlewood-Sobolev potential inequality we can prove that

$$||\mathcal{B}_i(u_1,u_2)||_2(T) \le c\Big(\varepsilon^{-n/2}||u_2^*||_2(T)+||u_1^*||_n(T)\Big)||u_2^{\#}||_2(T), \quad i=1,2. \tag{4.14}$$

Taking $\varepsilon^{-n/2}T^{1/2}$ and $\|u_1^*\|_n$ small enough, we can apply Lemma 3.2 to obtain the existence of u_2. □

Using the arguments in the proofs of Lemmas [1, III.3, III.4], one can similarly prove the next two lemmas.

Lemma 4.4. *Let $u_1(x,t)=(v_1,b_1)$ be the solution obtained in Lemma 4.1 solving (4.1). Then, for all $T>0$ and all $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, we have*

$$\begin{aligned} ||D^\alpha u_1||_n(S_T) &< \infty, \\ ||D_tD^\alpha u_1||_n(S_T) &< \infty. \end{aligned} \tag{4.15}$$

Lemma 4.5. *Consider the following integral equations of unknown $u_2=(v_2,b_2)$:*

$$\begin{aligned} v_2+\mathcal{B}_1(u_1,u_2) &= F_1(x,t), \\ b_2+\mathcal{B}_2(u_1,u_2) &= F_2(x,t), \end{aligned} \tag{4.16}$$

where $\mathcal{B}_i$, $i=1,2$ are defined in (4.12), u_1 is the solution of (4.1), and F_1, F_2 are functions satisfying

$$||D^\alpha F_i||_2(S_T) < \infty, \qquad ||D_tD^\alpha F_i||_2(S_T) < \infty. \tag{4.17}$$

If we denote $T > 0$ the existence interval for t of solution of (4.16) by the standard fixed point argument, then

$$||D^\alpha u_2||_2(S_T) < \infty, \tag{4.18}$$

$$||D_t D^\alpha u_2||_2(S_T) < \infty. \tag{4.19}$$

Using the above estimates, we can prove the following theorem.

Lemma 4.6. *The solution obtained in Lemma 4.3 can be extended to all time $t > 0$ and it satisfies (4.5) for all t.*

Proof. We only give a sketch of the proof here. The existence time T obtained in Lemma 4.3 by the standard fixed point argument depends only on the L^2 norm of the initial data and $\|u_1^*\|_n$. Lemma 4.2 implies that $\|u_2(t)\|_2$ is uniformly bounded by the corresponding norm of the initial data when u_2 satisfies the regularity conditions of Lemma 4.2, which is guaranteed by Lemmas 4.4 and 4.5. Therefore, (4.5) holds for all t by moving from $[0,T]$ to $[T,2T]$ to $[2T,3T]$ and so on. And then the interval of existence can be extended to $(0,\infty)$. □

5. The global existence theorem. In this section, we establish some a priori estimate for the solution of (4.11) and obtain, by following Calderón's procedure [1], the global existence and uniqueness of solution of (1.1) and (1.2).

To adapt Leray's argument [8] to prove Lemmas 5.2 and 5.3 that we state later, we need to establish the following a priori estimate.

Lemma 5.1. *For a C^∞ function $\beta(x)$ satisfying $\beta(x) = 1$, if $|x| > N$; $\beta(x) = 0$, if $|x| < N/2$, and $\|\nabla\beta\| \le C/N$, the solution u_2 of (4.11) satisfies the following inequality*

$$\begin{aligned}
&\frac{1}{2}\int_{R^n}\beta(x)\Big[|v_2|^2+\frac{1}{\rho\mu}|b_2|^2\Big]dx+\int_0^t\int_{R^n}\beta(x)\Big[\nu|\nabla v_2|^2+\frac{1}{\rho\mu}|\nabla b_2|^2\Big]dx\,dt\\
&\quad\le\frac{1}{2}\int_{R^n}\beta(x)\Big[\,|v_2(0)|^2+\frac{1}{\rho\mu}|b_2(0)|^2\Big]dx+C\Big(\frac{1+t}{N}\Big)\,|u_2(0)|_2^2\,dx\,dt\\
&\quad\quad+\frac{C}{N}\|u_1\|_{n,\infty}(T)\,|u_2(0)|_2^2+C\|\beta u_1\|_{n,\infty}(T)\,|u_2(0)|_2^2,
\end{aligned}\tag{5.1}$$

where $|u_2|_2$ is defined by (4.6).

Proof. Multiplying equations (3.12) and (3.14) by βv_2 and βb_2, respectively, and integrating over R^n, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{R^n}&\beta(x)|v_2|^2\,dx\nu(\nabla v_2,\nabla(\beta v_2))+((\nabla v_1)v_2^\#,\beta v_2)\\
&+((\nabla v_2)v_2^\#,\beta v_2)+((\nabla v_2)v_1^\#,\beta v_2)\\
&-\frac{1}{\rho\mu}\Big[((\nabla b_1)b_2^\#,\beta v_2)+((\nabla b_2)b_2^\#,\beta v_2)+((\nabla b_2)b_1^\#,\beta v_2)\Big]\\
&-\int_{R^n}(\nabla\beta,v_2)p_2\,dx=0,
\end{aligned}\tag{5.2}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{R^n}\beta(x)|b_2|^2\,dx+\lambda(\nabla b_2,\nabla(\beta b_2))+((\nabla b_1)v_2^{\#},\beta b_2)\\ +((\nabla b_2)v_2^{\#},\beta b_2)+((\nabla b_2)v_1^{\#},\beta b_2)\\ -\Big[((\nabla v_1)b_2^{\#},\beta b_2)+((\nabla v_2)b_2^{\#},\beta b_2)+((\nabla v_2)b_1^{\#},\beta b_2)\Big]\\ +\int_{R^n}(\nabla\beta,b_2)q_2\,dx=0.\end{aligned}\tag{5.3}$$

Let us now separately estimate terms on the left-hand sides of (5.2) and (5.3). First, we deal with the terms on the left side of (5.2). For the second term, we have

$$\begin{aligned}\nu(\nabla v_2,\nabla(\beta v_2))&\geq\nu\int_{R^n}\beta|\nabla v_2|^2\,dx-\nu(\nabla v_2,\nabla\beta v_2)\\ &\geq\nu\int_{R^n}\beta|\nabla v_2|^2\,dx-\frac{C}{N}\|\nabla v_2\|_2\|v_2\|_2.\end{aligned}\tag{5.4}$$

For the third term, we apply Hölder's inequality for exponents, r, 2, n, to get

$$\begin{aligned}((\nabla v_1)v_2^{\#},\beta v_2)&=-((\nabla(\beta v_2))v_2^{\#},v_1)=-((\nabla\beta v_2)v_2^{\#},v_1)-((\nabla v_2)v_2^{\#},\beta v_1)\\ &\leq\frac{C}{N}\|v_1\|_n\|v_2\|_r\|v_2^{\#}\|_2+\|\beta v_1\|_n\|\nabla v_2\|_2\|v_2^{\#}\|_r\\ &\leq\frac{C}{N}\|v_1\|_n\|\nabla v_2\|_2^2+\|\beta v_1\|_n\|\nabla v_2\|_2^2,\end{aligned}\tag{5.5}$$

where $1/r=(1/2)-(1/n)$. For the fourth term, integration by parts, Hölder's inequality, and then Sobolov's inequality yield

$$\begin{aligned}\big|((\nabla v_2)v_2^{\#},\beta v_2)\big|&=\Big|\tfrac{1}{2}((v_2\nabla\beta)v_2^{\#},v_2)\Big|\\ &\leq\frac{C}{N}\|v_2\|_2\Big(\|\nabla v_2\|_2^2+\|v_2\|_2^2\Big).\end{aligned}\tag{5.6}$$

For the fifth term, we have

$$\big|((\nabla v_2)v_1^{\#},\beta v_2)\big|\leq\|\beta v_1\|_n\|\nabla v_2\|\|v_2\|.\tag{5.7}$$

The estimates on the sixth and eighth terms can be obtained, respectively, as

$$\begin{aligned}\left|\frac{1}{\rho\mu}((\nabla b_1)b_2^{\#},\beta v_2)\right|&\leq\frac{C}{N}\|b_1\|_n\|\nabla v_2\|_2\|b_2\|_2+C\|\beta b_1\|_n\|\nabla v_2\|_2\|\nabla b_2\|_2,\\ \left|\frac{1}{\rho\mu}((\nabla b_2)b_1^{\#},\beta v_2)\right|&\leq C\|\beta b_1\|_n\|\nabla b_2\|_2\|\nabla v_2\|_2.\end{aligned}\tag{5.8}$$

We do not need to estimate the seventh term because it will be canceled with part of the seventh term in (5.3).

Now, let us check terms on the left-hand side of (5.3). Similarly, for the second term, we have

$$\lambda(\nabla b_2,\nabla(\beta b_2))\geq\lambda\int_{R^n}\beta|\nabla b_2|^2\,dx-\frac{C}{N}\|\nabla b_2\|_2\|b_2\|_2.\tag{5.9}$$

For the third term, we have

$$((\nabla b_1)v_2^{\#},\beta b_2)\leq\frac{C}{N}\|b_1\|_n\|\nabla v_2\|_2\|\nabla b_2\|_2+\|\beta b_1\|_n\|\nabla v_2\|_2\|b_2\|_2.\tag{5.10}$$

For the fourth term, we get

$$\big|\,((\nabla b_2)v_2^{\#},\beta b_2)\,\big| \le \frac{C}{N}\|b_2\|_2\Big(\|\nabla b_2\|_2^2+\|b_2\|_2^2\Big). \tag{5.11}$$

For the fifth and sixth terms, we have, respectively,

$$\begin{aligned}\big|\,((\nabla b_2)v_1^{\#},\beta b_2)\,\big| &\le \|\beta v_1\|_n\|\nabla b_2\|\|b_2\|,\\ \big|\,((\nabla v_1)b_2^{\#},\beta b_2)\,\big| &\le \frac{C}{N}\|v_1\|_n\|\nabla b_2\|_2\|b_2\|_2+C\|\beta v_1\|_n\|\nabla b_2\|_2\|b_2\|_2.\end{aligned} \tag{5.12}$$

The seventh term

$$\big|-((\nabla v_2)b_2^{\#},\beta b_2)-((\nabla b_2)b_2^{\#},\beta v_2)\,\big| \le \frac{C}{N}\|v_2\|_2\Big(\|\nabla b_2\|^2+\|b_2\|_2^2\Big). \tag{5.13}$$

A multiple of the second term on the left-hand side of the above inequality cancels out the seventh term on the left-hand side of (5.2). The estimate on the eighth term can be obtained as

$$\big|\,((\nabla v_2)b_1^{\#},\beta b_2)\,\big| \le C\|\beta b_1\|_n\|\nabla v_2\|_2\|\nabla b_2\|_2. \tag{5.14}$$

Now, multiplying (5.3) by $1/\rho\mu$, adding the resulting equation to (5.2), and applying the above estimates yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{R^n}\beta(x)\Big(|v_2|^2+\frac{1}{\rho\mu}|b_2|^2\Big)+\int_{R^n}\beta(x)\Big(\nu|\nabla v_2|^2+\frac{1}{\rho\mu}|b_2|^2\Big)\,dx\\ &\qquad\le \frac{C}{N}\|u_1\|_n\|\nabla u_2\|_2\|u_2\|_2+C\|\beta u_1\|_n\|\nabla u_2\|_2\|\nabla u_2\|_2\\ &\qquad\quad+\int_{R^n}(\nabla\beta,v_2)p_2\,dx+\int_{R^n}(\nabla\beta,v_2)q_2\,dx.\end{aligned} \tag{5.15}$$

We use Riesz transformation to express p_2 and q_2 as

$$\begin{aligned}p_2 &= R_iR_j\Big(v_{1i}(v_2)_j^{\#}+v_{2i}(v_2)_j^{\#}+v_{2i}(v_1)_j^{\#}\\ &\qquad -\frac{1}{\rho\mu}b_{1i}(b_2)_j^{\#}+b_{2i}(b_2)_j^{\#}+b_{2i}(b_1)_j^{\#}\Big),\\ q_2 &= R_iR_j\Big(b_{1i}(v_2)_j^{\#}+b_{2i}(v_2)_j^{\#}+b_{2i}(v_1)_j^{\#}-v_{1i}(b_2)_j^{\#}-v_{2i}(b_2)_j^{\#}-v_{2i}(b_1)_j^{\#}\Big).\end{aligned} \tag{5.16}$$

Since R_i is a continuous map from L^p to itself for $p>1$, we have

$$\begin{aligned}&\Big|\int_{R^n}(\nabla\beta,v_2)p_2\,dx+\int_{R^n}(\nabla\beta,v_2)q_2\,dx\Big|\\ &\qquad\le \frac{C}{N}\|u_1\|_n\|\nabla u_2\|_2\|u_2\|_2+C\|\beta u_1\|_n\|\nabla u_2\|_2\|\nabla u_2\|_2.\end{aligned} \tag{5.17}$$

Plugging this inequality into (5.15) and integrating over $[0,t]$, we complete the proof of the lemma. □

Applying Lemma 5.1 and following the procedure adapted in [1], we can similarly prove the next two lemmas.

Lemma 5.2. (1) *Let* $u_1'(0) = (v_1'(0), b_1'(0))$ *be the partition by Lemma 3.1 such that Lemma 4.2 holds. There is a* $T > 0$*, depending only on the norm of* $u_1'(0)$*,*

$$||u_1'(0)|| = ||u_1'(0)||_n + ||u_1'(0)||_{n/\beta}, \quad 0 < \beta < 1, \tag{5.18}$$

such that $u_1(x,t)$*, as a family depending on parameter* ε *and* $0 < t < T$*, is compact in* L^n*;*
(2) *the size of* T *is determined by*

$$T^{(1-\beta)/2}\left(s^{1-(\beta p/n)}\|u(0)\|_p^{\beta p/n} + s^{1-(\beta/n)}\|u(0)\|_p^{p/n}\right) < \varepsilon_0; \tag{5.19}$$

(3) *the following inequalities hold*

$$\begin{aligned} ||u_1^*||_n &\le c_1(\varepsilon_0)\left(s^{1-(\beta p/n)}\|u(0)\|_p^{\beta p/n} + s^{1-(p/n)}\|u(0)\|_p^{p/n}\right), \\ ||u_1^*||_p &\le c_2(\varepsilon_0)\left(s^{1-(\beta p/n)}\|u(0)\|_p^{\beta p/n} + \|u^s(0)\|_p\right). \end{aligned} \tag{5.20}$$

Lemma 5.3. *The solution* $u_2(x,t) = (v_2, b_2)$ *of (3.12), (3.14), (3.15), and (3.16), as a family depending on the parameter* ε*, contains a subfamily that converges in* L^2 *of any subset* S_T*, for* $n = 3, 4$*,* $T > 0$*.*

We are now ready to state and prove the main result of this paper.

Theorem 5.1. *Assume that the initial data* $(v_0, b_0) \in L^p(R^n)$*,* $2 < p < n$*,* $n = 3, 4$*,* $\operatorname{div} v_0 = \operatorname{div} b_0 = 0$*. Then there exists a weak solution* $u(x,t) = (v(x,t), b(x,t))$ *of (1.1) and (1.2) for all time* t*, such that, for* $0 < t < T$*, where* T *can be arbitrarily large, we have*

$$\|u\|_{p,2} < C, \tag{5.21}$$

where the constant C *depends on* T*,* $\|u_0\|_p$*.*

Proof. From Lemmas 5.2 and 5.3, we have a sequence of solutions u_{1m}, u_{2m} of (3.11), (3.12), (3.13), (3.14), (3.15), and (3.16) such that u_{1m}, u_{2m} converge in $L^n(S_T)$, $L^2(S_T)$ to u_1, u_2, respectively. Sending m to ∞, we see that $u_1 + u_2$ is a weak solution of (3.9) for some $T > 0$.

By Lemma 5.2, we have

$$||u_{1m}^*||_p \le c_2(\varepsilon_0)\left(s^{1-(\beta p/n)}\|u(0)\|_p^{\beta p/n} + \|u^s(0)\|_p\right), \quad [0,T]. \tag{5.22}$$

Fatou's theorem implies that

$$||u_1^*||_{p,2} \le T^{1/2} c_2(\varepsilon_0)\left(s^{1-(\beta p/n)}\|u(0)\|_p^{\beta p/n} + \|u^s(0)\|_p\right), \quad [0,T]. \tag{5.23}$$

Now, for u_{2m}, from a priori estimate for u_{2m}, we have

$$\begin{aligned} \|u_{2m}\|_{p,2} &\le c_1\left(\|\nabla u_{2m}\|_{2,2} + \|u_{2m}\|_2\right) \\ &\le c_2\|u_{2m}(0)\|_2 \le c_2 s^{1-(p/2)}\|u(0)\|_p^{p/2}. \end{aligned} \tag{5.24}$$

Fatou's theorem implies that

$$||u_2||_{p,2} \le c_2 s^{1-(p/2)}\|u(0)\|_p^{p/2}. \tag{5.25}$$

(5.23) and (5.25) implies (5.21).

Due to a priori estimates, we can extend the interval of existence of solution u from $[0,T]$ to $[T, T_1]$, from $[T, T_1]$ to $[T_1, T_2]$, and so on in such a way that, in each step, we make sure that $T_{k+1} - T_k > \delta_0$—a fixed constant. Therefore, we obtain the weak solution u for all t. □

For $n \geq 3$, adapting Calderón's approach [1], one can also prove the global existence result for system (1.1) and (1.2) as long as the L^n norm of the initial data is suitably small.

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Bates and Chen: Department of Mathematics, Brigham Young University, Provo, Utah 84602, USA

Wang: Department of Mathematics, Penn State University, Schuylkill Hven, PA 17972, USA