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REGULARITY OF CONSERVATIVE INDUCTIVE LIMITS

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ABSTRACT. A sequentially complete inductive limit of Fréchet spaces is regular, see [3]. With a minor modification, this property can be extended to inductive limits of arbitrary locally convex spaces under an additional assumption of conservativeness.

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Throughout the paper $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $\mathrm{id} : E_n \to E_{n+1}$, $n \in N$. Their respective topologies are denoted by τ_n. The topology of their inductive limit $\mathrm{ind}\, E_n$ is denoted by $\tau = \mathrm{ind}\, \tau_n$.

We will use a result from [1, Cor. IV. 6.5]. It reads:

If F as well as all spaces E_n are Fréchet and $T : F \to \mathrm{ind}\, E_n$ is a linear map with a closed graph, then there is $n \in N$ such that T is a continuous map of F into E_n.

According to [2, Sec. 5.2], the space $\mathrm{ind}\, E_n$ is called α-regular, resp. regular, if every set bounded in $\mathrm{ind}\, E_n$ is contained, resp. bounded, in some constituent space E_n. We will need a slightly modified notion of regularity.

DEFINITION 1. An inductive limit $\mathrm{ind}\, E_n$ is quasi α-regular, resp. quasi regular, if every set bounded in $\mathrm{ind}\, E_n$ is a subset of a τ-closure of a set contained, resp. bounded, in some constituent space E_n.

DEFINITION 2. An inductive limit $\mathrm{ind}\, E_n$ is called conservative if for every linear subspace $F \subset \mathrm{ind}\, E_n$, we have

$$\mathrm{ind}\,(F \cap E_n, \tau_n) = (F, \mathrm{ind}\, \tau_n). \tag{1}$$

LEMMA. *Let a locally convex (Hausdorff) space E be sequentially complete, and B be a balanced, bounded, closed, and convex set in E. Then the linear span F of B, equipped with the topology generated by the Minkowski functional of B, is a Banach space and the identity map* $\mathrm{id} : F \to E$ *is continuous.*

PROOF. Clearly F is a normed space and $\mathrm{id} : F \to E$ is continuous.

To prove the completeness of F, take a Cauchy sequence $\{x_n\}$ in F. Since $\mathrm{id} : F \to E$ is continuous, $\{x_n\}$ is Cauchy in E. Hence it converges to some $x \in E$. The set $\bigcup\{x_n; n \in N\}$, which is bounded in F, is contained in some αB. Since the set αB is closed in E, we have $x \in \alpha B \subset F$.

For any 0-nbhd λB, $\lambda > 0$, in F, there exists $k \in N$ such that $m, n > k$ imply $x_n - x_m \in \lambda B$. If we let $m \to \infty$, we get $x_n - x \in \lambda B$ for $n > k$, i.e., $x_n \to x$ in F. □

Proposition 1. *Any sequentially complete* ind E_n *is quasi α-regular.*

Proof. Let a set A be bounded in ind E_n. Denote by B its balanced, convex, τ-closed hull, and by F the linear span of B with the same topology γ as in the Lemma. We know that F is a Banach space.

For any $n \in N$, denote by G_n the completion of the normed space $(F \cap E_n, \gamma)$. Then $G_n \subset F$ and F equals strict inductive limit ind G_n. Since B is bounded in F, it is bounded in ind G_n. Hence, by [1, Cor. IV. 6.5], B is bounded in some G_n.

Finally, $A \subset B$ and B is a γ-closure of a set $V = \bigcup\{E_n \cap \lambda B; 0 < \lambda < 1\}$ in $F \cap E_n$. Hence A is also a subset of the τ-closure of V in ind E_n. □

Proposition 2. *Let* ind E_n *be sequentially complete and conservative. Then every set $A \subset E_1$, which is bounded in* ind E_n *is also bounded in some constituent space E_n.*

Proof. Take such A and assume that it is not bounded in any E_n. Then for any $n \in N$, there exists a balanced convex 0-nbhd U_n in E_n which does not absorb A. For any $m, n \in N$, choose $a_{m,n} \in A$ such that $a_{m,n} \notin mU_n$. Denote by B the τ-closure of the convex balanced hull of $\bigcup\{a_{m,n}; m, n \in N\}$ and by F the linear span of B. For any $m, n \in N$, there exists $f_{m,n} \in (\text{ind}\, E_n)'$, (the dual of ind E_n), such that $f_{m,n}(a_{m,n}) \neq 0$. Put $V_{m,n} = \{x \in F; |f_{m,n}(x)| \leq 1\}$ and denote by F_n the linear space F equipped with the topology generated by $\{U_m; m \geq n\} \bigcup\{V_{m,n}; m, n \in N\}$. Then each F_n is a metrizable Hausdorff locally convex space and its completion G_n is a Fréchet space.

Finally, let H be the space F equipped with the topology generated by the Minkowski functional of B. The set B is bounded in ind E_n, hence, by the Lemma, H is Banach space and the identity map id : $H \to$ ind E_n is continuous.

Since ind E_n is conservative and $F \subset$ ind E_n, we have

$$\text{ind}\,(F, \tau_n) = (F, \text{ind}\,\tau_n). \tag{2}$$

For any $n \in N$, the identity maps $(F, \tau_n) \to F_n \to G_n$ are continuous. Hence

$$\text{id} : \text{ind}\,(F, \tau_n) \longrightarrow \text{ind}\, G_n \tag{3}$$

is continuous, too. Then, the continuity of id : $H \to$ ind E_n implies the continuity of id : $H \to (F, \text{ind}\,\tau_n)$. By (2) and (3), we finally get the continuity of id : $H \to$ ind G_n.

By [1, Cor. IV. 6.5], there exists $n \in N$ such that id : $H \to G_n$ is continuous. Since the set B is bounded in H and contained in F_n, it is bounded in G_n, and also bounded in F_n. But then B, as well as its subset A, are absorbed by the 0-nbhd V_n in F_n, a contradiction. □

By combining Propositions 1 and 2, we get

Theorem. *Any sequentially complete conservative* ind E_n *is quasi regular.*

Corollary. *If moreover each space E_n in the above Theorem is closed in* ind E_n, *then* ind E_n *is regular.*

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