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# ON CHARACTERIZATIONS OF A CENTER GALOIS EXTENSION

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ABSTRACT. Let $B$ be a ring with 1, $C$ the center of $B$, $G$ a finite automorphism group of $B$, and $B^G$ the set of elements in $B$ fixed under each element in $G$. Then, it is shown that $B$ is a center Galois extension of $B^G$ (that is, $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$) if and only if the ideal of $B$ generated by $\{c-g(c) \mid c \in C\}$ is $B$ for each $g \neq 1$ in $G$. This generalizes the well known characterization of a commutative Galois extension $C$ that $C$ is a Galois extension of $C^G$ with Galois group $G$ if and only if the ideal generated by $\{c-g(c) \mid c \in C\}$ is $C$ for each $g \neq 1$ in $G$. Some more characterizations of a center Galois extension $B$ are also given.

Keywords and phrases. Galois extensions, center Galois extensions, central extensions, Galois central extensions, Azumaya algebras, separable extensions, $H$-separable extensions.

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**1. Introduction.** Let $C$ be a commutative ring with 1, $G$ a finite automorphism group of $C$ and $C^G$ the set of elements in $C$ fixed under each element in $G$. It is well known that a commutative Galois extension $C$ is characterized in terms of the ideals generated by $\{c-g(c) \mid c \in C\}$ for $g \neq 1$ in $G$, that is $C$ is a Galois extension with Galois group $G$ if and only if the ideal generated by $\{c-g(c) \mid c \in C\}$ is $C$ for each $g \neq 1$ in $G$ (see [3, Proposition 1.2, page 80]). A natural generalization of a commutative Galois extension is the notion of a center Galois extension, that is, a noncommutative ring $B$ with a finite automorphism group $G$ and center $C$ is called a center Galois extension of $B^G$ with Galois group $G$ if $C$ is a Galois extension of $C^G$ with Galois group $G|_C \cong G$. Ikehata (see [4, 5]) characterized a center Galois extension with a cyclic Galois group $G$ of prime order in terms of a skew polynomial ring. Then, the present authors generalized the Ikehata characterization to center Galois extensions with Galois group $G$ of any cyclic order [7] and to center Galois extensions with any finite Galois group $G$ [8]. The purpose of the present paper is to generalize the above characterization of a commutative Galois extension to a center Galois extension. We shall show that $B$ is a center Galois extension of $B^G$ if and only if the ideal of $B$ generated by $\{c-g(c) \mid c \in C\}$ is $B$ for each $g \neq 1$ in $G$. A center Galois extension $B$ is also equivalent to each of the following statements:

(i) $B$ is a Galois central extension of $B^G$, that is, $B = B^G C$ which is $G$-Galois extension of $B^G$.

(ii) $B$ is a Galois extension of $B^G$ with a Galois system $\{b_i \in B,\ c_i \in C,\ i = 1,2,\dots,m\}$ for some integer $m$.

(iii) the ideal of the subring $B^G C$ generated by $\{c-g(c) \mid c \in C\}$ is $B^G C$ for each $g \neq 1$ in $G$.

**2. Definitions and notations.** Throughout this paper, $B$ will represent a ring with 1, $G = \{g_1 = 1, g_2, \ldots, g_n\}$ an automorphism group of $B$ of order $n$ for some integer $n$, $C$ the center of $B$, $B^G$ the set of elements in $B$ fixed under each element in $G$, and $B * G$ a skew group ring in which the multiplication is given by $gb = g(b)g$ for $b \in B$ and $g \in G$.

$B$ is called a $G$-Galois extension of $B^G$ if there exist elements $\{a_i, b_i \in B, i = 1, 2, \ldots, m\}$ for some integer $m$ such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$. Such a set $\{a_i, b_i\}$ is called a $G$-Galois system for $B$. $B$ is called a center Galois extension of $B^G$ if $C$ is a Galois algebra over $C^G$ with Galois group $G|_C \cong G$. $B$ is called a central extension of $B^G$ if $B = B^G C$, and $B$ is called a Galois central extension of $B^G$ if $B = B^G C$ is a Galois extension of $B^G$ with Galois group $G$.

Let $A$ be a subring of a ring $B$ with the same identity 1. We denote $V_B(A)$ the commutator subring of $A$ in $B$. We call $B$ a separable extension of $A$ if there exist $\{a_i, b_i \in B, i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b \in B$ where $\otimes$ is over $A$. $B$ is called $H$-separable extension of $A$ if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of $B$ as a $B$-bimodule. $B$ is called centrally projective over $A$ if $B$ is a direct summand of a finite direct sum of $A$ as a $A$-bimodule.

**3. The characterizations.** In this section, we denote $J_j^{(C)} = \{c - g_j(c) \mid c \in C\}$. We shall show that $B$ is a center Galois extension of $B^G$ if and only if $B = BJ_j^{(C)}$, the ideal of $B$ generated by $J_j^{(C)}$, for each $g_j \neq 1$ in $G$. Some more characterizations of a center Galois extension $B$ are also given. We begin with a lemma.

**Lemma 3.1.** *If $B = BJ_j^{(C)}$ for each $g_j \neq 1$ in $G$ (that is, $j \neq 1$), then*

(1) *$B$ is a Galois extension of $B^G$ with Galois group $G$ and a Galois system $\{b_i \in B;\ c_i \in C,\ i = 1, 2, \ldots, m\}$ for some integer $m$.*

(2) *$B$ is a centrally projective over $B^G$.*

(3) *$B * G$ is $H$-separable over $B$.*

(4) *$V_{B*G}(B) = C$.*

**Proof.** (1) Since $B = BJ_j^{(C)}$ for each $j \neq 1$, there exist $\{b_i^{(j)} \in B,\ c_i^{(j)} \in C,\ i = 1, 2, \ldots, m_j\}$ for some integer $m_j$, $j = 2, 3, \ldots, n$ such that $\sum_{i=1}^{m_j} b_i^{(j)}(c_i^{(j)} - g_j(c_i^{(j)})) = 1$. Therefore, $\sum_{i=1}^{m_j} b_i^{(j)} c_i^{(j)} = 1 + \sum_{i=1}^{m_j} b_i^{(j)} g_j(c_i^{(j)})$. Let $b_{m_j+1}^{(j)} = -\sum_{i=1}^{m_j} b_i^{(j)} g_j(c_i^{(j)})$ and $c_{m_j+1}^{(j)} = 1$. Then $\sum_{i=1}^{m_j+1} b_i^{(j)} c_i^{(j)} = 1$ and $\sum_{i=1}^{m_j+1} b_i^{(j)} g_j(c_i^{(j)}) = 0$. Let $b_{i_2, i_3, \ldots, i_n} = b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{in}^{(n)}$ and $c_{i_2, i_3, \ldots, i_n} = c_{i_2}^{(2)} c_{i_3}^{(3)} \ldots c_{i_n}^{(n)}$ for $i_j = 1, 2, \ldots, m_j + 1$ and $j = 2, 3, \ldots, n$. Then

$$\begin{aligned}\sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2, i_3, \ldots, i_n} c_{i_2, i_3, \ldots, i_n} &= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)} \cdots b_{i_n}^{(n)} c_{i_2}^{(2)} c_{i_3}^{(3)} \cdots c_{i_n}^{(n)} \\ &= \sum_{i_2=1}^{m_2+1} \sum_{i_3=1}^{m_3+1} \cdots \sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} c_{i_2}^{(2)} b_{i_3}^{(3)} c_{i_3}^{(3)} \cdots b_{i_n}^{(n)} c_{i_n}^{(n)} \\ &= \sum_{i_2=1}^{m_2+1} b_{i_2}^{(2)} c_{i_2}^{(2)} \sum_{i_3=1}^{m_3+1} b_{i_3}^{(3)} c_{i_3}^{(3)} \cdots \sum_{i_n=1}^{m_n+1} b_{i_n}^{(n)} c_{i_n}^{(n)} = 1\end{aligned} \tag{3.1}$$

and for each $j \neq 1$

$$\begin{aligned}
&\sum_{i_2=1}^{m_2+1}\sum_{i_3=1}^{m_3+1}\cdots\sum_{i_n=1}^{m_n+1} b_{i2,i_3,\dots,i_n} g_j(c_{i_2,i_3,\dots,i_n}) \\
&\quad= \sum_{i_2=1}^{m_2+1}\sum_{i_3=1}^{m_3+1}\cdots\sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)}\cdots b_{i_n}^{(n)} g_j\left(c_{i_2}^{(2)} c_{i_3}^{(3)}\cdots c_{i_n}^{(n)}\right) \\
&\quad= \sum_{i_2=1}^{m_2+1}\sum_{i_3=1}^{m_3+1}\cdots\sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} b_{i_3}^{(3)}\cdots b_{i_n}^{(n)} g_j\left(c_{i_2}^{(2)}\right) g_j\left(c_{i_3}^{(3)}\right)\cdots g_j\left(c_{i_n}^{(n)}\right) \\
&\quad= \sum_{i_2=1}^{m_2+1}\sum_{i_3=1}^{m_3+1}\cdots\sum_{i_n=1}^{m_n+1} b_{i_2}^{(2)} g_j\left(c_{i_2}^{(2)}\right) b_{i_3}^{(3)} g_j\left(c_{i_3}^{(3)}\right)\cdots b_{i_n}^{(n)} g_j\left(c_{i_n}^{(n)}\right) \\
&\quad= \sum_{i_2=1}^{m_2+1} b_{i_2}^{(2)} g_j\left(c_{i_2}^{(2)}\right) \sum_{i_3=1}^{m_3+1} b_{i_3}^{(3)} g_j\left(c_{i_3}^{(3)}\right)\cdots \sum_{i_n=1}^{m_n+1} b_{i_n}^{(n)} g_j\left(c_{i_n}^{(n)}\right) = 0.
\end{aligned} \tag{3.2}$$

Thus, $\{b_{i_2,i_3,\dots,i_n} \in B;\ c_{i_2,i_3,\dots,i_n} \in C,\ i_j = 1,2,\dots,m_j+1$ and $j = 2,3,\dots,n\}$ is a Galois system for $B$. This complete the proof of (1).

(2) By (1), $B$ is a Galois extension of $B^G$ with a Galois system $\{b_i \in B,\ c_i \in C,\ i = 1,2,\dots,m\}$ for some integer $m$. Let $f_i : B \to B^G$ given by $f_i(b) = \sum_{j=1}^n g_j(c_i b)$ for all $b \in B$, $i = 1,2,\dots,m$. Then it is easy to check that $f_i$ is a homomorphism as $B^G$-bimodule and $b = \sum_{i=1}^m b_i c_i b = \sum_{j=1}^n \sum_{i=1}^m b_i g_j(c_i) g_j(b) = \sum_{i=1}^m b_i \sum_{j=1}^n g_j(c_i b) = \sum_{i=1}^m b_i f_i(b)$ for all $b \in B$. Hence $\{b_i; f_i,\ i = 1,2,\dots,m\}$ is a dual bases for $B$ as $B^G$-bimodule, and so $B$ is finitely generated and projective as $B^G$-bimodule. Therefore, $B$ is a direct summand of a finite direct sum of $B^G$ as a $B^G$-bimodule. Thus $B$ is centrally projective over $B^G$.

(3) By (1), $B$ is a Galois extension of $B^G$ with Galois group $G$. Hence $B * G \cong \mathrm{Hom}_{B^G}(B, B)$ [2, Theorem 1]. By (2), $B$ is centrally projective over $B^G$. Thus, $B * G (\cong \mathrm{Hom}_{B^G}(B,B))$ is $H$-separable over $B$ [6, Proposition 11].

(4) We first claim that $V_{B*G}(C) = B$. Clearly, $B \subset V_{B*G}(C)$. Let $\sum_{j=1}^n b_j g_j\ in\ V_{B*G}(C)$ for some $b_j \in B$. Then $c(\sum_{j=1}^n b_j g_j) = (\sum_{j=1}^n b_j g_j)c$ for each $c \in C$, so $cb_j = b_j g_j(c)$, that is, $b_j(c - g_j(c)) = 0$ for each $g_j \in G$ and $c \in C$. Since $B = BJ_j^{(C)}$ for each $g_j \neq 1$, there exist $b_i^{(j)} \in B$ and $c_i^{(j)} \in C$, $i = 1,2,\dots,m$ such that $\sum_{i=1}^m b_i^{(j)}(c_i^{(j)} - g_j(c_i^{(j)})) = 1$. Hence $b_j = \sum_{i=1}^m b_i^{(j)}(c_i^{(j)} - g_j(c_i^{(j)}))b_j = \sum_{i=1}^m b_i^{(j)} b_j (c_i^{(j)} - g_j(c_i^{(j)})) = 0$ for each $g_j \neq 1$. This implies that $\sum_{j=1}^n b_j g_j = b_1 \in B$. Hence $V_{B*G}(C) \subseteq B$, and so $V_{B*G}(C) = B$. Therefore, $V_{B*G}(B) \subset V_{B*G}(C) = B$. Thus $V_{B*G}(B) = V_B(B) = C$. □

We now show some characterizations of a center Galois extension $B$.

**Theorem 3.2.** *The following statements are equivalent.*

(1) *$B$ is a center Galois extension of $B^G$.*

(2) *$B = BJ_j^{(C)}$ for each $g_j \neq 1$ in $G$.*

(3) *$B$ is a Galois extension of $B^G$ with a Galois system $\{b_i \in B,\ c_i \in C,\ i = 1,2,\dots,m\}$ for some integer $m$.*

(4) *$B$ is a Galois central extension of $B^G$.*

(5) *$B^G C = B^G C J_j^{(C)}$ for each $g_j \neq 1$ in $G$.*

**Proof.** (1)$\Longrightarrow$(2). By hypothesis, $C$ is a Galois extension of $C^G$ with Galois group $G|_C \cong G$. Hence $C = CJ_j^{(C)}$ for each $g_j \neq 1$ in $G$ [3, Proposition 1.2, page 80]. Thus, $B = BJ_j^{(C)}$ for each $g_j \neq 1$ in $G$.

(2)$\Longrightarrow$(1). Since $B = BJ_j^C$ for each $g_j \neq 1$ in $G$, $B*G$ is $H$-separable over $B$ by Lemma 3.1(3) and $V_{B*G}(B) = C$ by Lemma 3.1(4). Thus $C$ is a Galois extension of $C^G$ with Galois group $G|_C \cong G$ by [1, Proposition 4].

(1)$\Longrightarrow$(3). This is Lemma 3.1(1).

(3)$\Longrightarrow$(1). Since $B$ is Galois extension of $B^G$ with a Galois system $\{b_i \in B,\ c_i \in C,\ i = 1,2,\dots,m\}$ for some integer $m$, we have $\sum_{i=1}^m b_i g_j(c_i) = \delta_{1,g}$. Hence $\sum_{i=1}^m b_i(c_i - g_j(c_i))$ $= 1$ for each $g_j \neq 1$ $in$ $G$. So for every $b \in B$, $b = \sum_{i=1}^m bb_i(c_i - g_j(c_i)) \in BJ_j^{(C)}$. Therefore, $B = BJ_i^{(C)}$ for each $g_i \neq 1$ $in$ $G$. Thus, $B$ is a center Galois extension of $B^G$ by (2)$\Longrightarrow$(1).

(1)$\Longrightarrow$(4). Since $C$ is a Galois algebra with Galois group $G|_C \cong G$, $B$ and $B^GC$ are Galois extensions of $B^G$ with Galois group $G|_{B^GC} \cong G$. Noting that $B^GC \subset B$, we have $B = B^GC$, that is, $B$ is a central extension of $B^G$. But $B$ is a Galois extension of $B^G$, so $B$ is a Galois central extension of $B^G$.

(4)$\Longrightarrow$(1). By hypothesis, $B = B^GC$ is a Galois extension of $B^G$. Hence there exists a Galois system $\{a_i; b_i \in B,\ i = 1,2,\dots,m\}$ for some integer $m$ such that $\sum_{i=1}^m a_i g_j(b_i) = \delta_{1,j}$. But $B = B^GC$, so $a_i = \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)}$ and $b_i = \sum_{l=1}^{n_{b_i}} b_l^{(b_i)} c_l^{(b_i)}$ for some $a_k^{(a_i)}$, $b_l^{(b_i)}$ in $B^G$ and $c_k^{(a_i)}$, $c_l^{(b_i)}$ $in$ $C$, $k = 1,2\dots,n_{a_i}$, $l = 1,2,\dots,n_{b_i}$, $i = 1,2,\dots,m$. Therefore,

$$\begin{aligned}\delta_{1,j} &= \sum_{i=1}^m a_i g_j(b_i) = \sum_{i=1}^m \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)} g_j\left(\sum_{l=1}^{n_{b_i}} b_l^{(b_i)} c_l^{(b_i)}\right) \\ &= \sum_{i=1}^m \sum_{k=1}^{n_{a_i}} b_k^{(a_i)} c_k^{(a_i)} \sum_{l=1}^{n_{b_i}} b_l^{b_i} g_j\left(c_l^{(b_i)}\right) = \sum_{i=1}^m \sum_{k=1}^{n_{a_i}} \sum_{l=1}^{n_{b_i}} \left(b_k^{(a_i)} c_k^{(a_i)} b_l^{(b_i)}\right) g_j\left(c_l^{(b_i)}\right).\end{aligned} \tag{3.3}$$

This shows that $\{b_{k,l}^{(a_i,b_i)} = b_k^{(a_i)} c_k^{(a_i)} b_l^{(b_i)} \in B;\ c_{k,l}^{(a_i,b_i)} = c_l^{(b_i)} \in C,\ k = 1,2,\dots,n_{a_i},\ l = 1,2,\dots,n_{b_i},\ i = 1,2,\dots,m\}$ is a Galois system for $B$. Thus, $B$ is a center Galois extension of $B^G$ by (3)$\Longrightarrow$(1).

(1)$\Longrightarrow$(5). Since $B$ is a center Galois extension of $B^G$, $B = BJ_j^{(C)}$ for each $g_j \neq 1$ in $G$ by (1)$\Longrightarrow$(2) and $B = B^GC$ by (1)$\Longrightarrow$(4). Thus, $B^GC = B^GCJ_j^{(C)}$ for each $g_j \neq 1$ in $G$.

(5)$\Longrightarrow$(1). Since $B^GC = B^GCJ_j^{(C)}$ for each $g_j \neq 1$ in $G$, $B = BJ_j^{(C)}$ for each $g_j \neq 1$ in $G$. Thus, $B$ is a center Galois extension of $B^G$ by (2)$\Longrightarrow$(1). □

The characterization of a commutative Galois extension $C$ in terms of the ideals generated by $\{c - g(c) \mid c \in C\}$ for $g \neq 1$ in $G$ is an immediate consequence of Theorem 3.2.

**Corollary 3.3.** *A commutative ring $C$ is a Galois extension of $C^G$ if and only if $C = CJ_j^{(C)}$, the ideal generated by $\{c - g_j(c) \mid c \in C\}$ is $C$ for each $g_j \neq 1$ in $G$.*

**Proof.** Let $B = C$ in Theorem 3.2. Then, the corollary is an immediate consequence of Theorem 3.2(2). □

By Theorem 3.2, we derive several characterizations of a Galois centeral extension $B$.

**Corollary 3.4.** *If $B$ is a central extension of $B^G$ (that is, $B = B^G C$), then the following statements are equivalent.*

(1) *$B$ is a Galois extension of $B^G$.*

(2) *$B$ is a center Galois extension of $B^G$.*

(3) *$B * G$ is $H$-separable over $B$.*

(4) *$B = CJ_j^{(B)}$ for each $g_j \neq 1$ in $G$.*

(5) *$B = BJ_j^{(B)}$ for each $g_j \neq 1$ in $G$.*

**Proof.** (1)$\Longleftrightarrow$(2). This is given by (1)$\Longleftrightarrow$(4) in Theorem 3.2.

(2)$\Longrightarrow$(3). This is Lemma 3.1(3).

(3)$\Longrightarrow$(1). Since $B * G$ is $H$-separable over $B$, $B$ is a Galois extension of $B^G$ [1, Proposition 2].

Since $B = B^G C$ by hypothesis, it is easy to see that $J_j^{(B)} = B^G J_j^{(C)}$ for each $g_j$ in $G$. Thus, $B = CJ_j^{(B)}$, $B = BJ_j^{(B)}$, and $B = BJ_j^{(C)}$ are equivalent. This implies that (2)$\Longleftrightarrow$(4)$\Longleftrightarrow$(5) by Theorem 3.2(2). □

We call a ring $B$ the DeMeyer-Kanzaki Galois extension of $B^G$ if $B$ is an Azumaya $C$-algebra and $B$ is a center Galois extension of $B^G$ (for more about the DeMeyer-Kanzaki Galois extensions, see [2]). Clearly, the class of center Galois extensions is broader than the class of the DeMeyer-Kanzaki Galois extensions. We conclude the present paper with two examples. (1) The DeMeyer-Kanzaki Galois extension of $B^G$ and (2) a center Galois extension of $B^G$, but not the DeMeyer-Kanzaki Galois extension of $B^G$.

**Example 3.5.** Let $\mathbb{C}$ be the field of complex numbers, that is, $\mathbb{C} = \mathbb{R} + \mathbb{R}\sqrt{-1}$ where $\mathbb{R}$ is the field of real numbers, $B = \mathbb{C}[i,j,k]$ the quaternion algebra over $\mathbb{C}$, and $G = \{1, g \mid g(c_1 + c_i i + c_j j + c_k k) = g(c_1) + g(c_i)i + g(c_j)j + g(c_k)k$ for each $b = c_1 + c_i i + c_j j + c_k k \in \mathbb{C}[i,j,k]$ and $g(u + v\sqrt{-1}) = u - v\sqrt{-1}$ for each $c = u + v\sqrt{-1} \in \mathbb{C}\}$. Then

(1) The center of $B$ is $\mathbb{C}$.

(2) $B$ is an Azumaya $C$-algebra.

(3) $\mathbb{C}$ is a Galois extension of $\mathbb{C}^G$ with Galois group $G|_{\mathbb{C}} \cong G$ and a Galois system $\{a_1 = 1/\sqrt{2},\ a_2 = (1/\sqrt{2})\sqrt{-1};\ b_1 = 1/\sqrt{2},\ b_2 = -(1/\sqrt{2})\sqrt{-1}\}$.

(4) $B$ is the DeMeyer-Kanzaki Galois extension of $B^G$ by (2) and (3).

(5) $B^G = \mathbb{R}[i,j,k]$.

(6) $B = B^G\mathbb{C}$, so $B$ is a centeral extension of $B^G$.

(7) $J_g^{(\mathbb{C})} = \mathbb{R}\sqrt{-1}$.

(8) $B = BJ_g^{(C)}$ since $1 = -\sqrt{-1}\sqrt{-1} \in BJ_g^{(C)}$.

(9) $J_g^{(B)} = \mathbb{R}\sqrt{-1} + \mathbb{R}\sqrt{-1}i + \mathbb{R}\sqrt{-1}j + \mathbb{R}\sqrt{-1}k$.

(10) $B = \mathbb{C}J_g^{(B)}$.

**Example 3.6.** By replacing in Example 3.5 the field of complex numbers $\mathbb{C}$ with the ring $C = \mathbb{Z} \oplus \mathbb{Z}$ where $\mathbb{Z}$ is the ring of integers, $g(a,b) = (b,a)$ for all $(a,b) \in C$, and $G = \{1, g \mid g(c_1 + c_i i + c_j j + c_k k) = g(c_1) + g(c_i)i + g(c_j)j + g(c_k)k$ for each $b = c_1 + c_i i + c_j j + c_k k \in B = C[i,j,k]\}$. Then

(1) The center of $B$ is $C$.

(2) $C$ is a Galois extension of $C^G$ with Galois group $G|_C \cong G$ and a Galois system $\{a_1 = (1,0),\ a_2 = (0,1);\ b_1 = (1,0),\ b_2 = (0,1)\}$.

(3) $B$ is not an Azumaya $C$-algebra (for $1/2 \notin C$), and so $B$ is not the DeMeyer-Kanzaki Galois extension of $B^G$.

(4) $C^G = \{(a,a) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$.

(5) $B^G = C^G[i,j,k]$.

(6) $B = B^G C$, so $B$ is a central extension of $B^G$.

(7) $J_g^{(C)} = \{(a,-a) \mid a \in \mathbb{Z}\} = \mathbb{Z}(1,-1)$.

(8) $B = BJ_g^{(C)}$ since $1 = (1,1) = (1,-1)(1,-1) \in BJ_G^{(C)}$.

(9) $J_g^{(B)} = \mathbb{Z}(1,-1) + \mathbb{Z}(1,-1)i + \mathbb{Z}(1,-1)j + \mathbb{Z}(1,-1)k$.

(10) $B = CJ_g^{(B)}$.

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