Internat. J. Math. & Math. Sci.
Vol. 23, No. 2 (2000) 123–129
S0161171200000430

AN IRREDUCIBLE HEEGAARD DIAGRAM OF THE REAL PROJECTIVE 3-SPACE P^3

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(Received 19 December 1997 and in revised form 15 June 1998)

ABSTRACT. We give a genus 3 Heegaard diagram H of the real projective space P^3, which has no waves and pairs of complementary handles. So Negami's result that every genus 2 Heegaard diagram of P^3 is reducible cannot be extended to Heegaard diagrams of P^3 with genus 3.

Keywords and phrases. Heegaard diagram, wave move, crystallization.

2000 Mathematics Subject Classification. Primary 57M50; Secondary 57M15.

1. Introduction. In the study of 3-manifolds, the construction of an algorithm for recognizing the 3-sphere S^3 among all 3-manifolds is a very important problem. The first work in this direction was done by Whitehead [12], and later Volodin, Kuznetsov, and Fomenko [11] conjectured that Heegaard diagrams for S^3 are reducible, except for the canonical one.

Homma, Ochiai, and Takahashi [4] proved that the conjecture is true for the case of genus 2. But for the case of genera greater than two it is not true anymore. Morikawa [5] gave a counterexample for the case of genus 3, and Ochiai [8, 9] gave counterexamples for the case of genera 3 and 4. Negami [6, 7] proved that every 3-bridge projection of a link can be transformed into a minimum crossing one by a finite sequence of wave moves if and only if the link is equivalent to one of a trivial knot, a splittable link, and the Hopf link. Consequently, any genus 2 Heegaard diagrams of S^3, $S^2\times S^1\#L(p,q)$ and P^3 are reducible.

In this paper, we give a genus 3 Heegaard diagram H of the real projective space P^3, which has no waves and pairs of complementary handles. Moreover, we construct a crystallization Γ corresponding to the Heegaard diagram H and show that at least one among the Heegaard diagrams associated with Γ is transformed into a Heegaard diagram with some pairs of complementary handles by a finite sequence of wave moves, and so it is reducible to the canonical diagram of P^3.

2. Preliminaries. Let M be a closed orientable 3-manifold and let T_n, $\bar{T}_n$ be solid tori of genera n and $h:\partial T_n\to\partial\bar{T}_n$ a homeomorphism of the boundary surface. Then the triad $(T_n,\bar{T}_n;M)$ is called a Heegaard splitting of genus n for M when $M=T_n\cup_h\bar{T}_n$.

A collection of mutually disjoint n meridian disks $m_1,\dots,m_n$ in a solid torus T of genus n is called a complete system of meridian disks of T if $\mathrm{Cl}(T-\cup_{i=1}^{n}N(m_i,T))$ is a 3-ball, where $N(m_i,T)$ is a regular neighborhood of m_i in T. We call a collection of mutually disjoint $(n+1)$ meridian disks in T an extended complete system of meridian disks of T provided that any n subcollection is a complete system of meridian disks of T.

Let $\{m_1,\dots,m_n\}$ (respectively, $\{m_1,\dots,m_{n+1}\}$) be a complete system of meridian disks (respectively, an extended complete system of meridian disks) of T_n, and let $\{\bar{m}_1,\dots,\bar{m}_n\}$ (respectively, $\{\bar{m}_1,\dots,\bar{m}_{n+1}\}$) be a complete system of meridian disks (respectively, an extended complete system of meridian disks) of $\bar{T}_n$, where $(T_n,\bar{T}_n;M)$ is a Heegaard splitting of genus n for M; and let $u_j = \partial m_j$, $v_j = \partial \bar{m}_j$ for $j = 1,2,\dots,n$, $n+1$. We call the triad $H = (F;u,v)$ a Heegaard diagram for M, where $F = \partial T_n = \partial \bar{T}_n$ and $u = u_1 \cup \cdots \cup u_n$, $v = v_1 \cup \cdots \cup v_n$. Moreover, we call the triad $\tilde{H} = (F;\tilde{u},\tilde{v})$ an extended Heegaard diagram for M, where $\tilde{u} = u \cup u_{n+1}$ and $\tilde{v} = v \cup v_{n+1}$.

Next, we give the concept of wave of Heegaard diagrams. Let $H = (F;u,v)$ be a Heegaard diagram for M, and w an arc on F such that for a meridian or a longitude of H, say u_1,

$$w \cap (u_1 \cup \cdots \cup u_n \cup v_1 \cup \cdots \cup v_n) = w \cap u_1 = \partial w \tag{2.1}$$

and both ends of w attach to the same side of u_1. Then one of two circles in $u_1 \cup w$, different from u_1, bounds a meridian disk of H, say u_1', and $H' = (F;u',v)$ is a new Heegaard diagram, where $u' = u_1' \cup u_2 \cup \cdots \cup u_n$. We call w a wave for H, and the replacement of u_1 with u_1' a wave move with w if $C(H') < C(H)$, where $C(H)$ is the complexity of H which is defined as the cardinality of $u \cap v$.

Let H be a Heegaard diagram of the real projective space P^3 other than the canonical one $\bar{H}$ associated with Figure 5. Then H is said to be reducible if there is a finite sequence of (normal) Heegaard diagrams, $H_n,\dots,H_0$, with $H_n = H$ and $H_0 = \bar{H}$, such that H_{i-1} is a wave move of H_i $(i = 1,2,\dots,n)$.

Wave moves are also defined for n-bridge decompositions of links; the relations between two wave theories are investigated in [7]. In particular, for 3-bridge decomposition of links, we have the following theorem.

Theorem 2.1 [6]. *Every 3-bridge projection of a link can be transformed into a minimum crossing one by a finite sequence of wave moves if and only if the link is equivalent to one of a trivial knots, a splittable link, and the Hopf link.*

By a 4-colored graph $G = (\Gamma,\gamma)$, we mean a regular graph Γ (with possibly multiple edges, but no loops) of degree 4, endowed with a proper edge coloration; a coloration $\gamma : E(\Gamma) \to \Delta_3 = \{0,1,2,3\}$, where $E(\Gamma)$ is the set of edges of Γ, such that $\gamma(e_1) \neq \gamma(e_2)$ for any two adjacent edges e_1, e_2.

A 3-dimensional pseudocomplex $K(G)$ is associated with $G(\Gamma,\delta)$. For details, see [2]. G is said to represent $|K(G)|$ and every homeomorphic polyhedron.

A 4-colored graph G representing a PL manifold M is called a crystallization if, for each colour $c \in \Gamma_3$, the subgraph obtained by deleting all coloured edges c is connected. Crystallizations exist for all PL manifolds (see [10]).

3. A Heegaard diagram of P^3. As mentioned in Section 2, genus 2 Heegaard splittings of closed orientable 3-manifolds are closely related to 3-bridge decompositions of links. In fact, Birman and Hilden [1] proved that there is a bijective correspondence between the equivalence classes of 3-bridge projections and those of genus 2

Heegaard diagrams. By Theorem 2.1, every Heegaard diagram of genus 2 of P^3, other than the canonical one, contain at least one wave.

In this section, we give a Heegaard diagram of genus 3 of P^3 which has no waves and pairs of complementary handles.

In Figure 1, it is easily checked that this Heegaard diagram H has no waves and pairs of complementary handles.

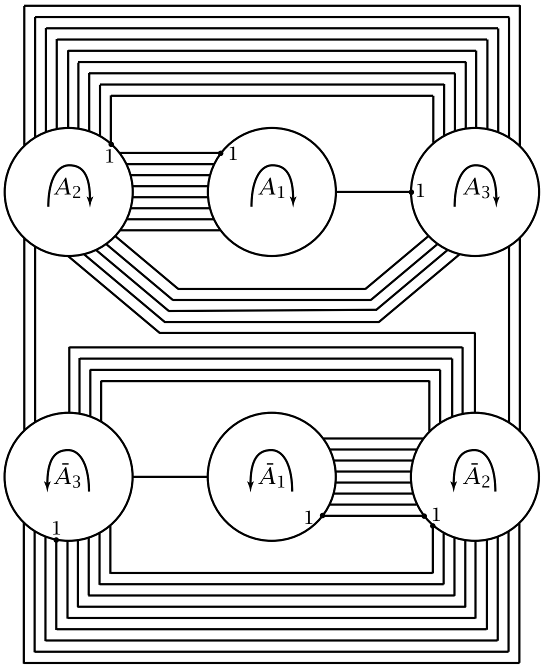

FIGURE 1.

Now, we need to show that this Heegaard diagram H represents the real projective space P^3.

PROPOSITION 3.1. *Let M^3 be a manifold with the above Heegaard diagram. Then M^3 is the real projective space P^3.*

PROOF. Construct a crystallization Γ associated with the above Heegaard diagram via Gagliardi's method [3].

In Figure 2, colorations $\{0,1,2,3\}$ are given as follows: edges consisting of circles $C_i(i = 1,2,3,4)$ are $\{1,2\}$-colored alternatively, edges connecting vertices of C_i and $C_j(i \neq j)$ are 3-colored, and edges connecting small and capital letters are remaining 0-colored.

Since the dotted lines in Figure 2 are axes for an involution, this crystallization represents a 2-fold branched covering of S^3 branched over the following link (Figure 3) by Ferris' construction of 2-fold branched coverings of S^3 [2].

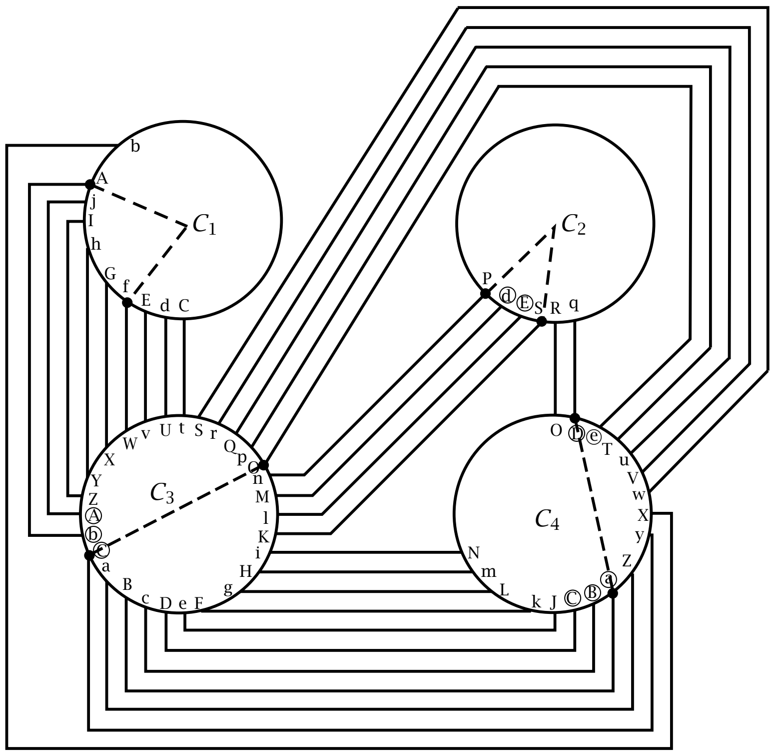

FIGURE 2.

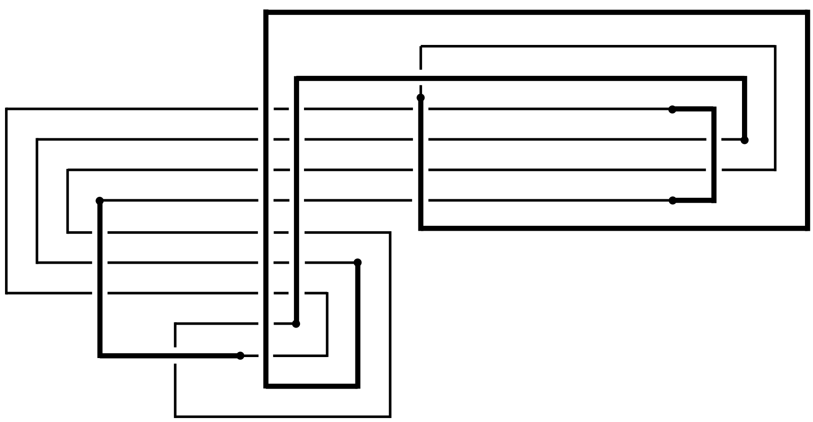

FIGURE 3.

In Figure 4, dotted lines are eliminated overbridges by jump moves [7]. By a couple of jump moves about underbridge, it is not hard to see that this link is equivalent to the standard Hopf link (Figure 5). Therefore, M^3 is the same as P^3. □

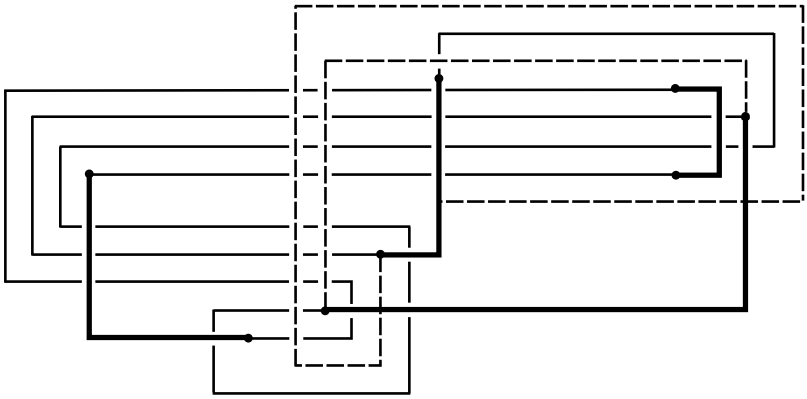

FIGURE 4.

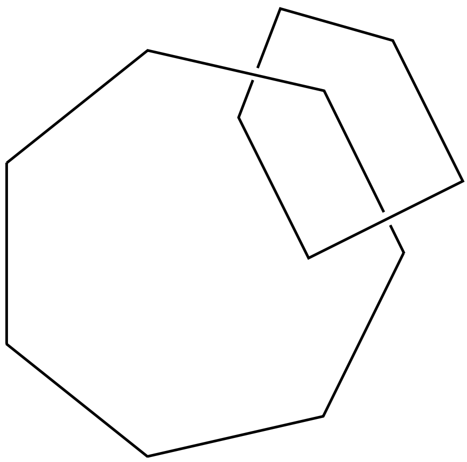

FIGURE 5.

REMARK. In Figure 3, this link represents a 4-bridge projection which has no waves.

Now, we construct an extended Heegaard diagram $\tilde{H}$ associated with $H = (F; u, v)$. The extended Heegaard diagram $\tilde{H}$ contains 16 Heegaard diagrams for P^3. At least one of them can be transformed into a Heegaard diagram H' with a pair of complementary handles by a finite sequence of wave moves (Figure 6).

By Singer moves on H', we have a Heegaard diagram of genus 2 of P^3 and so it is transformed into the canonical one [6]. In Figure 6, a pair of complementary handles occurs at black dot.

ACKNOWLEDGEMENT. The present study was supported by Basic Science Research Institute Program, Ministry of Education, 1997, Project No. BSRI-97-1433.

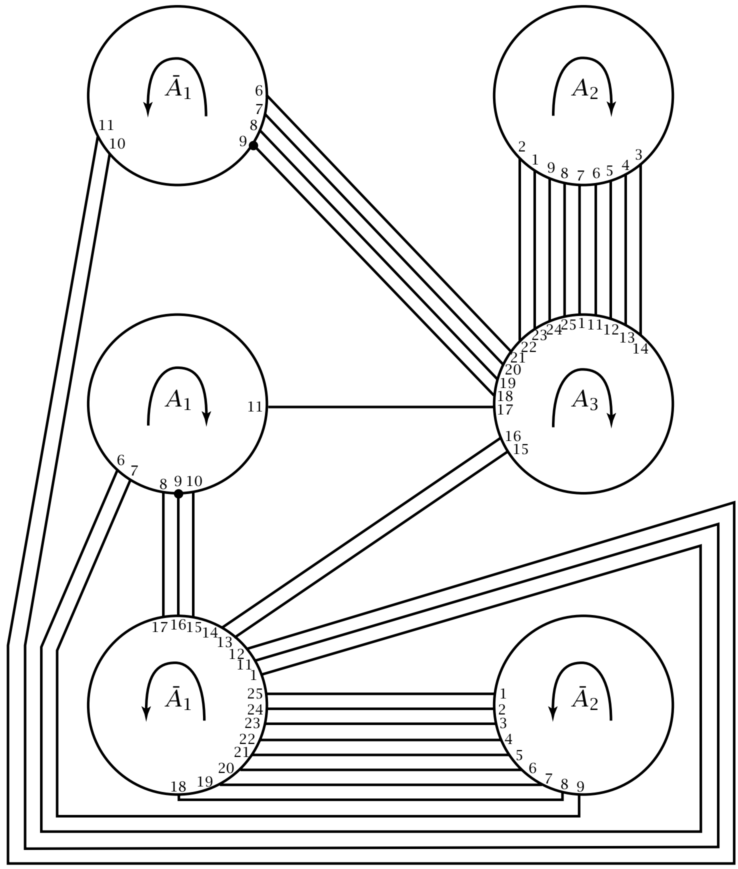

FIGURE 6.

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