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# A NOTE ON A RESULT OF SINGH AND KULKARNI

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ABSTRACT. We prove that if $f$ is a transcendental meromorphic function of finite order and $\sum_{a\neq\infty}\delta(a,f)+\delta(\infty,f)=2$, then

$$K(f^{(k)})=\frac{2k(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)},$$

where

$$K(f^{(k)})=\underline{\lim}_{r\to\infty}\frac{N(r,1/f^{(k)})+N(r,f^{(k)})}{T(r,f^{(k)})}.$$

This result improves a result by Singh and Kulkarni.

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**1. Introduction and the main result.** Let $f(z)$ be a meromorphic function in the complex plane. We use the following notations of value distribution theory (see [2])

$$T(r,f),m(r,f),N(r,f),\overline{N}(r,f),\ldots \tag{1.1}$$

and denote by $S(r,f)$ a function with the property that $S(r,f)=o(T(r,f))$, $r\to\infty$ (outside an exceptional set of finite linear measure, if $f$ is of infinite order). The Nevanlinna's deficiency of $f$ with respect to a finite complex number $a$ is defined by

$$\delta(a,f)=\underline{\lim}_{r\to\infty}\frac{m(r,1/(f-a))}{T(r,f)}. \tag{1.2}$$

If $a=\infty$, then one should replace $m(r,1/(f-a))$ in the above formula by $m(r,f)$. The well known Nevanlinna's deficiency relation states that

$$\sum_{a\neq\infty}\delta(a,f)+\delta(\infty,f)\leq 2. \tag{1.3}$$

If the above inequality holds, then we say that $f$ has maximum deficiency sum.

In [3], Singh and Kulkarni proved the following result.

**THEOREM 1.1.** *Suppose that $f$ is a transcendental meromorphic function of finite order and $\sum_{a\neq\infty}\delta(a,f)+\delta(\infty,f)=2$, then*

$$\frac{1-\delta(\infty,f)}{2-\delta(\infty,f)}\leq K(f')\leq\frac{2(1-\delta(\infty,f))}{2-\delta(\infty,f)}, \tag{1.4}$$

*where*

$$K(f') = \varlimsup_{r\to\infty} \frac{N(r,(1/f')) + N(r,f')}{T(r,f')}. \tag{1.5}$$

In this note, we prove the following.

**Theorem 1.2.** *Suppose that $f$ is a transcendental meromorphic function of finite order and $\sum_{a\neq\infty}\delta(a,f)+\delta(\infty,f)=2$, then*

$$K(f^{(k)}) = \frac{2k(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)}, \tag{1.6}$$

*where*

$$K(f^{(k)}) = \lim_{r\to\infty} \frac{N(r,1/f^{(k)}) + N(r,f^{(k)})}{T(r,f^{(k)})}. \tag{1.7}$$

## 2. An important lemma

**Lemma 2.1** [1]. *Let $f(z)$ be a transcendental meromorphic function, then for each positive number $\epsilon$ and each positive integer $k$, we have*

$$k\overline{N}(r,f) \le N(r,1/f^{(k)}) + N(r,f) + \epsilon T(r,f) + S(r,f). \tag{2.1}$$

**Proof of Theorem 1.2.** First, we prove that

$$\lim_{r\to\infty} \frac{T(r,f^{(k)})}{T(r,f)} = 1+k-k\delta(\infty,f), \quad r \longrightarrow \infty. \tag{2.2}$$

Without loss of generality, we assume that $f$ has infinitely many finite deficient values $a_1, a_2, \ldots$. It follows from Littlewood's inequality

$$\begin{aligned}\sum_{n=1}^{p} m\left(r,\frac{1}{f-a_n}\right) &\le m\left(r,\frac{1}{f'}\right) + S(r,f)\\ &\le T(r,f) + \overline{N}(r,f) + S(r,f),\end{aligned} \tag{2.3}$$

that

$$\sum_{n=1}^{p} \delta(a_n,f) \le 1 + \varliminf_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)} \le 1 + \varlimsup_{r\to\infty}\frac{N(r,f)}{T(r,f)} = 2-\delta(\infty,f). \tag{2.4}$$

By the assumption, we have

$$\sum_{n=1}^{\infty} \delta(a_n,f) = 2-\delta(\infty,f). \tag{2.5}$$

Let $p \to \infty$ in (2.4) and use (2.5) to obtain

$$\lim_{r\to\infty}\frac{\overline{N}(r,f)}{T(r,f)} = \lim_{r\to\infty}\frac{N(r,f)}{T(r,f)} = 1-\delta(\infty,f). \tag{2.6}$$

Replacing $f'$ in (2.3) by $f^{(k)}$, we get

$$\begin{aligned}\sum_{n=1}^{P} m\left(r,\frac{1}{f-a_n}\right) &\le m\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ &\le T(r,f^{(k)})-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).\end{aligned} \tag{2.7}$$

It follows from (2.7) and (2.1) that

$$\sum_{n=1}^{P} m\left(r,\frac{1}{f-a_n}\right) \le T(r,f^{(k)})+N(r,f)-k\overline{N}(r,f)+\epsilon T(r,f)+S(r,f). \tag{2.8}$$

Consequently, because of (2.6), we have

$$\varliminf_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)} \ge (k-1)(1-\delta(\infty,f))+\sum_{n=1}^{P}\delta(a_n,f)-\epsilon. \tag{2.9}$$

Now, let $p\to\infty$ and $\epsilon\to 0$ and use (2.5) to obtain

$$\varliminf_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)} \ge 1+k-k\delta(\infty,f). \tag{2.10}$$

On the other side,

$$T(r,f^{(k)}) \le T(r,f)+k\overline{N}(r,f)+S(r,f). \tag{2.11}$$

Therefore, because of (2.6),

$$\varlimsup_{r\to\infty}\frac{T(r,f^{(k)})}{T(r,f)} \le 1+k-k\delta(\infty,f). \tag{2.12}$$

Equation (2.2) follows from the above estimates.

Next, we prove that

$$\lim_{r\to\infty}\frac{N(r,1/f^{(k)})}{T(r,f^{(k)})} = \frac{(k-1)(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)}. \tag{2.13}$$

From the first inequality of (2.7), we have

$$\varliminf_{r\to\infty}\frac{m(r,1/f^{(k)})}{T(r,f)} \ge \sum_{n=1}^{P}\delta(a_n,f). \tag{2.14}$$

Consequently, if we let $p\to+\infty$ and use (2.5), we get

$$\varliminf_{r\to\infty}\frac{m(r,1/f^{(k)})}{T(r,f)} \ge 2-\delta(\infty,f). \tag{2.15}$$

On the other side, from (2.1) and (2.7), we have

$$\begin{aligned} m\left(r,\frac{1}{f^{(k)}}\right) &\le T(r,f^{(k)})-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ &\le T(r,f)+k\overline{N}(r,f)-N\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\\ &\le T(r,f)+N(r,f)+\epsilon T(r,f)+S(r,f),\end{aligned} \tag{2.16}$$

hence,

$$\overline{\lim_{r\to\infty}} \frac{m(r,1/f^{(k)})}{T(r,f)} \le 2-\delta(\infty,f)+\epsilon, \tag{2.17}$$

if we let $\epsilon \to 0$, we get

$$\overline{\lim_{r\to\infty}} \frac{m(r,1/f^{(k)})}{T(r,f)} \le 2-\delta(\infty,f). \tag{2.18}$$

Thus, from (2.15) and (2.18), we obtain

$$\lim_{r\to\infty} \frac{m(r,1/f^{(k)})}{T(r,f)} = 2-\delta(\infty,f). \tag{2.19}$$

Hence, from (2.2), (2.18), and (2.19), we have

$$\begin{aligned} \lim_{r\to\infty} \frac{N(r,1/f^{(k)})}{T(r,f^{(k)})} &= 1-\lim_{r\to\infty} \frac{m(r,1/f^{(k)})}{T(r,f^{(k)})} \\ &= 1-\lim_{r\to\infty} \frac{m(r,(1/f^{(k)}))}{T(r,f)} \lim_{r\to\infty} \frac{T(r,f)}{T(r,f^{(k)})} \\ &= 1-\frac{2-\delta(\infty,f)}{1+k-k\delta(\infty,f)} = \frac{(k-1)(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)}. \end{aligned} \tag{2.20}$$

Finally, from (2.2) and (2.6), we have

$$\begin{aligned} \lim_{r\to\infty} \frac{N(r,f^{(k)})}{T(r,f^{(k)})} &= \lim_{r\to\infty} \frac{N(r,f)+k\overline{N}(r,f)}{T(r,f^{(k)})} \\ &= \lim_{r\to\infty} \frac{N(r,f)+k\overline{N}(r,f)}{T(r,f)} \lim_{r\to\infty} \frac{T(r,f)}{T(r,f^{(k)})} \\ &= \frac{(k+1)(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)}. \end{aligned} \tag{2.21}$$

Therefore, we deduce, from (2.20) and (2.21), that

$$\lim_{r\to\infty} \frac{N(r,1/f^{(k)})+N(r,f^{(k)})}{T(r,f^{(k)})} = \frac{2k(1-\delta(\infty,f))}{1+k-k\delta(\infty,f)}. \tag{2.22}$$

Thus, the proof of Theorem 1.2 is complete. □

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