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RAYLEIGH WAVES IN A THERMOELASTIC GRANULAR MEDIUM UNDER INITIAL STRESS

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ABSTRACT. We study the effect of initial stress on the propagation of Rayleigh waves in a granular medium under incremental thermal stresses. We also obtain the frequency equation, in the form of a twelfth-order determinantal expression, which is in agreement with the corresponding classical results.

Keywords and phrases. Rayleigh waves, thermoelastic waves, granular medium.

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1. Introduction. The propagation of thermoelastic waves in a granular medium under initial stress has some applications in soil mechanics, earthquake science, geophysics, mining engineering, etc. The theoretical outline of the development of the subject from the mid-thirties was given by Paria [9]. The present paper, however, is based on the dynamics of granular media as propounded by Oshima [7, 8].

The medium under consideration is discontinuous such as one composed numerous large or small grains. Unlike a continuous body, each element or grain cannot only translate but also rotate about its centre of gravity. This motion is the characteristics of the medium and has an important effect upon the equations of motion to produce internal friction. It is assumed that the medium contains so many grains that they will never be separated from each other during the deformation and that the grain has perfect elasticity. The frequency equation of Rayleigh waves in a granular layer over a granular half-space was given by Bhattacharyya [2]. In [4], Elnaggar investigated the influence of initial stress of the waves propagation in a thermoelastic granular infinite cylinder. Recently [1], Ahmed discussed the influence of gravity on the propagation of waves in granular medium.

This paper is devoted to the study of the effect of initial stress on the propagation of Rayleigh waves in a granular medium under incremental thermal stresses. The medium under consideration is granular half-space overlain by a different granular layer and initial stresses present in this medium have considerable effect in the propagation of Rayleigh waves [3]. The wave velocity equation has been derived in the form of twelfth-order determinant. The roots of this equation are in general complex and the imaginary part of an appropriate root measures the attenuation of the waves. It is shown that the frequency of Rayleigh waves contains terms involving thermal coefficients and other terms involving initial stress and so the phase velocity changes with respect to this thermal coefficients and the initial stress. When there is no coupling between the temperature and the strain field in the absence of the initial stress, the derived frequency equation reduces to an equation in the form of ninth-order determinant similar to that obtained by Bhattacharyya [2]. Also, the classical frequency equation when both media are elastic and the other effects are absent is obtained.

2. Formulation of the problem. Consider a system of orthogonal cartesian axes x_1, x_2, x_3 such that the interface and the free surface of the granular layer resting on the granular half-space of different material are the planes $x_3 = H$ and $x_3 = 0$, respectively, the origin O is any point on the free surface, x_3-axis is positive in the direction towards the exterior of the half-space, and the x_1-axis is positive along the direction of Rayleigh waves propagation.

The state of deformation in the granular medium is described by the displacement vector $\underline{\mathbf{U}}(u_1, 0, u_3)$ of the centre of gravity of a grain and the rotation vector $\underline{\xi}(\xi, \eta, \zeta)$ of the grain about its centre of gravity.

In this problem the stress tensor and the stress couple are nonsymmetric, i.e., $\tau_{ij} \neq \tau_{ji}$ and $M_{ij} \neq M_{ji}$. The stress tensor τ_{ij} can be expressed as the sum of symmetric and antisymmetric tensors

$$\tau_{ij} = \sigma_{ij} + \sigma'_{ij}, \tag{2.1}$$

where

$$\sigma_{ij} = \frac{1}{2}(\tau_{ij} + \tau_{ji}) \qquad \text{and} \qquad \sigma'_{ij} = \frac{1}{2}(\tau_{ij} - \tau_{ji}). \tag{2.2}$$

The symmetric tensor $\sigma_{ij} = \sigma_{ji}$ is related to the symmetric strain tensor

$$e_{ij} = e_{ji} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.3}$$

by Hooke's law.

The antisymmetric stresses σ'_{ij} are given by

$$\sigma'_{23} = -F\frac{\partial \xi}{\partial t}, \qquad \sigma'_{31} = -F\frac{\partial \eta}{\partial t}, \qquad \sigma'_{12} = -F\frac{\partial \zeta}{\partial t}, \qquad \sigma'_{11} = \sigma'_{22} = \sigma'_{33} = 0, \tag{2.4}$$

where F is the coefficient of friction.

The stress couple M_{ij} is given by

$$M_{ij} = M\nu_{ij}, \tag{2.5}$$

where M is the third elastic constant,

$$\begin{gathered} \nu_{11} = \frac{\partial \xi}{\partial x_1}, \qquad \nu_{22} = 0, \qquad \nu_{33} = \frac{\partial \zeta}{\partial x_3}, \qquad \nu_{23} = 0, \qquad \nu_{31} = \frac{\partial \xi}{\partial x_3}, \\ \nu_{12} = \frac{\partial}{\partial x_1}(\omega_2 + \eta), \qquad \nu_{32} = \frac{\partial}{\partial x_3}(\omega_2 + \eta), \qquad \nu_{13} = \frac{\partial \xi}{\partial x_3}, \qquad \nu_{21} = 0, \end{gathered} \tag{2.6}$$

where $\omega_2 = 1/2(\partial u_1/\partial x_3 - \partial u_3/\partial x_1)$ is the component of rotation.

The heat conduction equation is given by (see [6])

$$K\nabla^2 T = \rho C_e \frac{\partial T}{\partial t} + \gamma T_0 \nabla \cdot \frac{\partial \underline{U}}{\partial t}, \tag{2.7}$$

where K is the thermal conductivity, T is the temperature change about the initial temperature T_0, ρ is the density, C_e is the specific heat per unit mass at constant strain, γ is equal to $\alpha(3\lambda+2\mu)$, α is the thermal expansion coefficient, and λ and μ are Lame's constants and t is the time.

The components of incremental stress in terms of the displacement are given by (see [3, 6])

$$\begin{aligned}
\sigma_{11} &= (\lambda+2\mu+p)\frac{\partial u_1}{\partial x_1}+(\lambda+p)\frac{\partial u_3}{\partial x_3}-\gamma T,\\
\sigma_{33} &= \lambda\frac{\partial u_1}{\partial x_1}+(\lambda+2\mu)\frac{\partial u_3}{\partial x_3}-\gamma T,\\
\sigma_{13} &= \mu\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right).
\end{aligned} \tag{2.8}$$

The dynamical equations of motion are

$$\begin{aligned}
\frac{\partial \tau_{11}}{\partial x_1}+\frac{\partial \tau_{13}}{\partial x_3}+P\frac{\partial \omega_2}{\partial x_3} &= \rho\frac{\partial^2 u_1}{\partial t^2},\\
\frac{\partial \tau_{12}}{\partial x_1}+\frac{\partial \tau_{32}}{\partial x_3} &= 0,\\
\frac{\partial \tau_{13}}{\partial x_1}+\frac{\partial \tau_{33}}{\partial x_3}+P\frac{\partial \omega_2}{\partial x_1} &= \rho\frac{\partial^2 u_3}{\partial t^2},
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\tau_{23}-\tau_{32}+\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{31}}{\partial x_1} &= 0,\\
\tau_{31}-\tau_{13}+\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{32}}{\partial x_3} &= 0,\\
\tau_{12}-\tau_{21}+\frac{\partial M_{13}}{\partial x_1}+\frac{\partial M_{33}}{\partial x_3} &= 0.
\end{aligned} \tag{2.10}$$

Equations (2.9) and (2.10) take the forms, when the stresses are substituted,

$$\begin{aligned}
&(\lambda+2\mu+P)\frac{\partial^2 u_1}{\partial x_1^2}+\left(\mu+\frac{P}{2}\right)\frac{\partial^2 u_1}{\partial x_3^2}\\
&\quad+\left(\lambda+\mu+\frac{P}{2}\right)\frac{\partial^2 u_3}{\partial x_1\partial x_3}-\gamma\frac{\partial T}{\partial x_1}-F\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x_3}\right)=\rho\frac{\partial^2 u_1}{\partial t^2},
\end{aligned} \tag{2.11}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x_3}-\frac{\partial \zeta}{\partial x_1}\right)=0, \tag{2.12}$$

$$\begin{aligned}
&\left(\lambda+\mu+\frac{P}{2}\right)\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\left(\mu-\frac{P}{2}\right)\frac{\partial^2 u_3}{\partial x_1^2}\\
&\quad+(\lambda+2\mu)\frac{\partial^2 u_3}{\partial x_3^2}-\gamma\frac{\partial T}{\partial x_3}+F\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x_1}\right)=\rho\frac{\partial^2 u_3}{\partial t^2},
\end{aligned} \tag{2.13}$$

$$-F\frac{\partial \xi}{\partial t}+M\nabla^2\xi=0, \tag{2.14}$$

$$-F\frac{\partial \eta}{\partial t}+M\nabla^2(\eta+\omega_2)=0, \tag{2.15}$$

$$-F\frac{\partial \zeta}{\partial t}+M\nabla^2\zeta=0. \tag{2.16}$$

3. Solution of the problem. Let the constants λ, μ, ρ, F, M, γ and $\overline{\lambda}$, $\overline{\mu}$, $\overline{\rho}$, $\overline{F}$, $\overline{M}$, $\overline{\gamma}$ be characteristics of the layer and the half-space, respectively. Let us introduce the displacement potentials $\phi(x_1,x_3,t)$ and $\psi(x_1,x_3,t)$ which are related to the displacement components u_1 and u_3 by the relations

$$u_1=\frac{\partial \phi}{\partial x_1}-\frac{\partial \psi}{\partial x_3}, \qquad u_3=\frac{\partial \phi}{\partial x_3}-\frac{\partial \psi}{\partial x_1}. \tag{3.1}$$

Substituting from (3.1) into (2.11), (2.13), and (2.15), we see that ϕ and ψ satisfy the wave equations

$$\alpha^2\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}-\frac{\gamma}{\rho}T=0, \tag{3.2}$$

$$\beta^2\nabla^2\psi-\frac{\partial^2\psi}{\partial t^2}+s\frac{\partial \eta}{\partial t}=0, \tag{3.3}$$

$$-s'\frac{\partial \eta}{\partial t}+\nabla^2\eta-\nabla^4\psi=0, \tag{3.4}$$

where

$$\alpha^2=\frac{\lambda+2\mu+p}{\rho}, \qquad \beta^2=\frac{\mu-(p/2)}{\rho}, \qquad S=\frac{F}{\rho}, \qquad S'=\frac{F}{M}. \tag{3.5}$$

From (3.1), the heat conduction equation (2.7) becomes

$$k\nabla^2T=\rho C_e\frac{\partial T}{\partial t}+\gamma T_0\nabla^2\left(\frac{\partial \phi}{\partial t}\right). \tag{3.6}$$

Elimination of T from (3.2) and (3.6), gives

$$\left(\nabla^2-\frac{1}{\chi}\frac{\partial}{\partial t}\right)\left(\alpha^2\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}\right)-\epsilon\nabla^2\frac{\partial \phi}{\partial t}=0, \tag{3.7}$$

where

$$\chi=\frac{k}{\rho c_e}, \qquad \epsilon=\frac{\gamma^2T_0}{\rho k}. \tag{3.8}$$

Also, η can be eliminated by the use of equations (3.3) and (3.4) as follows:

$$\left(\nabla^2-s'\frac{\partial}{\partial t}\right)\left(\beta^2\nabla^2\psi-\frac{\partial^2\psi}{\partial t^2}\right)+S\nabla^4\frac{\partial \psi}{\partial t}=0. \tag{3.9}$$

For a plane harmonic wave propagation in the x_1-direction, we assume

$$\phi=\phi_1(x_3)\exp\left(i(Lx_1-bt)\right), \tag{3.10}$$

$$\psi=\psi_1(x_3)\exp\left(i(Lx_1-bt)\right), \tag{3.11}$$

$$(\xi,\eta,\zeta)=(\xi_1(x_3),\eta_1(x_3),\zeta_1(x_3))\exp\left(i(Lx_1-bt)\right), \tag{3.12}$$

where b is real positive and L is in general complex.

Substituting from (3.12) into (2.12), (2.14), and (2.16), gives

$$D\xi_1 - iL\zeta_1 = 0, \tag{3.13}$$

$$D^2\xi_1 + h^2\xi_1 = 0, \tag{3.14}$$

$$D^2\zeta_1 + h^2\zeta_1 = 0, \tag{3.15}$$

where $h^2 = ibs' - L^2$, $D \equiv d/dx_3$.

Solutions of (3.14) and (3.15) are

$$\xi_1 = A_1 e^{ihx_3} + A_2 e^{-ihx_3}, \qquad \zeta_1 = B_1 e^{ihx_3} + B_2 e^{-ihx_3}, \tag{3.16}$$

respectively.

From (3.13) and (3.16), we obtain

$$h(A_1 e^{ihx_3} - A_2 e^{-ihx_3}) - L(B_1 e^{ihx_3} - B_2 e^{-ihx_3}) = 0. \tag{3.17}$$

Equating the coefficients of e^{ihx_3} and e^{-ihx_3} to zero in (3.17), gives

$$A_1 = \frac{L}{h} B_1, \qquad A_2 = \frac{-L}{h} B_2. \tag{3.18}$$

Also, substitution from (3.10) and (3.11) into (3.7) and (3.9), we obtain

$$\begin{aligned}\Bigg[\alpha^2 D^4 + \left(b^2 - 2L^2\alpha^2 + ib\varepsilon + \frac{ib\alpha^2}{\chi}\right) D^2 + \alpha^2 L^4 \\ - b^2 L^2 - ibL^2\varepsilon - \frac{ibL^2\alpha^2}{\chi} + \frac{ib^3}{\chi}\Bigg]\phi_1 = 0,\end{aligned} \tag{3.19}$$

$$\begin{aligned}[(\beta^2 - ibs)D^4 + (b^2 - 2L^2\beta^2 + ibs'\beta^2 + 2ibsL^2)D^2 \\ + (\beta^2 - ibs)L^4 - (b + is'\beta^2)bL^2 + ib^3 S']\psi_1 = 0.\end{aligned} \tag{3.20}$$

The solution of (3.19) and (3.20) has the form

$$\phi_1 = A_3 e^{m_3 x_3} + B_3 e^{-m_3 x_3} + A_4 e^{m_4 x_3} + B_4 e^{-m_4 x_3}, \tag{3.21}$$

$$\psi_1 = E_3 e^{n_3 x_3} + F_3 e^{-n_3 x_3} + E_4 e^{n_4 x_3} + F_4 e^{-n_4 x_3}, \tag{3.22}$$

where

$$\begin{aligned}(m_3^2, m_4^2) &= L^2 - \frac{b(b+i\varepsilon)}{2\alpha^2} - \frac{ib}{2\chi} \pm \frac{b}{2\alpha^2}\left[(b+i\varepsilon)^2 - 2i\alpha^2(b+i\varepsilon) - \frac{\alpha^4}{\chi}\right]^{1/2}, \\ (n_3^2, n_4^2) &= \frac{2L^2\beta^2 - b^2 - ib\beta^2 s' - 2ibL^2 s \pm b\left[(b - i\beta^2 s')^2 - 4b^2 ss'\right]^{1/2}}{2(\beta^2 - ibs)}.\end{aligned} \tag{3.23}$$

Using (3.3), (3.11), (3.12), and (3.22), we get

$$\eta_1 = \Omega_3(E_3 e^{n_3 x_3} + F_3 e^{-n_3 x_3}) + \Omega_4(E_4 e^{n_4 x_3} + F_4 e^{-n_4 x_3}), \tag{3.24}$$

where

$$\Omega_3 = \frac{-i\beta^2}{bS}\left(n_3^2 - L^2 + \frac{b^2}{\beta^2}\right), \qquad \Omega_4 = \frac{-i\beta^2}{bS}\left(n_4^2 - L^2 + \frac{b^2}{\beta^2}\right). \tag{3.25}$$

From (3.2), (3.10), and (3.21), we have

$$T = \left[\Omega_3'\left(A_3 e^{m_3 x_3} + B_3 e^{-m_3 x_3}\right) + \Omega_4'\left(A_4 e^{m_4 x_3} + B_4 e^{-m_4 x_3}\right)\right]\exp\left[i(Lx_1 - bt)\right], \tag{3.26}$$

where

$$\Omega_3' = \frac{\rho\alpha^2}{\gamma}\left(m_3^2 - L^2 + \frac{b^2}{\alpha^2}\right), \qquad \Omega_4' = \frac{\rho\alpha^2}{\gamma}\left(m_4^2 - L^2 + \frac{b^2}{\alpha^2}\right). \tag{3.27}$$

The functions ξ_1, ζ_1, η_1, ϕ_1, and ψ_1 in the state of the lower medium must vanish as $x_3 \to \infty$ and using the symbols with a bar for the quantities in the lower medium (except x_3, L, b, p) and assuming the real parts of $\overline{m}_3, \overline{m}_4, \overline{n}_3, \overline{n}_4$ are positive while the imaginary part of $\overline{h}$ is negative, we obtain, for $x_3 > H$,

$$\begin{aligned}
\overline{\xi}_1 &= -\frac{L}{\overline{h}}\overline{B}_2 e^{-i\overline{h}x_3}, \\
\overline{\zeta}_1 &= \overline{B}_2 e^{-ihx_3}, \\
\overline{\eta}_1 &= \overline{\Omega}_3\overline{F}_3 e^{-\overline{n}_3 x_3} + \overline{\Omega}_4\overline{F}_4 e^{-\overline{n}_4 x_3}, \\
\overline{\phi}_1 &= \overline{B}_3 e^{-\overline{m}_3 x_3} + \overline{B}_4 e^{-\overline{m}_4 x_3}, \\
\overline{\Psi}_1 &= \overline{F}_3 e^{-\overline{n}_3 x_3} + \overline{F}_4 e^{-\overline{n}_4 x_3}, \\
\overline{T} &= \overline{\Omega}_3'\overline{B}_3 e^{-\overline{m}_3 x_3} + \overline{\Omega}_4'\overline{B}_4 e^{-\overline{m}_4 x_3}.
\end{aligned} \tag{3.28}$$

4. Boundary conditions and frequency equation. The boundary conditions on the interface $x_3 = H$ are

$$\begin{aligned}
&\text{(i)}\ u_1 = \overline{u}_1, && \text{(ii)}\ u_3 = \overline{u}_3, && \text{(iii)}\ \xi = \overline{\xi}, \\
&\text{(iv)}\ \eta = \overline{\eta}, && \text{(v)}\ \zeta = \overline{\zeta}, && \text{(vi)}\ M_{33} = \overline{M}_{33}, \\
&\text{(vii)}\ M_{31} = \overline{M}_{31}, && \text{(viii)}\ M_{32} = \overline{M}_{32}, && \text{(ix)}\ \tau_{33} = \overline{\tau}_{33}, \\
&\text{(x)}\ \tau_{31} = \overline{\tau}_{31}, && \text{(xi)}\ \tau_{32} = \overline{\tau}_{32}, && \text{(xii)}\ T = \overline{T}, \\
&\text{(xiii)}\ \frac{\partial T}{\partial x_3} + \theta T = \frac{\partial \overline{T}}{\partial x_3} + \overline{\theta}\overline{T}.
\end{aligned} \tag{4.1}$$

The boundary conditions on the free surface $x_3 = 0$ are

$$\begin{aligned}
&\text{(xiv)}\ M_{33} = 0, \quad \text{(xv)}\ M_{31} = 0, \quad \text{(xvi)}\ M_{32} = 0, \quad \text{(xvii)}\ \tau_{33} = 0, \\
&\text{(xviii)}\ \tau_{31} = 0, \quad \text{(xxi)}\ \tau_{32} = 0, \quad \text{(xx)}\ \frac{\partial T}{\partial x_3} + \theta T = 0,
\end{aligned} \tag{4.2}$$

where

$$\begin{gathered}
M_{33} = M\frac{\partial \zeta}{\partial x_3}, \qquad M_{32} = M\frac{\partial}{\partial x_3}\left(\eta - \nabla^2\Psi\right), \qquad M_{31} = M\frac{\partial \xi}{\partial x_3}, \\
\tau_{33} = \lambda\nabla^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial x_3^2} - \frac{\partial^2\psi}{\partial x_1\partial x_3}\right) - \gamma T, \qquad \tau_{32} = -F\frac{\partial \xi}{\partial t}, \\
\tau_{31} = \mu\left(2\frac{\partial^2\phi}{\partial x_1\partial x_3} - \frac{\partial^2\psi}{\partial x_3^2} + \frac{\partial^2\psi}{\partial x_1^2}\right) - F\frac{\partial \eta}{\partial t},
\end{gathered}$$

θ is the ratio of the coefficients of heat transfer to the thermal conductivity.

From the boundary conditions (iii), (v), (vi), and (vii), we get

$$\begin{aligned} B_1 e^{ihH} - B_2 e^{-ihH} &= -\overline{B}_2 e^{-i\overline{h}H}, \\ B_1 e^{ihH} + B_2 e^{-ihH} &= -\overline{B}_2 e^{i\overline{h}H}, \\ M(B_1 e^{ihH} - B_2 e^{-ihH}) &= -\overline{M}\,\overline{B}_2 e^{-i\overline{h}H}, \\ M(B_1 e^{ihH} + B_2 e^{-ihH}) &= -\overline{M}\,\overline{B}_2 e^{-i\overline{h}H}. \end{aligned} \tag{4.3}$$

Whence

$$B_1 = B_2 = \overline{B}_2 = 0, \qquad \xi = \zeta = \overline{\xi} = \overline{\zeta} = 0. \tag{4.4}$$

The other significant boundary conditions are responsible for the following relations:

(xvi) $q_1(E_3 - F_3) + q_2(E_4 - F_4) = 0,$

(xvii) $q_3(A_3 + B_3) + q_4(A_4 + B_4) + q_5(E_3 - F_3) + q_6(E_4 - F_4) = 0,$

(xviii) $q_7(A_3 - B_3) + q_8(A_4 - B_4) + q_9(E_3 - F_3) + q_{10}(E_4 - F_4) = 0,$

(i) $$iL(A_3 e^{m_3H} + B_3 e^{-m_3H} + A_4 e^{m_4H} + B_4 e^{-m_4H}) - n_3(E_3 e^{n_3H} + F_3 e^{-n_3H}) - n_4(E_3 e^{n_4H} + F_4 e^{-n_4H}) = iL\overline{B}_3 e^{-\overline{m}_3H} + iL\overline{B}_4 e^{-\overline{m}_4H} + \overline{n}_3\overline{F}_3 e^{-\overline{n}_3H} + \overline{n}_4\overline{F}_4 e^{-\overline{n}_4H},$$

(ii) $$m_3(A_3 e^{m_3H} - B_3 e^{-m_3H}) + m_4(A_4 e^{m_4H} - B_4 e^{-m_4H}) + iL(E_3 e^{n_3H} - F_3 e^{-n_3H} + E_4 e^{n_4H} + F_4 e^{-n_4H}) = -\overline{m}_3\overline{B}_3 e^{-\overline{m}_3H} - \overline{m}_4\overline{B}_4 e^{-\overline{m}_4H} + iL\overline{F}_3 e^{-\overline{n}_3H} + iL\overline{F}_4 e^{-\overline{n}_4H},$$

(iv) $$\Omega_3(E_3 e^{n_3H} + F_3 e^{-n_3H}) + \Omega_4(E_3 e^{n_4H} + F_4 e^{-n_4H}) = \overline{\Omega}_3\overline{F}_3 e^{\overline{n}_3H} + \overline{\Omega}_4\overline{F}_4 e^{-\overline{n}_3H},$$

(viii) $$M[q_1(E_3 e^{n_3H} - F_3 e^{-n_3H}) + q_2(E_4 e^{n_4H} - F_4 e^{-n_4H})] = -\overline{M}(\overline{q}_1\overline{F}_3 e^{-\overline{n}_3H} + \overline{q}_2\overline{F}_4 e^{-\overline{n}_4H}),$$

(ix) $$q_3(A_3 e^{m_3H} + B_3 e^{-m_3H}) + q_4(A_4 e^{m_4H} + B_4 e^{-m_4H}) + q_5(E_3 e^{n_3H} - F_3 e^{-n_3H}) + q_6(E_4 e^{n_4H} - F_4 e^{-n_4H}) = \overline{q}_3\overline{B}_3 e^{-\overline{m}_3H} + \overline{q}_4\overline{B}_4 e^{-\overline{m}_4H} - \overline{q}_5\overline{F}_3 e^{-\overline{n}_3H} - \overline{q}_6\overline{F}_4 e^{-\overline{n}_4H},$$

(x) $$q_7(A_3 e^{m_3H} - B_3 e^{-m_3H}) + q_8(A_4 e^{m_4H} - B_4 e^{-m_4H}) + q_9(E_3 e^{n_3H} - F_3 e^{-n_3H}) + q_{10}(E_4 e^{n_4H} - F_4 e^{-n_4H}) = -\overline{q}_7\overline{B}_3 e^{-\overline{m}_3H} - \overline{q}_8\overline{B}_4 e^{-\overline{m}_4H} - \overline{q}_9\overline{F}_3 e^{-\overline{n}_3H} - \overline{q}_{10}\overline{F}_4 e^{-\overline{n}_4H},$$

(xii) $$\Omega_3'(A_3 e^{m_3H} + B_3 e^{-m_3H}) + \Omega_4'(A_4 e^{m_4H} + B_4 e^{-m_4H}) = \overline{\Omega}_3'\overline{B}_3 e^{-\overline{m}_3H} + \overline{\Omega}_4'\overline{B}_4 e^{-\overline{m}_4H},$$

$$\text{(xiii)}\quad q_{11}A_3e^{m_3H}+q_{12}B_3e^{-m_3H}+q_{13}A_4e^{m_4H}+q_{14}B_4e^{-m_4H} = \overline{q}_{12}\overline{B}_3e^{-\overline{m}_3H}+\overline{q}_{14}\overline{B}_4e^{-\overline{m}_4H},$$

$$\text{(xx)}\quad q_{11}A_3+q_{12}B_3+q_{13}A_4+q_{14}B_4=0, \tag{4.5}$$

where

$$\begin{aligned}
q_1 &= n_3(\Omega_3+L^2-n_3^2), & \overline{q}_1 &= \overline{n}_3(\overline{\Omega}_3+L^2-\overline{n}_3^2),\\
q_2 &= n_4(\Omega_4+L^2-n_4^2), & \overline{q}_2 &= \overline{n}_4(\overline{\Omega}_4+L^2-\overline{n}_4^2),\\
q_3 &= (2\mu+p)L^2-\rho b^2-pm_3^2, & \overline{q}_3 &= (2\overline{\mu}+p)L^2-\overline{\rho}b^2-p\overline{m}_3^2,\\
q_4 &= (2\mu+p)L^2-\rho b^2-pm_4^2, & \overline{q}_4 &= (2\overline{\mu}+p)L^2-\overline{\rho}b^2-p\overline{m}_4^2,\\
q_5 &= -2iL\mu n_3, & \overline{q}_5 &= -2iL\overline{\mu}\,\overline{n}_3,\\
q_6 &= -2iL\mu n_4, & \overline{q}_6 &= 2iL\overline{\mu}\,\overline{n}_4,\\
q_7 &= 2iL\mu m_3, & \overline{q}_7 &= 2iL\overline{\mu}\,\overline{m}_3,\\
q_8 &= 2iL\mu m_4, & \overline{q}_8 &= 2iL\overline{\mu}\,\overline{m}_4,\\
q_9 &= ibF\Omega_3-\mu L^2-\mu n_3^2, & \overline{q}_9 &= ib\overline{F}\,\overline{\Omega}_3-\overline{\mu}L^2-\overline{\mu}\,\overline{n}_3^2,\\
q_{10} &= ibF\Omega_4-\mu L^2-\mu n_4^2, & \overline{q}_{10} &= ib\overline{F}\,\overline{\Omega}_4-\overline{\mu}L^2-\overline{\mu}\,\overline{n}_4^2,\\
q_{11} &= \Omega_3'(\theta+m_3), & \overline{q}_{11} &= \overline{\Omega}_3'(\overline{\theta}+\overline{m}_3),\\
q_{12} &= \Omega_3'(\theta-m_3), & \overline{q}_{12} &= \overline{\Omega}_3'(\overline{\theta}-\overline{m}_3),\\
q_{13} &= \Omega_4'(\theta+m_4), & \overline{q}_{13} &= \overline{\Omega}_4'(\overline{\theta}+\overline{m}_4),\\
q_{14} &= \Omega_4'(\theta-m_4), & \overline{q}_{14} &= \overline{\Omega}_4'(\overline{\theta}-\overline{m}_4),
\end{aligned} \tag{4.6}$$

Elimination of $A_3, B_3, A_4, B_4, E_3, F_3, E_4, F_4, \overline{B}_3, \overline{B}_4, \overline{F}_3, \overline{F}_4$ gives the wave velocity equation in the form of

$$\det d_{ij}=0, \tag{4.7}$$

where the non-vanishing entries of the twelfth-order determinant of d_{ij} are given by

$$\begin{aligned}
&d_{15}=q_1e^{-n_3H}, && d_{16}=-q_1e^{n_3H}, && d_{17}=q_2e^{-n_4H}, && d_{18}=-q_2e^{n_4H},\\
&d_{21}=q_3e^{-m_3H}, && d_{22}=q_3e^{m_3H}, && d_{23}=q_4e^{-m_4H}, && d_{24}=q_4e^{m_4H},\\
&d_{25}=q_5e^{-n_3H}, && d_{26}=-q_5e^{n_3H}, && d_{27}=q_6e^{-n_4H}, && d_{28}=-q_6e^{n_4H},\\
&d_{31}=q_7e^{-m_3H}, && d_{32}=-q_7e^{m_3H}, && d_{33}=q_8e^{-m_4H}, && d_{34}=-q_8e^{m_4H},\\
&d_{35}=q_9e^{-n_3H}, && d_{36}=-q_9e^{n_3H}, && d_{37}=q_{10}e^{-n_4H}, && d_{38}=-q_{10}e^{n_4H},\\
&d_{41}=iL, && d_{42}=iL, && d_{43}=iL, && d_{44}=iL,\\
&d_{45}=-n_3, && d_{46}=-n_3, && d_{47}=-n_4, && d_{48}=n_4,\\
&d_{49}=-iL, && d_{410}=iL, && d_{411}=-\overline{n}_3, && d_{412}=-\overline{n}_4,\\
&d_{51}=m_3, && d_{52}=-m_3, && d_{53}=m_4, && d_{54}=-m_4,\\
&d_{55}=iL, && d_{56}=iL, && d_{57}=iL, && d_{58}=iL,\\
&d_{59}=\overline{m}_3, && d_{510}=\overline{m}_4, && d_{511}=iL, && d_{512}=iL,\\
&d_{65}=\Omega_3, && d_{66}=\Omega_3, && d_{67}=\Omega_4, && d_{68}=\Omega_4,
\end{aligned}$$

$$
\begin{array}{llll}
d_{611} = -\overline{\Omega}_3, & d_{612} = -\overline{\Omega}_4, & d_{75} = Mq_1, & d_{76} = -Mq_1, \\
d_{77} = Mq_2, & d_{78} = -Mq_2, & d_{711} = \overline{M}\overline{q}_1, & d_{712} = -\overline{M}\overline{q}_2, \\
d_{81} = q_3, & d_{82} = q_3, & d_{83} = q_4, & d_{84} = q_4, \\
d_{85} = q_5, & d_{86} = -q_5, & d_{87} = q_6, & d_{88} = -q_6, \\
d_{89} = -\overline{q}_3, & d_{810} = -\overline{q}_4, & d_{811} = \overline{q}_5, & d_{812} = \overline{q}_6, \\
d_{91} = q_7, & d_{92} = -q_7, & d_{93} = q_8, & d_{94} = -q_8, \\
d_{95} = q_9, & d_{96} = -q_9, & d_{97} = q_{10}, & d_{98} = -q_{10}, \\
d_{99} = \overline{q}_7, & d_{910} = \overline{q}_8, & d_{911} = \overline{q}_9, & d_{912} = \overline{q}_{10}, \\
d_{101} = \Omega_3', & d_{102} = \Omega_3', & d_{103} = \Omega_4', & d_{104} = \Omega_4', \\
d_{109} = -\overline{\Omega}_3', & d_{1010} = -\overline{\Omega}_4', & d_{111} = q_{11}, & d_{112} = q_{12}, \\
d_{113} = q_{13}, & d_{114} = q_{14}, & d_{119} = -\overline{q}_{12}, & d_{1110} = -\overline{q}_{14}, \\
d_{121} = q_{11}e^{-m_3H}, & d_{122} = q_{12}e^{m_3H}, & d_{123} = q_{13}e^{-m_4H}, & d_{124} = q_{14}e^{m_4H}.
\end{array}
\tag{4.8}
$$

Equation (4.7) determines the wave velocity equation for the Rayleigh waves in a thermoelastic granular medium under initial stress.

5. Discussion. The transcendental equation (4.7), in the determinant form, has complex roots. The real part gives the velocity of Rayleigh waves and the imaginary part gives the attenuation due to the granular nature of the medium. It is clear from the frequency equation (4.7) that the phase velocity depends on the initial stress P, the friction F, and the coupling factor ϵ.

When there is no coupling between the temperature and strain fields, we have θ vanishes,

$$
\lim_{\epsilon\to 0}\left(m_3^2, m_4^2\right) = \left(L^2 - \frac{b^2}{\alpha^2}, L^2\right), \qquad \lim_{\gamma\to 0}\left(\gamma\cdot\Omega_3'\right) = 0, \qquad \lim_{\gamma\to 0}\left(\gamma\cdot\Omega_4'\right) = b^2, \tag{5.1}
$$

where

$$
\lim_{\gamma\to 0} q_{11} = 0, \qquad \lim_{\gamma\to 0} q_{12} = 0, \qquad \lim_{\gamma\to 0} q_{13} = b^2L, \qquad \lim_{\gamma\to 0} q_{14} = -b^2L. \tag{5.2}
$$

Similar results hold for the lower medium. Multiplying the rows 10, 11 and 12 of the determinant $|d_{ij}|$ by γ and then taking $\lim_{\gamma\to 0}$, equation (4.7) reduces, after some computation, to the following ninth-order determinantal equation:

$$
\begin{vmatrix}
0 & 0 & q_1e^{-n_3H} & -q_1e^{n_3H} & q_2e^{-n_4H} & -q_2e^{n_4H} & 0 & 0 & 0 \\
q_3e^{-m_3H} & q_3e^{m_3H} & q_5e^{-n_3H} & -q_5e^{n_3H} & q_6e^{-n_4H} & -q_6e^{n_4H} & 0 & 0 & 0 \\
q_7e^{-m_3H} & -q_7e^{m_3H} & q_9e^{-n_3H} & -q_9e^{n_3H} & q_{10}e^{-n_4H} & -q_{10}e^{n_4H} & 0 & 0 & 0 \\
iL & iL & -n_3 & -n_3 & -n_4 & n_4 & -iL & -\overline{n}_3 & -\overline{n}_4 \\
m_3 & -m_3 & iL & iL & iL & iL & \overline{m}_3 & -iL & -iL \\
0 & 0 & \Omega_3 & \Omega_3 & \Omega_4 & \Omega_4 & 0 & -\overline{\Omega}_3 & -\overline{\Omega}_4 \\
0 & 0 & Mq_1 & -Mq_1 & Mq_2 & -Mq_2 & 0 & \overline{M}\overline{q}_1 & \overline{M}\overline{q}_2 \\
q_3 & q_3 & q_5 & -q_5 & q_6 & -q_6 & -\overline{q}_3 & \overline{q}_5 & \overline{q}_6 \\
q_7 & -q_7 & q_9 & -q_9 & q_{10} & -q_{10} & \overline{q}_7 & \overline{q}_9 & \overline{q}_{10}
\end{vmatrix} = 0,
\tag{5.3}
$$

where

$$
\begin{aligned}
q_1 &= n_3(\Omega_3 + L^2 - n_3^2), & \overline{q}_1 &= \overline{n}_3(\overline{\Omega}_3 + L^2 - \overline{n}_3^2),\\
q_2 &= n_4(\Omega_4 + L^2 - n_4^2), & \overline{q}_2 &= \overline{n}_4(\overline{\Omega}_4 + L^2 - \overline{n}_4^2),\\
q_3 &= 2\mu L^2 - b^2\left(\rho - \frac{P}{\alpha^2}\right), & \overline{q}_3 &= 2\overline{\mu}L^2 - b^2\left(\overline{\rho} - \frac{P}{\overline{\alpha}^2}\right),\\
q_4 &= 2\mu L^2 - \rho b^2, & \overline{q}_4 &= 2\overline{\mu}L^2 - \overline{\rho} b^2,\\
q_5 &= -2iL\mu n_3, & \overline{q}_5 &= -2iL\overline{\mu}\,\overline{n}_3,\\
q_6 &= -2iL\mu n_4, & \overline{q}_6 &= -2iL\overline{\mu}\,\overline{n}_4,\\
q_7 &= 2iL\mu m_3, & \overline{q}_7 &= 2iL\overline{\mu}\,\overline{m}_3,\\
q_9 &= ibF\Omega_3 - \mu L^2 - \mu n_3^2, & \overline{q}_9 &= ib\overline{F}\,\overline{\Omega}_3 - \overline{\mu}L^2 - \overline{\mu}\,\overline{n}_3^2,\\
q_{10} &= ibF\Omega_4 - \mu L^2 - \mu n_4^2, & \overline{q}_{10} &= ib\overline{F}\,\overline{\Omega}_4 - \overline{\mu}L^2 - \overline{\mu}\,\overline{n}_4^2.
\end{aligned}
\tag{5.4}
$$

The frequency equation (5.3) determines the wave velocity equation for the Rayleigh waves in a granular medium under initial stress.

When the initial stress is absent, we have

$$
\alpha^2 = \frac{\lambda + 2\mu}{\rho}, \qquad B^2 = \frac{\mu}{\rho}, \qquad q_3 = 2\mu L^2 - \rho b^2, \qquad \overline{q}_3 = 2\overline{\mu}L^2 - \overline{\rho} b^2. \tag{5.5}
$$

Thus, equation (5.3) with the relations (5.5) reduces to the frequency equation obtained by Bhattacharyya [2].

If the granular rotations vanish, we get

$$
\begin{gathered}
\lim_{M\to 0}\lim_{S\to 0}(n_3^2, n_4^2) = \left(L^2, L^2 - \frac{b^2}{\beta^2}\right), \qquad \lim_{M\to 0}\lim_{S\to 0}(S\cdot\Omega_3) = -ib,\\
\lim_{M\to 0}\lim_{S\to 0}(S\cdot\Omega_4) = 0, \qquad \lim_{M\to 0}\lim_{S\to 0}(\Omega_4) = -\frac{b^2}{\beta^2}\\
\lim_{M\to 0}\lim_{S\to 0} q_9 = \rho b^2 - 2\mu L^2, \qquad \lim_{M\to 0}\lim_{S\to 0} q_{10} = -\mu\left(2L^2 - \frac{b^2}{\beta^2}\right).
\end{gathered}
\tag{5.6}
$$

Similar results also hold for the lower medium. Multiplying the columns 5, 6 and 11 of the determinant $|d_{ij}|$ by S and then taking $\lim_{M\to 0}\lim_{S\to 0}$, we get after some computation, the following ninth-order determinantal equation:

$$
\begin{vmatrix}
q_3e^{-m_3H} & q_3e^{m_3H} & q_4e^{-m_4H} & q_4e^{m_4H} & q_6e^{-n_4H} & -q_6e^{n_4H} & 0 & 0 & 0\\
q_7e^{-m_3H} & q_7e^{m_3H} & q_8e^{-m_4H} & -q_8e^{m_4H} & q_{10}e^{-n_4H} & q_{10}e^{n_4H} & 0 & 0 & 0\\
iL & iL & iL & iL & -n_4 & n_4 & -iL & -iL & -\overline{n}_4\\
m_3 & -m_3 & m_4 & -m_4 & iL & iL & \overline{m}_3 & \overline{m}_4 & -iL\\
q_3 & q_3 & q_4 & q_4 & -q_6 & q_6 & -\overline{q}_3 & -\overline{q}_4 & \overline{q}_6\\
q_7 & -q_7 & q_8 & -q_8 & -q_{10} & q_{10} & \overline{q}_7 & \overline{q}_8 & \overline{q}_{10}\\
\Omega_3' & \Omega_3' & \Omega_4' & -\Omega_4' & 0 & 0 & -\overline{\Omega}_3' & -\overline{\Omega}_4' & 0\\
q_{11} & q_{12} & q_{13} & q_{14} & 0 & 0 & \overline{q}_{12} & -\overline{q}_{14} & 0\\
q_{11}e^{-m_3H} & q_{12}e^{m_3H} & q_{13}e^{-m_4H} & q_{14}e^{m_4H} & 0 & 0 & 0 & 0 & 0
\end{vmatrix} = 0.
\tag{5.7}
$$

Equation (5.7) is the velocity equation of an initially stressed thermoelastic granular layer of thickness H overlaying semi-infinite elastic isotropic medium.

Finally, in the absence of initial stress and when there is no coupling between the temperature and strain fields, as well as the vanishing of granular rotations, equation (5.7) takes the form

$$\begin{vmatrix} R^2e^{m_3H} & 2Lm_4e^{m_4H} & R^2e^{-m_3H} & -2Lm_4e^{-m_4H} & 0 & 0 \\ 2Lm_3e^{m_3H} & R^2e^{m_4H} & -2Lm_3e^{-m_3H} & R^2e^{-m_4H} & 0 & 0 \\ -L & -m_4 & -L & m_4 & L & -\overline{m}_4 \\ -m_3 & -L & m_3 & -L & -\overline{m}_3 & L \\ 2Lm_3 & R^2 & -2Lm_3 & R^2 & -2L\frac{\overline{\mu}}{\mu}\overline{m}_3 & \frac{\overline{\mu}}{\mu}\overline{R}^2 \\ R^2 & 2Lm_4 & R^2 & -2L\overline{m}_4 & -\frac{\overline{\mu}}{\mu}\overline{R}^2 & -2L\frac{\overline{\mu}}{\mu}\overline{m}_4 \end{vmatrix} = 0, \tag{5.8}$$

where

$$m_3^2 = L^2 - \frac{\rho b^2}{\lambda + 2\mu}, \qquad \overline{m}_3^2 = L^2 - \frac{\overline{\rho} b^2}{\overline{\lambda} + b\overline{\mu}},$$
$$\beta^2 = \frac{\mu}{\rho}, \qquad \overline{\beta}^2 = \frac{\overline{\mu}}{\overline{\rho}}, \tag{5.9}$$
$$m_4^2 = L^2 - \frac{b^2}{\beta^2}, \qquad \overline{m}_4^2 = L^2 - \frac{b^2}{\overline{\beta}^2}, \qquad R^2 = \left(2L^2 - \frac{b^2}{\beta^2}\right), \qquad \overline{R}^2 = \left(2L^2 - \frac{b^2}{\overline{\beta}^2}\right)$$

Equation (5.8) is identical to [5, equation (4.195)].

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