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A NOTE ON CENTRALIZERS

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ABSTRACT. For prime rings R, we characterize the set $U \cap C_R([U,U])$, where U is a right ideal of R; and we apply our result to obtain a commutativity-or-finiteness theorem. We include extensions to semiprime rings.

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Let R be an arbitrary ring with center Z. For $x,y \in R$, denote by $[x,y]$ the commutator $xy - yx$; and for an arbitrary nonempty subset S of R, denote by $[S,S]$ the set $\{[x,y] \mid x,y \in S\}$. Denote by $C_R(S)$ the centralizer of S in R—i.e., the set $\{x \in R \mid [x,s] = 0 \text{ for all } s \in S\}$.

It is proved in [2] that if R is semiprime and I is a nonzero ideal of R, then $C_R([I,I]) \subseteq C_R(I)$. It follows that $C([I,I]) \cap I \subseteq Z$, since in a semiprime ring R the center of a nonzero right ideal is contained in the center of R. The first goal of this note is to study the subring $H = C_R([U,U]) \cap U$, where R is prime or semiprime and U is a nonzero right ideal. The information obtained is used to prove commutativity-or-finiteness results extending [1, Theorem 3].

1. Preliminaries.

We shall use standard notation for annihilators—that is, for a nonempty subset S of R, $A_l(S)$ and $A(S)$ will be the left and two-sided annihilators of S. A subring S will be said to have finite index in R if $(S,+)$ is of finite index in $(R,+)$. We shall use without explicit mention the commutator identities $[xy,z] = x[y,z] + [x,z]y$ and $[x,yz] = y[x,z] + [x,y]z$.

We begin with a revealing example.

EXAMPLE 1.1. Let F be an arbitrary field, let R be the ring of 2×2 matrices over F, and let $U = e_{11}R$. Then R is prime, U is a right ideal, and $[U,U] = F_{e_{12}}$. Note that $C_R([U,U]) \cap U = Fe_{12} = A([U,U]) \cap U$, and note that this set does not centralize U. Thus, the result in [2] for two-sided ideals does not hold for one-sided ideals, even in the case of prime rings.

2. The case of R prime

THEOREM 2.1. *Let R be a prime ring, U a right ideal of R, and $H = C_R([U,U]) \cap U$. Then either $H = U \cap Z$, or H is a zero ring and $H = A([U,U]) \cap U$. In any case, H is a commutative subring of R.*

Proof. We begin as in the proof of [2, Lemma 1]. Let $z \in C_R([U,U])$. Then for all $x, y \in U$, $z[x,xy] = [x,xy]z$; hence $zx[x,y] = x[x,y]z = xz[x,y]$ and therefore $[z,x][x,y] = 0$. Replacing y by yz, we get $[z,x]U[z,x] = \{0\}$ for all $x \in U$; and since $[z,x]U$ is a nilpotent right ideal, we have $[z,x]U = \{0\}$ for all $z \in C_R([U,U])$ and $x \in U$. Taking $z \in H$, we obtain $[z,x]z = 0 = z[z,x]$ for all $z \in H$ and $x \in U$; and replacing x by xr for arbitrary $r \in R$ yields $zU[z,r] = \{0\}$, hence

$$zUR[z,r] = \{0\} \quad \text{for all } z \in H \text{ and } r \in R. \tag{2.1}$$

Since R is prime, (2.1) shows that either $z \in Z$ or $zU = \{0\}$; hence $H = (H \cap Z) \cup (H \cap A_l(U))$. Since the abelian group H cannot be the union of two proper subgroups, we have $H = H \cap Z$ or $H = H \cap A_l(U)$, so that $H \subseteq Z$ or $H \subseteq A_l(U)$. In the first case, H is clearly equal to $U \cap Z$, so suppose $H \subseteq A_l(U)$. Since $H \subseteq U$, $H^2 = \{0\}$; moreover, $H \subseteq A_l([U,U]) \cap C_R([U,U])$, so $H \subseteq A([U,U])$ and hence $H = A([U,U]) \cap U$.

We now proceed to a commutativity-or-finiteness result. □

Theorem 2.2. *Let R be a prime ring and U a right ideal of finite index in R. If $[U,U]$ is finite, then R is either finite or commutative.*

Proof. Suppose that $[U,U] = \{x_1, x_2, \dots, x_m\}$. For each $i = 1, 2, \dots, m$ define $\Phi_i : U \to U$ by $\Phi_i(x) = [x_i, x]$ for all $x \in U$. Then $\Phi_i(U)$ is finite, hence $\operatorname{Ker}\Phi_i$ is of finite index in U. Letting $H = \bigcap_{i=1}^{m} \operatorname{Ker}\Phi_i$, we see that $H = U \cap C_R([U,U])$ and that H is of finite index in U. Now U is of finite index in R, so H is of finite index in R. It follows by a theorem of Lewin [3] that H contains an ideal I of R which is also of finite index in R. If $I = \{0\}$, then R is finite; if $I \neq \{0\}$, Theorem 2.1 implies that R has a nonzero commutative ideal and hence R is commutative. □

3. The case of R semiprime. Let R be semiprime, U a right ideal, and $H = U \cap C_R([U,U])$. Let $\{P_\alpha \mid \alpha \in \Lambda\}$ be a collection of prime ideals such that $\cap P_\alpha = \{0\}$. Now (2.1) holds in R, hence for each $\alpha \in \Lambda$ and each $z \in H$, either $[z,R] \subseteq P_\alpha$ or $zU \subseteq P_\alpha$. Since each of these conditions defines an additive subgroup of H, we see that $[H,R] \subseteq P_\alpha$ or $HU \subseteq P_\alpha$; therefore $[H,H] \subseteq P_\alpha$ for all $\alpha \in \Lambda$. Thus $[H,H] = \{0\}$—that is, H is a commutative subring of R.

Revisiting the proof of Theorem 2.2, we see that in the semiprime case, either R is finite or R contains a nonzero commutative ideal I. But in a semiprime ring, a commutative ideal is central; hence we have the following extension of Theorem 2.2.

Theorem 3.1. *Let R be a semiprime ring and U a right ideal of finite index in R. If $[U,U]$ is finite, then either R is finite or R contains a nonzero central ideal.*

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