Internat. J. Math. & Math. Sci.
Vol. 24, No. 10 (2000) 673–689
S0161171200004671

AN EXAMPLE OF NONSYMMETRIC SEMI-CLASSICAL FORM OF CLASS $s = 1$; GENERALIZATION OF A CASE OF JACOBI SEQUENCE

MOHAMED JALEL ATIA

(Received 24 February 2000)

Abstract. We give explicitly the recurrence coefficients of a nonsymmetric semi-classical sequence of polynomials of class $s = 1$. This sequence generalizes the Jacobi polynomial sequence, that is, we give a new orthogonal sequence $\{\hat{P}_n^{(\alpha,\alpha+1)}(x,\mu)\}$, where μ is an arbitrary parameter with $\Re(1-\mu) > 0$ in such a way that for $\mu = 0$ one has the well-known Jacobi polynomial sequence $\{\hat{P}_n^{(\alpha,\alpha+1)}(x)\}$, $n \geq 0$.

Keywords and phrases. Orthogonal polynomials, semi-classical polynomials.

2000 Mathematics Subject Classification. Primary 33C45; Secondary 42C05.

1. Introduction. Many authors [1, 2, 3] have studied semi-classical sequences of polynomials of class $s = 1$. In particular, Bachène [2, page 87] gave the system fulfilled by such sequences using the structure relation and Belmehdi [3, page 272] gave the same system (in a more simple way) using directly the functional equation. This system is not linear and has not been sorted out before. The aim of this paper is to present a method that may give us some solutions.

In Section 2, we recall the general features which are needed in what follows. Section 3 is devoted to the setting of the problem, to give an integral representation and the expressions of the moments of the form $\mathcal{J}(\alpha,\alpha+1)(\mu)$ which generalizes the form $\mathcal{J}(\alpha,\alpha+1)$, where $\mathcal{J}(\alpha,\beta)$ is the Jacobi functional.

In Section 4, the recurrence coefficients of the semi-classical sequence of polynomials orthogonal with respect to $\mathcal{J}(\alpha,\alpha+1)(\mu)$ are explicitly given using the Laguerre-Freud equation of semi-classical orthogonal sequences of class $s = 1$ given in [3, page 272].

2. Preliminaries. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. Let us define the following operations on $\mathcal{P}'$:

- the left-multiplication of a linear functional by a polynomial

$$\langle gu, f\rangle := \langle u, gf\rangle, \quad f, g \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.1}$$

- the derivative of a linear functional

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.2}$$

- the homothetic of a linear functional

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle, \quad a \in \mathbb{C}-\{0\}, \tag{2.3}$$

where

$$(h_a f)(x) = f(ax), \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.4}$$

- the translation of a linear functional

$$\langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle, \quad b \in \mathbb{C}, \tag{2.5}$$

where

$$(\tau_b f)(x) = f(x-b), \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.6}$$

- the division of a linear functional by a polynomial of first degree

$$\langle (x-c)^{-1} u, f\rangle := \langle u, \theta_c f\rangle, \quad c \in \mathbb{C}, \tag{2.7}$$

where

$$(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}, \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.8}$$

- using (2.1) and (2.2) we can easily prove

$$(fu)' = f'u + fu', \quad f \in \mathcal{P},\ u \in \mathcal{P}'. \tag{2.9}$$

Definition 2.1 (see [4]). A sequence of polynomials $\{\hat{P}_n\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence with respect to the linear functional u if

(i) $\deg \hat{P}_n = n$ *and the leading coefficient of* $\hat{P}_n(x)$ *is equal to* 1.

(ii) $\langle u, \hat{P}_n \hat{P}_m\rangle = r_n \delta_{n,m}$, $n,m \geq 0$, $r_n \neq 0$, $n \geq 0$.

It is well known that a sequence of monic orthogonal polynomial satisfies a three-term recurrence relation

$$\begin{gathered}\hat{P}_0(x) = 1, \qquad \hat{P}_1(x) = x - \beta_0, \\ \hat{P}_{n+2}(x) = (x - \beta_{n+1})\hat{P}_{n+1}(x) - \gamma_{n+1}\hat{P}_n(x), \quad n \geq 0,\end{gathered} \tag{2.10}$$

with $(\beta_n, \gamma_{n+1}) \in \mathbb{C}\times\mathbb{C}-\{0\}$, $n \geq 0$.

In such conditions, we say that u is regular or quasi-definite (see [4]). In what follows, we assume that the linear functionals are regular.

A shifting leaves invariant the orthogonality for the sequence $\{\tilde{P}_n\}_{n\geq 0}$. In fact, $\tilde{P}_n(x) = a^{-n}\hat{P}_n(ax+b)$, $n \geq 0$, fulfills the recurrence relation [6] and [8, page 265]

$$\begin{gathered}\tilde{P}_0(x) = 1, \qquad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x), \quad n \geq 0\end{gathered} \tag{2.11}$$

with $\tilde{\beta}_n = (\beta_n - b)/a$, $\tilde{\gamma}_{n+1} = (\gamma_{n+1})/a^2$, $n \geq 0$, $a \in \mathbb{C}-\{0\}$.

Definition 2.2 (see [4]). $\{\hat{P}_n\}_{n\geq 0}$ (respectively, the linear functional u) is semi-classical of class s, if and only if the following statement holds: [6] and [7, pages 143–144].

There exist two polynomials ψ of degree $p \geq 1$ and ϕ of degree $t \geq 0$, such that

$$(\phi u)' + \psi u = 0,$$
$$\prod_{c\in Z_\phi} \left(\left|\psi(c) + \phi'(c)\right| + \left|\left\langle u, \theta_c(\psi) + \theta_c^2(\phi)\right\rangle\right| \right) \neq 0, \tag{2.12}$$

where Z_ϕ is the set of zeros of ϕ. The class of $\{P_n\}_{n\geq 0}$ or u is given by $s = \max(p-1, t-2)$ [7, pages 143–144].

If u is a semi-classical functional of class s, then $v = (h_{a^{-1}} \circ \tau_{-b})u$ is also semi-classical of the same class and it verifies the equation $(\phi_1 v)' + \psi_1 v = 0$, where

$$\phi_1(x) = a^{-t}\phi(ax+b), \qquad \psi_1(x) = a^{1-t}\psi(ax+b). \tag{2.13}$$

3. Generalization of $\mathcal{J}(\alpha, \alpha+1)$ as a semi-classical sequence of class $s = 1$

3.1. Problem setting. If u is a classical linear function, that is,

$$(\phi(x)u)' + \psi(x)u = 0, \quad \deg\phi \leq 2,\ \deg\psi = 1, \tag{3.1}$$

from (2.9) the multiplication by x gives

$$(x\phi(x)u)' - \phi(x)u + x\psi(x)u = 0, \quad \deg(x\phi) \leq 3,\ \deg(x\psi - \phi) \leq 2. \tag{3.2}$$

If we consider the following perturbed equation

$$\begin{gathered}(x\phi(x)u(\mu))' + ((\mu-1)\phi(x) + x\psi(x))u(\mu) = 0,\\ \deg(x\phi) \leq 3, \qquad \deg(x\psi + (\mu-1)\phi) \leq 2,\end{gathered} \tag{3.3}$$

we obtain, under some conditions on μ, a linear functional $u(\mu)$ of class $s = 1$ which generalizes the classical linear functional u.

Examples

(1) The Hermite case. One knows that the functional equation for the Hermite linear functional, noted $\mathcal{H}$, is [6, page 117]

$$\mathcal{H}' + 2x\mathcal{H} = 0 \tag{3.4}$$

multiplied by x gives

$$(x\mathcal{H})' + (2x^2 - 1)\mathcal{H} = 0. \tag{3.5}$$

Thus, we consider the functional equation

$$(x\mathcal{H}(\mu))' + (2x^2 - 2\mu - 1)\mathcal{H}(\mu) = 0 \tag{3.6}$$

which is the functional equation of the well-known generalized-Hermite linear functional, noted $\mathcal{H}(\mu)$, which is regular for $\mu \neq -n-1/2$, $n \geq 0$, and semi-classical of class $s=1$ for $\mu \neq 0$ [4] and [5, page 243]. Notice that $\mathcal{H}(0)=\mathcal{H}$.

(2) The Jacobi case. Let us consider the functional equation for the Jacobi form, $\mathcal{J}(\alpha,\beta)$:

$$((x^2-1)\mathcal{J}(\alpha,\beta))' + (-(\alpha+\beta+2)x+\alpha-\beta)\mathcal{J}(\alpha,\beta) = 0 \tag{3.7}$$

multiplication by x gives the following equation:

$$((x^3-x)\mathcal{J}(\alpha,\beta))' - (x^2-1)\mathcal{J}(\alpha,\beta) + (-(\alpha+\beta+2)x^2+(\alpha-\beta)x)\mathcal{J}(\alpha,\beta) = 0. \tag{3.8}$$

Thus, consider

$$((x^3-x)\mathcal{J}(\alpha,\beta)(\mu))' + ((\mu-\alpha-\beta-3)x^2+(\alpha-\beta)x+1-\mu)\mathcal{J}(\alpha,\beta)(\mu) = 0. \tag{3.9}$$

Notice that $\mathcal{J}(\alpha,\beta)(0)=\mathcal{J}(\alpha,\beta)$.

(a) The Gegenbauer case ($\alpha=\beta$). In this case (3.9) becomes

$$((x^3-x)\mathcal{J}(\alpha,\alpha)(\mu))' + ((\mu-2\alpha-3)x^2+1-\mu)\mathcal{J}(\alpha,\alpha)(\mu) = 0 \tag{3.10}$$

which is the functional equation of the symmetric semi-classical functional, regular for $\mu \neq 2n+2\alpha+1$, $\mu \neq 2n+1$, $n \geq 0$, of class $s=1$ for $\mu \neq 0$, and $\mathcal{J}(\alpha,\alpha)(0)=\mathcal{J}(\alpha,\alpha)$.

In fact, in [1, page 317], we have

$$((x^3-x)u)' + 2(-(\tilde{\alpha}+\tilde{\beta}+2)x^2+\tilde{\beta}+1)u = 0 \tag{3.11}$$

and if we denote by $\{P_n\}_{n\geq 0}$ the sequence of monic polynomials orthogonal with respect to u, then $\{P_n\}_{n\geq 0}$ fulfills (2.10) such that

$$\begin{aligned} \beta_n &= 0, \\ \gamma_{2n+1} &= \frac{(n+\tilde{\beta}+1)(n+\tilde{\alpha}+\tilde{\beta}+1)}{(2n+\tilde{\alpha}+\tilde{\beta}+1)(2n+\tilde{\alpha}+\tilde{\beta}+2)}, \\ \gamma_{2n+2} &= \frac{(n+1)(n+\tilde{\alpha}+1)}{(2n+\tilde{\alpha}+\tilde{\beta}+2)(2n+\tilde{\alpha}+\tilde{\beta}+3)}, \end{aligned} \tag{3.12}$$

for $n \geq 0$. Put

$$-2(\tilde{\alpha}+\tilde{\beta}+2) = \mu-(2\alpha+3), \qquad 2(\tilde{\beta}+1) = 1-\mu, \tag{3.13}$$

we obtain $((x^3-x)u)' + ((\mu-2\alpha-3)x^2+1-\mu)u = 0$ with

$$\begin{aligned} \beta_n &= 0, \\ \gamma_{2n+1} &= \frac{(2n+2\alpha+1-\mu)(2n+1-\mu)}{(4n+2\alpha+1-\mu)(4n+2\alpha+3-\mu)}, \\ \gamma_{2n+2} &= \frac{4(n+1)(n+\alpha+1)}{(4n+2\alpha+3-\mu)(4n+2\alpha+5-\mu)}, \end{aligned} \tag{3.14}$$

for $n \geq 0$.

(b) $\mathcal{J}(\alpha,\alpha+1)$ case. If in (3.9), $\beta=\alpha+1$ we get

$$((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))'+((\mu-2\alpha-4)x^2-x+1-\mu)\mathcal{J}(\alpha,\alpha+1)(\mu)=0. \tag{3.15}$$

In what follows, we will look for the regular linear functional, $\mathcal{J}(\alpha,\alpha+1)(\mu)$ which is a solution of (3.15) and we denote by $\{P_n\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to $\mathcal{J}(\alpha,\alpha+1)(\mu)$ and by $\beta_n,\gamma_n,\ n\geq 0$ the recurrence coefficients of P_n.

Remark 3.1. The solutions of the functional equation (3.15) depend on the value of $(\mathcal{J}(\alpha,\alpha+1)(\mu))_1=\beta_0$, in fact,

$$\langle((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))'+((\mu-2\alpha-4)x^2-x+1-\mu)\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle=0, \tag{3.16}$$

then, using (2.2), one has

$$\begin{aligned}\langle((\mu-2\alpha-4)x^2&-x+1-\mu)\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle\\&=\langle\mathcal{J}(\alpha,\alpha+1)(\mu),((\mu-2\alpha-4)x^2-x+1-\mu)\rangle\\&=(\mu-2\alpha-4)(\mathcal{J}(\alpha,\alpha+1)(\mu))_2-(\mathcal{J}(\alpha,\alpha+1)(\mu))_1+1-\mu=0,\end{aligned} \tag{3.17}$$

but $(\mathcal{J}(\alpha,\alpha+1)(\mu))_2=\gamma_1+\beta_0^2$ and $(\mathcal{J}(\alpha,\alpha+1)(\mu))_1=\beta_0$ then

$$(\mu-2\alpha-4)\gamma_1+(\mu-2\alpha-4)\beta_0^2-\beta_0+1-\mu=0. \tag{3.18}$$

First we search an integral representation in order to obtain β_0.

3.2. An integral representation

Proposition 3.2. *An integral representation of a linear functional* $\mathcal{J}(\alpha,\alpha+1)(\mu)$ *is*

$$\langle\mathcal{J}(\alpha,\alpha+1)(\mu),f(x)\rangle=\frac{\Gamma((2\alpha+3-\mu)/2)}{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)f(x)\,dx \tag{3.19}$$

with $\operatorname{Re}(1-\mu)>0$*, that is,* $\operatorname{Re}(-u)>-1$ *and* $\operatorname{Re}(\alpha+1)>0$*.*

Proof. A solution of (3.15) has the integral representation

$$\langle\mathcal{J}(\alpha,\alpha+1)(\mu),f\rangle=\int_C U(x)f(x)\,dx,\quad f\in\mathcal{P} \tag{3.20}$$

if the following conditions hold [5]:

$$\begin{gathered}((x^3-x)U(x))'+((\mu-2\alpha-4)x^2-x+1-\mu)U(x)=0,\\(x^3-x)U(x)f(x)]_C=0,\quad f\in\mathcal{P},\end{gathered} \tag{3.21}$$

where C is an acceptable integration path. We solve the first condition as a differential equation:

$$((x^3-x)U(x))'+((\mu-2\alpha-4)x^2-x+1-\mu)U(x)=0 \tag{3.22}$$

or, equivalently,

$$(x^3-x)U'(x)+((\mu-2\alpha-1)x^2-x-\mu)U(x)=0,$$
$$\frac{U'(x)}{U(x)}=-\frac{(\mu-2\alpha-1)x^2-x-\mu}{x^3-x}=-\frac{(\mu-2\alpha-1)x^2-x-\mu}{x(x-1)(x+1)}. \tag{3.23}$$

Thus

$$\frac{U'(x)}{U(x)}=-\frac{\mu}{x}+\frac{(\alpha+1)}{(x-1)}+\frac{\alpha}{(x+1)} \tag{3.24}$$

and

$$U(x)=\begin{cases}k|x|^{-\mu}(1-x^2)^{\alpha}(1-x), & |x|<1,\\ 0, & |x|>1.\end{cases} \tag{3.25}$$

If we assume $\mathrm{Re}(1-\mu)>0$, $\mathrm{Re}(\alpha+1)>0$, then

$$(x^3-x)U(x)f(x)]_C=k(x^3-x)|x|^{-\mu}(1-x^2)^{\alpha}(1-x)f(x)]_{-1}^{+1}=0 \tag{3.26}$$

holds.

Determination of the normalisation factor.

$$\begin{aligned}\langle\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle &= k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)\,dx\\ &= k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}\,dx\\ &= 2k_1\int_{0}^{+1}(x)^{-\mu}(1-x^2)^{\alpha}\,dx\\ &= 2k_1\frac{1}{2}B\left(\frac{1-\mu}{2},\alpha+1\right)=1,\end{aligned} \tag{3.27}$$

where $B(p,q)$ is the beta function. Thus, from

$$\langle\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle=k_1B\left(\frac{1-\mu}{2},\alpha+1\right)=k_1\frac{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}{\Gamma((2\alpha+3-\mu)/2)}=1 \tag{3.28}$$

we get

$$k_1=\frac{\Gamma((2\alpha+3-\mu)/2)}{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}. \tag{3.29}$$

Conversely, using this integral representation, we give explicitly the expressions of the moments and the functional equation (3.15). □

3.3. The expressions of the moments. Using the integral representation we have a relation between $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}$ and $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}$ and a relation between $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}$ and $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n}$. Then, using these two relations, we obtain the functional equation.

Lemma 3.3. *Using the integral representation we have*

$$(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}=-(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2},\quad n\geq 0. \tag{3.30}$$

Proof.

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+1}+x^{2n+2}\rangle &= k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)(x^{2n+1}+x^{2n+2})\,dx \\ &= k_1\int_{-1}^{+1}x^{2n+1}|x|^{-\mu}(1-x^2)^{\alpha+1}\,dx = 0\end{aligned} \tag{3.31}$$

because $x^{2n+1}|x|^{-\mu}(1-x^2)^{\alpha+1}$ is an odd function. □

Lemma 3.4. *Using the integral representation we have*

$$(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2} = \frac{\Gamma((2n+3-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+5-\mu)/2)} \tag{3.32}$$

and, in particular,

$$(2n+2\alpha+3-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2} = (2n+1-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n}, \quad n\geq 0. \tag{3.33}$$

Proof. From

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)x^{2n+2}\,dx \\ &= k_1\int_{-1}^{+1}x^{2n+2}|x|^{-\mu}(1-x^2)^{\alpha}\,dx\end{aligned} \tag{3.34}$$

taking into account that $x^{2n+3}|x|^{-\mu}(1-x^2)^{\alpha}$ is an odd function,

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= 2k_1\int_{0}^{+1}x^{2n+2-\mu}(1-x^2)^{\alpha}\,dx \\ &= 2k_1\frac{1}{2}B\left(\frac{2n+3-\mu}{2},\alpha+1\right),\end{aligned} \tag{3.35}$$

where $B(p,q)$ is the beta function

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= \frac{\Gamma((2n+3-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+5-\mu)/2)} \\ &= \frac{2n+1-\mu}{2n+2\alpha+3-\mu}\,\frac{\Gamma((2n+1-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+3-\mu)/2)} \\ \langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= \frac{2n+1-\mu}{2n+2\alpha+3-\mu}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n}\rangle, \quad n\geq 0.\end{aligned} \tag{3.36}$$

Using (3.30) and (3.33) we can find the functional equation (3.15).

From (3.33), we have, for $n\geq 0$,

$$(2n+2\alpha+3-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2} = (2n+1-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n}, \tag{3.37}$$

with (3.30), one has

$$\begin{aligned}&(2n+2\alpha+4-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2} \\ &\quad = -(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1} + (2n+1-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n}, \quad n\geq 0.\end{aligned} \tag{3.38}$$

Using (2.1) and (2.2), we get, for $n \geq 0$,

$$\langle ((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n}\rangle = 0. \quad (3.39)$$

From (3.15) and (3.33), we have, for $n \geq 0$,

$$(2n+2\alpha+5-\mu)\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+3} = (2n+3-\mu)\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+1}. \quad (3.40)$$

Thus, taking into account (3.30), one has

$$\begin{aligned}(2n+2\alpha+5-\mu)\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+3} \\ = -\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+2} + (2n+2-\mu)\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+1}, \quad n \geq 0.\end{aligned} \quad (3.41)$$

From

$$\langle ((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+1}\rangle = 0, \quad n \geq 0. \quad (3.42)$$

equations (3.39) and (3.42) give

$$\langle ((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu), x^{n}\rangle = 0, \quad n \geq 0. \quad (3.43)$$

Hence

$$((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu) = 0. \quad (3.44)$$

□

Corollary 3.5. *From (3.30) and (3.33) we deduce the expressions of the moments:*

$$\begin{aligned}\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+1} &= -\prod_{i=0}^{n}\frac{(2i+1-\mu)}{(2\alpha+2i+3-\mu)}, \quad n \geq 0, \\ \big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+2} &= -\big(\mathcal{J}(\alpha,\alpha+1)(\mu)\big)_{2n+1}, \quad n \geq 0.\end{aligned} \quad (3.45)$$

4. The recurrence coefficients β_n, γ_n, $n \geq 0$

4.1. The system satisfied by recurrence coefficients of semi-classical sequences of class $s = 1$. Assuming that u is semi-classical of class $s = 1$, then u satisfies

$$(\phi u)' + \psi u = 0 \quad (4.1)$$

with

$$\phi(x) = \sum_{k=0}^{3} c_k x^k, \qquad \sum_{k=0}^{3} |c_k| \neq 0, \qquad \psi(x) = \sum_{k=0}^{2} a_k x^k, \qquad |a_2| + |a_1| \neq 0 \quad (4.2)$$

(see [3, page 272]). Furthermore, the nonlinear system satisfied by the recurrence coefficients of semi-classical orthogonal sequences of class $s = 1$ is

$$\begin{gathered}(a_2 - 2nc_3)(\gamma_n + \gamma_{n+1}) = 4c_3 \sum_{k=1}^{n-1} \gamma_k + 2 \sum_{k=0}^{n-1} (\theta_{\beta_n}\phi)(\beta_k) - \psi(\beta_n), \quad n \geq 2, \\ (a_2 - 2c_3)(\gamma_1 + \gamma_2) = 2(\theta_{\beta_1}\phi)(\beta_0) - \psi(\beta_1), \\ a_2\gamma_1 = -\psi(\beta_0),\end{gathered} \tag{4.3}$$

$$\begin{aligned}(a_2 - (2n+1)c_3)\gamma_{n+1}\beta_{n+1} = {} & \sum_{k=0}^{n} \phi(\beta_k) + c_3\left(2\gamma_n\left(n\beta_n + \sum_{k=0}^{n}\beta_k\right) + 3\sum_{k=1}^{n}\gamma_k(\beta_k + \beta_{k-1})\right) \\ & + c_2\left((2n+1)\gamma_{n+1} + 2\sum_{k=1}^{n}\gamma_k\right) - (a_2\beta_n + a_1)\gamma_{n+1}, \quad n \geq 1, \\ (a_2 - c_3)\gamma_1\beta_1 = {} & \phi(\beta_0) + \gamma_1(2c_3\beta_0 + c_2 - a_2\beta_0 - a_1).\end{aligned} \tag{4.4}$$

In our case, since $c_3 = -c_1 = 1$, $c_2 = c_0 = 0$, the first equation of (4.3) becomes

$$(\mu - 2n - 2\alpha - 4)(\gamma_n + \gamma_{n+1}) = 4\sum_{k=1}^{n-1}\gamma_k + 2\sum_{k=0}^{n-1}(\theta_{\beta_n}\phi)(\beta_k) - \psi(\beta_n), \quad n \geq 2. \tag{4.5}$$

Using (2.7), we get

$$\begin{aligned}(\mu - 2n - 2\alpha - 4)\gamma_{n+1} &= -(\mu - 2n - 2\alpha - 4)\gamma_n + 4\sum_{k=1}^{n-1}\gamma_k + 2\sum_{k=0}^{n-1}(\beta_n^2 + \beta_k^2 + \beta_n\beta_k - 1) \\ &\quad - (\mu - 2\alpha - 4)\beta_n^2 + \beta_n - (1-\mu) \\ &= -(\mu - 2n - 2\alpha - 4)\gamma_n + 4\sum_{k=1}^{n-1}\gamma_k + 2\sum_{k=0}^{n-1}\beta_k^2 + 2\beta_n\sum_{k=0}^{n-1}\beta_k \\ &\quad + (2n + 2\alpha + 4 - \mu)\beta_n^2 + \beta_n + \mu - 2n - 1, \quad n \geq 2\end{aligned} \tag{4.6}$$

then

$$\begin{aligned}(\mu - 2n - 2\alpha - 6)\gamma_{n+2} = {} & -(\mu - 2n - 2\alpha - 6)\gamma_{n+1} + 4\sum_{k=1}^{n}\gamma_k + 2\sum_{k=0}^{n}\beta_k^2 + 2\beta_{n+1}\sum_{k=0}^{n}\beta_k \\ & + (2n + 2\alpha + 6 - \mu)\beta_{n+1}^2 + \beta_{n+1} + \mu - 2n - 3, \quad n \geq 1.\end{aligned} \tag{4.7}$$

If we subtract both identities,

$$\begin{aligned}(\mu - 2n - 2\alpha - 6)\gamma_{n+2} = {} & -(\mu - 2n - 2\alpha - 6)\gamma_{n+1} + (\mu - 2n - 2\alpha - 4)\gamma_{n+1} \\ & + (\mu - 2n - 2\alpha - 4)\gamma_n + 4\gamma_n + 2\beta_n^2 \\ & + 2\beta_{n+1}\sum_{k=0}^{n}\beta_k - 2\beta_n\sum_{k=0}^{n-1}\beta_k + (2n + 2\alpha + 6 - \mu)\beta_{n+1}^2 \\ & - (2n + 2\alpha + 4 - \mu)\beta_n^2 + \beta_{n+1} - \beta_n - 2, \quad n \geq 1.\end{aligned} \tag{4.8}$$

Thus the first equation of (4.3) becomes

$$\begin{aligned}(\mu-2n-2\alpha-6)\gamma_{n+2} &= 2\gamma_{n+1}+(\mu-2n-2\alpha)\gamma_n+2\beta_{n+1}\sum_{k=0}^{n}\beta_k-2\beta_n\sum_{k=0}^{n-1}\beta_k \\ &\quad+(2n+2\alpha+6-\mu)\beta_{n+1}^2-(2n+2\alpha+2-\mu)\beta_n^2 \\ &\quad+(\beta_{n+1}-\beta_n)-2,\quad n\geq 1.\end{aligned}\tag{4.9}$$

On the other hand, (4.4) becomes

$$\begin{aligned}(\mu-2n-2\alpha-5)\gamma_{n+1}\beta_{n+1} &= \sum_{k=0}^{n}\phi(\beta_k)+\left(2\gamma_n\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\right) \\ &\quad+c_2\left((2n+1)\gamma_{n+1}+2\sum_{k=1}^{n}\gamma_k\right)-((\mu-2\alpha-4)\beta_n-1)\gamma_{n+1} \\ &= \sum_{k=0}^{n}(\beta_k^3-\beta_k)+\left(2\gamma_n\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\right) \\ &\quad-((\mu-2\alpha-4)\beta_n-1)\gamma_{n+1},\quad n\geq 1.\end{aligned}\tag{4.10}$$

Shifting the indices and subtracting, we get

$$\begin{aligned}(\mu-2n-2\alpha-7)\gamma_{n+2}\beta_{n+2} &= (\mu-2n-2\alpha-5)\gamma_{n+1}\beta_{n+1} \\ &\quad+\beta_{n+1}^3-\beta_{n+1}+3\gamma_{n+1}(\beta_{n+1}+\beta_n) \\ &\quad+\left(2\gamma_{n+2}\left((n+1)\beta_{n+1}+\sum_{k=0}^{n+1}\beta_k\right)\right) \\ &\quad-\left(2\gamma_{n+1}\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)\right)-((\mu-2\alpha-4)\beta_{n+1}-1)\gamma_{n+2} \\ &\quad+((\mu-2\alpha-4)\beta_n-1)\gamma_{n+1},\quad n\geq 0.\end{aligned}\tag{4.11}$$

Thus, from (4.9) and (4.11) we have the following.

Proposition 4.1.

$$\begin{aligned}&(\mu-2n-2\alpha-6)\gamma_{n+2} \\ &\quad= 2\gamma_{n+1}+(\mu-2n-2\alpha)\gamma_n+2\beta_{n+1}\sum_{k=0}^{n}\beta_k-2\beta_n\sum_{k=0}^{n-1}\beta_k \\ &\qquad+(2n+2\alpha+6-\mu)\beta_{n+1}^2-(2n+2\alpha+2-\mu)\beta_n^2+(\beta_{n+1}-\beta_n)-2,\quad n\geq 1\end{aligned}\tag{4.12}$$

$$(\mu-2\alpha-6)(\gamma_1+\gamma_2)=2(\beta_1^2+\beta_0\beta_1+\beta_0^2-1)-(\mu-2\alpha-4)\beta_1^2+\beta_1-(1-\mu)\tag{4.13}$$

$$(\mu-2\alpha-4)\gamma_1=-(\mu-2\alpha-4)\beta_0^2+\beta_0-(1-\mu).\tag{4.14}$$

$$\begin{aligned}&(\mu-2n-2\alpha-7)\gamma_{n+2}\beta_{n+2} \\ &\quad= \beta_{n+1}^3-\beta_{n+1}+(2n+2\alpha+8-\mu)\gamma_{n+2}\beta_{n+1}+(\mu-2n-2\alpha-2)\gamma_{n+1}\beta_{n+1} \\ &\qquad+(\mu-2n-2\alpha-1)\gamma_{n+1}\beta_n+\left(2\sum_{k=0}^{n}\beta_k+1\right)(\gamma_{n+2}-\gamma_{n+1}),\quad n\geq 0\end{aligned}\tag{4.15}$$

$$(\mu-2\alpha-5)\gamma_1\beta_1=\beta_0^3-\beta_0+\gamma_1(2\beta_0-(\mu-2\alpha-4)\beta_0+1).\tag{4.16}$$

Next, we will find the expressions of the recurrence parameters β_n, γ_n, $n \geq 0$. Since $\beta_0 = -(\mu-1)/(\mu-2\alpha-3)$ and from (4.14) we have

$$\begin{aligned}\gamma_1 &= -\frac{(\mu-2\alpha-4)\beta_0^2+\beta_0-(1-\mu)}{\mu-2\alpha-4}\\ &= -\beta_0^2+\frac{\beta_0}{\mu-2\alpha-4}+\frac{\mu-1}{\mu-2\alpha-4}\\ &= -\left(\frac{\mu-1}{\mu-2\alpha-3}\right)^2-\frac{\mu-1}{(\mu-2\alpha-3)(\mu-2\alpha-4)}+\frac{\mu-1}{\mu-2\alpha-4}\\ &= -\left(\frac{\mu-1}{\mu-2\alpha-3}\right)^2+\frac{\mu-1}{\mu-2\alpha-3}=2\frac{(\alpha+1)(1-\mu)}{(2\alpha+3-\mu)^2}.\end{aligned} \tag{4.17}$$

Using (4.16), (4.17) gives

$$\beta_1 = \frac{\beta_0^3-\beta_0+\gamma_1(-(\mu-2\alpha-6)\beta_0+1)}{(\mu-2\alpha-5)\gamma_1} = \frac{\mu(\mu-2\alpha-4)-(2\alpha+1)}{(2\alpha+3-\mu)(2\alpha+5-\mu)}. \tag{4.18}$$

With β_0, β_1, and γ_1, (4.13) gives

$$\gamma_2 = -\gamma_1+\frac{2(\beta_1^2+\beta_0\beta_1+\beta_0^2-1)-(\mu-2\alpha-4)\beta_1^2+\beta_1-(1-\mu)}{\mu-2\alpha-6} = \frac{2(2\alpha+3-\mu)}{(2\alpha+5-\mu)^2}. \tag{4.19}$$

With β_0, β_1, γ_1, and γ_2, (4.15) and some easy computations

$$\beta_2 = -\frac{\mu(\mu-2\alpha-6)+(2\alpha+1)}{(2\alpha+5-\mu)(2\alpha+7-\mu)}. \tag{4.20}$$

Proposition 4.2. *Assuming*

$$\begin{aligned}\beta_0 &= -\frac{\mu-1}{\mu-2\alpha-3},\\ \beta_{n+1} &= (-1)^n\frac{\mu(\mu-2n-2\alpha-4)+(-1)^{n+1}(2\alpha+1)}{(2n+2\alpha+3-\mu)(2n+2\alpha+5-\mu)},\\ \gamma_{2n+1} &= 2\frac{(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2},\\ \gamma_{2n+2} &= \frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2},\end{aligned} \tag{4.21}$$

for $n \geq 0$ *and assume* $\mu \neq 2n+1$, $\mu \neq 2n+2\alpha+1$, $\alpha \neq -n-1$, $n \geq 0$.

Lemma 4.3. *If* $E_n = \sum_{k=0}^{n}\beta_k$, $n \geq 0$, *then*

$$E_{2n} = -\left(\frac{2n+1-\mu}{4n+2\alpha+3-\mu}\right), \quad E_{2n+1} = -\frac{2n+2}{4n+2\alpha+5-\mu}, \quad n \geq 0. \tag{4.22}$$

Proof. $E_0 = \beta_0$. For $n \geq 0$, we have

$$\begin{aligned}
E_{2n+1} &= \sum_{k=0}^{n} (\beta_{2k} + \beta_{2k+1}) \\
&= \sum_{k=0}^{n} -\frac{\mu(\mu-4k-2\alpha-2)+2\alpha+1}{(4k+2\alpha+1-\mu)(4k+2\alpha+3-\mu)} \\
&\quad + \frac{\mu(\mu-4k-2\alpha-4)-2\alpha-1}{(4k+2\alpha+3-\mu)(4k+2\alpha+5-\mu)} \\
&= \sum_{k=0}^{n} -\frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-1+\mu)} - \frac{1}{2}\frac{\mu+2\alpha+1}{(-4k-2\alpha-3+\mu)} \\
&\quad + \frac{1}{2}\frac{\mu+2\alpha+1}{(-4k-2\alpha-3+\mu)} + \frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-5+\mu)} \qquad (4.23) \\
&= \sum_{k=0}^{n} -\frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-1+\mu)} + \frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-5+\mu)} \\
&= -\frac{1}{2}\frac{\mu-2\alpha-1}{(-2\alpha-1+\mu)} + \frac{1}{2}\frac{\mu-2\alpha-1}{(-4n-2\alpha-5+\mu)} \\
&= -\frac{\mu-2\alpha-1}{2}\left(\frac{1}{-2\alpha-1+\mu} - \frac{1}{-4n-2\alpha-5+\mu}\right) \\
&= -\frac{(\mu-2\alpha-1)(-4n-2\alpha-5+\mu+2\alpha+1-\mu)}{2(-2\alpha-1+\mu)(-4n-2\alpha-5+\mu)} \\
&= -\frac{\mu-2\alpha-1}{2}\left(\frac{-4n-4}{(-2\alpha-1+\mu)(-4n-2\alpha-5+\mu)}\right)
\end{aligned}$$

$$E_{2n+1} = -\frac{2n+2}{(4n+2\alpha+5-\mu)}, \quad \mu \neq 4n+2\alpha+5,\ n \geq 0. \tag{4.24}$$

Calculus of

$$\begin{aligned}
E_{2n+2} &= E_{2n+1} + \beta_{2n+2} \\
&= -\frac{2n+2}{4n+2\alpha+5-\mu} - \frac{\mu(\mu-4n-2\alpha-6)+2\alpha+1}{(4n+2\alpha+5-\mu)(4n+2\alpha+7-\mu)} \\
&= -\frac{1}{4n+2\alpha+5-\mu}\left(2n+2+\frac{\mu(\mu-2\alpha-4n-6)+2\alpha+1}{4n+2\alpha+7-\mu}\right) \\
&= -\frac{1}{4n+2\alpha+5-\mu} \\
&\quad \times \left(\frac{(2n+2)(4n+2\alpha+7)-(2n+2)\mu+\mu(\mu-2\alpha-4n-6)+2\alpha+1}{4n+2\alpha+7-\mu}\right) \\
&= -\frac{1}{4n+2\alpha+5-\mu}\left(\frac{\mu^2-(6n+2\alpha+8)\mu+(2n+2)(4n+2\alpha+7)+2\alpha+1}{4n+2\alpha+7-\mu}\right) \\
&= -\frac{1}{4n+2\alpha+5-\mu}\left(\frac{(4n+2\alpha+5-\mu)(2n+3-\mu)}{4n+2\alpha+7-\mu}\right),
\end{aligned} \tag{4.25}$$

$$E_{2n+2} = -\frac{2n+3-\mu}{4n+2\alpha+7-\mu}, \quad \mu \neq 4n+2\alpha+7,\ n \geq 0. \tag{4.26}$$

□

Proof of Proposition 4.2. Suppose that we have

$$\begin{aligned}
\beta_0 &= -\frac{\mu-1}{\mu-2\alpha-3},\\
\beta_{2k+1} &= \frac{\mu(\mu-4k-2\alpha-4)-(2\alpha+1)}{(4k+2\alpha+3-\mu)(4k+2\alpha+5-\mu)}, \quad 0\le k\le n,\\
\beta_{2k} &= -\frac{\mu(\mu-4k-2\alpha-2)+(2\alpha+1)}{(4k+2\alpha+1-\mu)(4k+2\alpha+3-\mu)}, \quad 1\le k\le n,\\
\gamma_{2k+1} &= 2\frac{(k+\alpha+1)(2k+1-\mu)}{(4k+2\alpha+3-\mu)^2}, \quad 0\le k\le n,\\
\gamma_{2k+2} &= \frac{(2k+2)(2k+2\alpha+3-\mu)}{(4k+2\alpha+5-\mu)^2}, \quad 0\le k\le n-1,
\end{aligned} \tag{4.27}$$

and, using (4.10), (4.13), we prove by induction β_{2n+2}, β_{2n+3}, γ_{2n+2}, and γ_{2n+3}. The substitution $n\to 2n$ in (4.10) gives

$$\begin{aligned}
(\mu-2\alpha-4n-6)\gamma_{2n+2} = {}& 2\gamma_{2n+1}+(\mu-2\alpha-4n)\gamma_{2n}+2\beta_{2n+1}E_{2n}-2\beta_{2n}E_{2n-1}\\
&+(4n-\mu+2\alpha+6)\beta_{2n+1}^2-(4n-\mu+2\alpha+2)\beta_{2n}^2\\
&+(\beta_{2n+1}-\beta_{2n})-2, \quad n\ge 1.
\end{aligned} \tag{4.28}$$

We suppose known γ_{2n+1}, γ_{2n}, β_{2n+1}, β_{2n}, E_{2n}, and E_{2n-1} and then we evaluate γ_{2n+2} for the proof by recurrence; because of cumbersome computation, using Maple. The substitution $n\to 2n+1$ in (4.10) gives (see appendix)

$$\begin{aligned}
(\mu-2\alpha-4n-8)\gamma_{2n+3} = {}& 2\gamma_{2n+2}+(\mu-2\alpha-4n-2)\gamma_{2n+1}+2\beta_{2n+2}E_{2n+1}-2\beta_{2n+1}E_{2n}\\
&+(4n-\mu+2\alpha+8)\beta_{2n+2}^2-(4n-\mu+2\alpha+4)\beta_{2n+1}^2\\
&+(\beta_{2n+2}-\beta_{2n+1})-2, \quad n\ge 0.
\end{aligned} \tag{4.29}$$

The substitution $n\to 2n+1$ in (4.13) gives (see appendix)

$$\begin{aligned}
(\mu-2\alpha-4n-7)\gamma_{2n+2}\beta_{2n+2} = {}& \beta_{2n+1}^3-\beta_{2n+1}+(-\mu+2\alpha+4n+5)\beta_{2n+1}\gamma_{2n+2}\\
&-(-\mu+2\alpha+4n+2)\beta_{2n+1}\gamma_{2n+1}\\
&-(-\mu+2\alpha+4n+1)\beta_{2n}\gamma_{2n+1}\\
&+(2E_{2n}+1)(\gamma_{2n+2}-\gamma_{2n+1}), \quad n\ge 0.
\end{aligned} \tag{4.30}$$

Finally, the substitution $n\to 2n+2$ in (4.13) gives (see appendix)

$$\begin{aligned}
(\mu-2\alpha-4n-5)\gamma_{2n+3}\beta_{2n+3} = {}& \beta_{2n+2}^3-\beta_{2n+2}+(-\mu+2\alpha+4n+10)\beta_{2n+2}\gamma_{2n+3}\\
&-(-\mu+2\alpha+4n+4)\beta_{2n+2}\gamma_{2n+2}\\
&-(-\mu+2\alpha+4n+3)\beta_{2n+1}\gamma_{2n+2}\\
&+(2E_{2n+1}+1)(\gamma_{2n+3}-\gamma_{2n+2}), \quad n\ge 0.
\end{aligned} \tag{4.31}$$

□

Remarks. (1) An homothetie of rapport -1 gives a generalization of $\mathcal{J}(\alpha+1,\alpha)$, with (2.11), (2.13) we have

$$((x^3-x)u)'+((\mu-2\alpha-4)x^2+x-(\mu-1))u=0, \tag{4.32}$$

$$\begin{aligned}
\beta_0 &= \frac{\mu-1}{\mu-2\alpha-3},\\
\beta_{n+1} &= (-1)^{n+1}\frac{\mu(\mu-2n-2\alpha-4)+(-1)^{n+1}(2\alpha+1)}{(2n+2\alpha+3-\mu)(2n+2\alpha+5-\mu)},\\
\gamma_{2n+1} &= 2\frac{(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2},\\
\gamma_{2n+2} &= \frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2},
\end{aligned} \tag{4.33}$$

for $n \ge 0$.

(2) For $\mu = 2\alpha+4$, we have an apparent particular case

$$((x^3-x)u)'+(x-(2\alpha+3))u=0, \tag{4.34}$$

$$\begin{aligned}
\beta_0 &= -(2\alpha+3),\\
\beta_{n+1} &= (-1)^n\frac{(2\alpha+4)(-2n)+(-1)^{n+1}(2\alpha+1)}{(2n-1)(2n+1)},\\
\gamma_{2n+1} &= 2\frac{(n+\alpha+1)(2n-2\alpha-3)}{(4n-1)^2},\\
\gamma_{2n+2} &= \frac{(2n+2)(2n-1)}{(4n+1)^2},
\end{aligned} \tag{4.35}$$

for $n \ge 0$.

5. Appendix. In this appendix, we give both the input and output of the Maple programme used to carry out the computations of Section 4.

```
> restart;
> beta0:=-(mu-1)/(mu-2*alpha-3);
```

$$\beta 0 := -\frac{\mu-1}{\mu-2\alpha-3}$$

```
> gamma1:=factor(simplify(1/(mu-2*alpha-4)*((2*alpha+4-mu)*beta0^2
  +beta0+mu-1)));
```

$$\gamma 1 := -2\frac{(1+\alpha)(\mu-1)}{(\mu-2\alpha-3)^2}$$

```
> beta1:=collect(factor(simplify(1/((mu-2*alpha-5)*gamma1)*(beta0^3
  -beta0+gamma1*(-(mu-2*alpha-6)*beta0+1)))),mu);
  E1:=collect(simplify(beta0+beta1),mu);
```

$$\beta 1 := \frac{\mu^2+(-2\alpha-4)\mu-2\alpha-1}{(\mu-2\alpha-3)(\mu-2\alpha-5)}$$

$$E1 := \frac{2}{\mu-2\alpha-5}$$

```
> gamma2:=factor(simplify(-gamma1+1/(mu-2*alpha-6)*(2*beta1^2
  +2*beta0*beta1+2*beta0^2-2-(mu-2*alpha-4)*beta1^2+beta1-1+mu)));
```

$$\gamma 2 := -2\,\frac{\mu-2\alpha-3}{(\mu-2\alpha-5)^2}$$

```
> beta2:=collect(factor(simplify(1/((mu-2*alpha-4-3)*gamma2)
  *(beta1^3-beta1+(4-mu+2*alpha+4)*beta1*gamma2+(2+mu-2*alpha-4)
  *beta1*gamma1+(3+mu-2*alpha-4)*beta0*gamma1+(2*beta0+1)
  *(gamma2-gamma1)))),mu);E2:=collect(simplify(beta2+E1),mu);
```

$$\beta 2 := -\frac{\mu^2+(-2\alpha-6)\mu+2\alpha+1}{(\mu-2\alpha-5)(\mu-2\alpha-7)}$$

$$E2 := -\frac{\mu-3}{\mu-2\alpha-7}$$

```
> gamma3:=factor(simplify(1/(mu-2*alpha-8)*(2*gamma2+(mu-2*alpha-2)
  *gamma1+2*beta2*E1-2*beta1*beta0+(8-mu+2*alpha)*beta2^2
  -(2*alpha+4-mu)*beta1^2+beta2-beta1-2)));
```

$$\gamma 3 := -2\,\frac{(\alpha+2)(\mu-3)}{(\mu-2\alpha-7)^2}$$

```
> beta3:=collect(factor(simplify(1/((mu-2*alpha-4-5)*gamma3)
  *(beta2^3-beta2+(6-mu+2*alpha+4)*beta2*gamma3+(mu-2*alpha-4)
  *beta2*gamma2+(1+mu-2*alpha-4)*beta1*gamma2+(2*E1+1)
  *(gamma3-gamma2)))),mu);E3:=collect(simplify(beta3+E2),mu);
```

$$\beta 3 := \frac{\mu^2+(-2\alpha-8)\mu-2\alpha-1}{(\mu-2\alpha-7)(\mu-2\alpha-9)}$$

$$E3 := \frac{4}{\mu-2\alpha-9}$$

```
> gamma4:=factor(simplify(1/(mu-2*alpha-10)*(2*gamma3+(mu-2*alpha-4)
  *gamma2+2*beta3*E2-2*beta2*E1+(10-mu+2*alpha)*beta3^2
  -(2*alpha+6-mu)*beta2^2+beta3-beta2-2)));
```

$$\gamma 4 := -4\,\frac{\mu-2\alpha-5}{(\mu-2\alpha-9)^2}$$

```
> beta4:=collect(factor(simplify(1/((mu-2*alpha-4-4*1-3)*gamma4)
  *(beta3^3-beta3+(4*1+4-mu+2*alpha+4)*beta3*gamma4
  +(-4*1+2+mu-2*alpha-4)*beta3*gamma3+(-4*1+3+mu-2*alpha-4)
  *beta2*gamma3+(2*E2+1)*(gamma4-gamma3)))),mu);
```

$$\beta 4 := -\frac{\mu^2+(-10-2\alpha)\mu+2\alpha+1}{(\mu-2\alpha-9)(\mu-2\alpha-11)}$$

```
> gamma2n:=2*n*(2*n+2*alpha+1-mu)/(4*n+2*alpha+1-mu)^2;
```

$$\gamma 2n := 2\,\frac{n(2n+2\alpha+1-\mu)}{(4n+2\alpha+1-\mu)^2}$$

```
> gamma2np1:=2*(n+alpha+1)*(2*n+1-mu)/(4*n+2*alpha+3-mu)^2;
```

$$\gamma 2np1 := 2\,\frac{(n+\alpha+1)\,(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2}$$

```
> beta2n:=-(mu*(mu-4*n-2*alpha-2)+2*alpha+1)/((4*n+2*alpha+1-mu)
  *(4*n+2*alpha+3-mu));
```

$$\beta 2n := -\frac{\mu(\mu-4n-2\alpha-2)+2\alpha+1}{(4n+2\alpha+1-\mu)\,(4n+2\alpha+3-\mu)}$$

```
> convert(beta2n, parfrac,n);
```

$$-1/2\,\frac{-2\alpha-1+\mu}{-4n-2\alpha-1+\mu}-1/2\,\frac{2\alpha+1+\mu}{-4n-2\alpha-3+\mu}$$

```
> beta2np1:=(mu*(mu-4*n-2*alpha-4)-2*alpha-1)/((4*n+2*alpha+3-mu)
  *(4*n+2*alpha+5-mu));
```

$$\beta 2np1 := \frac{\mu(\mu-4n-2\alpha-4)-2\alpha-1}{(4n+2\alpha+3-\mu)\,(4n+2\alpha+5-\mu)}$$

```
> convert(beta2np1, parfrac,n);
```

$$1/2\,\frac{2\alpha+1+\mu}{-4n-2\alpha-3+\mu}+1/2\,\frac{-2\alpha-1+\mu}{-4n-2\alpha-5+\mu}$$

```
> E2n:=-(2*n+1-mu)/(4*n+2*alpha+3-mu);
```

$$E2n := -\frac{2n+1-\mu}{4n+2\alpha+3-\mu}$$

```
> E2np1:=-(2*n+2)/(4*n+2*alpha+5-mu);
  E2nm1:=-(2*n)/(4*n+2*alpha+1-mu);
```

$$E2np1 := -\frac{2n+2}{4n+2\alpha+5-\mu}$$

$$E2nm1 := -2\,\frac{n}{4n+2\alpha+1-\mu}$$

```
> gamma2np2:=factor(simplify(1/(mu-2*alpha-4*n-6)
  *(2*gamma2np1+(mu-2*alpha-4*n)*gamma2n+2*beta2np1*E2n-2
  *beta2n*E2nm1+(4*n+6-mu+2*alpha)*beta2np1^2
  -(4*n+2*alpha+2-mu)*beta2n^2+beta2np1-beta2n-2)));
```

$$\gamma 2np2 := -2\,\frac{(n+1)\,(-2n+\mu-3-2\alpha)}{(-4n-2\alpha-5+\mu)^2}$$

```
> beta2np2:=factor(simplify(1/((mu-2*alpha-4-4*n-3)*gamma2np2)
  *(beta2np1^3-beta2np1+(4*n+4-mu+2*alpha+4)*beta2np1*gamma2np2
  +(-4*n+2+mu-2*alpha-4)*beta2np1*gamma2np1+(-4*n+3+mu-2*alpha-4)
  *beta2n*gamma2np1+(2*E2n+1)*(gamma2np2-gamma2np1))));
```

$$\beta 2np2 := -\frac{\mu^2-2\mu\alpha-4\mu n-6\mu+1+2\alpha}{(-4n-2\alpha-5+\mu)\,(\mu-2\alpha-7-4n)}$$

```
> gamma2np3:=factor(simplify(1/(mu-2*alpha-4*n-8)
  *(2*gamma2np2+(mu-2*alpha-4*n-2)*gamma2np1+2*beta2np2*E2np1-2
  *beta2np1*E2n+(4*n+8-mu+2*alpha)*beta2np2^2-(4*n+2*alpha+4-mu)
  *beta2np1^2+beta2np2-beta2np1-2)));
```

$$\gamma 2np3 := -2\frac{(n+2+\alpha)(\mu-2n-3)}{(\mu-2\alpha-7-4n)^2}$$

```
> beta2np3:=(factor(simplify(1/((mu-2*alpha-4-4*n-5)*gamma2np3)
  *(beta2np2^3-beta2np2+(4*n+6-mu+2*alpha+4)*beta2np2*gamma2np3
  +(-4*n+mu-2*alpha-4)*beta2np2*gamma2np2+(-4*n+1+mu-2*alpha-4)
  *beta2np1*gamma2np2+(2*E2np1+1)*(gamma2np3-gamma2np2)))));
```

$$\beta 2np3 := \frac{\mu-4\mu n-8\mu-2\mu\alpha-2\alpha-1}{(\mu-2\alpha-7-4n)(\mu-2\alpha-9-4n)}$$

References

[1] J. Alaya and P. Maroni, *Symmetric Laguerre-Hahn forms of class* $s = 1$, Integral Transform. Spec. Funct. **4** (1996), no. 4, 301–320. MR 98m:42032. Zbl 865.42021.

[2] M. Bachene, *Les polynômes orthogonaux semi-classiques de classe zéro et de classe un*, Tech. report, Universite de Paris (Pierre et Marie Curie), Paris, France, 1985, Thèse de troisième cycle.

[3] S. Belmehdi, *On semi-classical linear functionals of class* $s = 1$*. Classification and integral representations*, Indag. Math. (N.S.) **3** (1992), no. 3, 253–275. MR 94e:33038. Zbl 783.33003.

[4] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications, vol. 13, Gordon and Breach Science Publishers, New York, London, Paris, 1978. MR 58#1979. Zbl 389.33008.

[5] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda(x-c)^{-1}L$ *[On the sequence of orthogonal polynomials associated with the form* $u = \delta_c + \lambda(x-c)^{-1}L$*]*, Period. Math. Hungar. **21** (1990), no. 3, 223–248 (French). MR 92c:42025. Zbl 732.42015.

[6] ______, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques [An algebraic theory of orthogonal polynomials. Application to semiclassical orthogonal polynomials]*, Orthogonal Polynomials and their Applications (Erice, 1990) (Basel), IMACS Ann. Comput. Appl. Math., vol. 9, Baltzer, 1991, pp. 95–130 (French). MR 95i:42018. Zbl 950.49938.

[7] ______, *Variations around classical orthogonal polynomials. Connected problems*, J. Comput. Appl. Math. **48** (1993), no. 1-2, 133–155, Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA) (Granada, 1991). MR 94k:33013. Zbl 790.33006.

[8] ______, *Modified classical orthogonal polynomials associated with oscillating functions—open problems*, Appl. Numer. Math. **15** (1994), no. 2, 259–283, Innovative methods in numerical analysis (Bressanone, 1992). MR 96d:42037. Zbl 826.42020.

Mohamed Jalel Atia: Faculté des Sciences de Gabès, 6029 Route de Mednine Gabès, Tunisia

E-mail address: jalel.atia@fsg.rnu.tn