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STABLE FINITE ELEMENT METHODS FOR THE STOKES PROBLEM

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ABSTRACT. The mixed finite element scheme of the Stokes problem with pressure stabilization is analyzed for the cross-grid $P_k - P_{k-1}$ elements, $k \geq 1$, using discontinuous pressures. The $P_k^+ - P_{k-1}$ elements are also analyzed. We prove the stability of the scheme using the macroelement technique. The order of convergence follows from the standard theory of mixed methods. The macroelement technique can also be applicable to the stability analysis for some higher order methods using continuous pressures such as Taylor-Hood methods, cross-grid methods, or iso-grid methods.

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1. Introduction. For the finite element approximation of the stationary Stokes equations several approaches appear in the literature [6, 9, 13]. The purpose of this paper is to analyze the mixed finite element scheme with pressure stabilization for some higher order triangular elements.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. We consider the approximation of the stationary Stokes problem: find $\mathbf{u} = (u_1, u_2)$ and p satisfying

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, $\mathbf{f}$ is the given body force per unit mass, and $\nu > 0$ is the viscosity. For the sake of simplicity, we take the viscosity equal to one. With the usual notation (see Section 2 for details) the standard variational formulation of this problem is: find $\mathbf{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla\mathbf{u}, \nabla\mathbf{v}) - (\operatorname{div}\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in H_0^1(\Omega)^2, \\ (\operatorname{div}\mathbf{u}, q) &= 0, \quad q \in L_0^2(\Omega), \end{aligned} \tag{1.2}$$

where $(\cdot,\cdot)$ denote the usual L^2 inner products. For $\mathbf{f} \in H^{-1}(\Omega)^2$ this problem has a unique solution (cf. [9]). The standard mixed method based on (1.2) reads as follows: find $\mathbf{u}_h \in \mathbf{V}_h \subset H_0^1(\Omega)^2$ and $p \in P_h \subset L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla\mathbf{u}_h, \nabla\mathbf{v}) - (\operatorname{div}\mathbf{v}, p_h) &= (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div}\mathbf{u}_h, q) &= 0, \quad q \in P_h. \end{aligned} \tag{1.3}$$

Suppose that the finite element spaces $\mathbf{V}_h$ and P_h indexed by the parameter h, $0<h<1$, satisfy the inf-sup condition or the Babuška-Brezzi stability condition

$$\inf_{0\neq p\in P_h}\sup_{\mathbf{0}\neq \mathbf{v}\in \mathbf{V}_h}\frac{(\operatorname{div}\mathbf{v},p)}{\|\mathbf{v}\|_1\|p\|_0}\geq C, \tag{1.4}$$

where C is a positive constant independent of h. Then the theory of mixed methods states that the system (1.3) has a unique solution $(\mathbf{u}_h,p_h)$ satisfying

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|p-p_h\|_0\leq C\Big(\inf_{\mathbf{v}\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}\|_1+\inf_{q\in P_h}\|p-q\|_0\Big), \tag{1.5}$$

where $(\mathbf{u},p)$ is the solution to (1.2). Introducing the associated bounded bilinear form

$$B(\mathbf{u},p;\mathbf{v},q)=(\nabla\mathbf{u},\nabla\mathbf{v})-(\operatorname{div}\mathbf{v},p)+\epsilon(\operatorname{div}\mathbf{u},q),\quad \epsilon=\pm1, \tag{1.6}$$

and the linear functional

$$L(\mathbf{v},q)=(\mathbf{f},\mathbf{v}), \tag{1.7}$$

we can recast the formulation (1.3) as follows: find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times P_h$ such that

$$B(\mathbf{u}_h,p_h;\mathbf{v},q)=L(\mathbf{v},q),\quad (\mathbf{v},q)\in\mathbf{V}_h\times P_h. \tag{1.8}$$

Then the main result of [1, 2] says that (1.5) holds provided

$$\sup_{\mathbf{0}\neq(\mathbf{v},q)\in\mathbf{V}_h\times P_h}\frac{B(\mathbf{u},p;\mathbf{v},q)}{\|\mathbf{v}\|_1+\|q\|_0}\geq C\big(\|\mathbf{u}\|_1+\|p\|_0\big),\quad (\mathbf{u},p)\in\mathbf{V}_h\times P_h, \tag{1.9}$$

where C is a positive constant independent of h. Equation (1.9) will be referred to the stability condition for a bilinear form B in general.

Since it could be a difficult task to verify (1.4) for a particular choice of velocity and pressure approximations, several methods have been developed aiming at stabilizing the discrete solution. We introduce an approximation scheme with pressure stabilization as follows: find $(\mathbf{u}_h,p_h)\in\mathbf{V}_h\times P_h$ such that

$$B(\mathbf{u}_h,p_h;\mathbf{v},q)=L(\mathbf{v},q),\quad (\mathbf{v},q)\in\mathbf{V}_h\times P_h \tag{1.10}$$

with

$$\begin{aligned}B(\mathbf{u},p;\mathbf{v},q)&=(\nabla\mathbf{u},\nabla\mathbf{v})-(\operatorname{div}\mathbf{v},p)+\epsilon(\operatorname{div}\mathbf{u},q)\\&\quad+\epsilon\beta\sum_{T\in\Gamma_h}h_T([[p]],[[q]])_T,\quad \epsilon=\pm1,\\ L(\mathbf{v},q)&=(\mathbf{f},\mathbf{v}).\end{aligned} \tag{1.11}$$

Note that this method was considered as a special case of the stabilization procedures in [7, 10].

It is the purpose of this paper to show that the mixed finite element method (1.10) with pressure stabilization converges for some higher order elements using discontinuous pressures. More specifically, we establish the convergence of the cross-grid

$P_k - P_{k-1}$ elements, $k \geq 1$ and the $P_k^+ - P_{k-1}$ elements, $k \geq 2$ (see Section 3). To verify the stability condition (1.9), we will combine the ideas of macroelement technique in [15, 16, 17] and the arguments in [8] for Galerkin least squares methods. Then the error estimate (1.5) follows in the usual manner. The macroelement technique can also be applicable to the stability analysis for some higher order methods using continuous pressures, such as Taylor-Hood methods, cross-grid methods, or iso-grid methods.

An outline of the paper is as follows. In Section 2, we develop the stability analysis and the macroelement technique together with the necessary preliminaries. The stability and convergence of various elements mentioned above are shown in Section 3 by an application of the results in Section 2.

2. Macroelement technique and weak stability. Let $\mathcal{C}_h$ be a partitioning of $\bar{\Omega}$ into triangles for a bounded polygonal domain $\Omega \subset \mathbb{R}^2$. The triangulation is assumed to be regular in the usual sense, that is, for some $\sigma > 1$,

$$h_K \leq \sigma \rho_K, \quad K \in \mathcal{C}_h, \tag{2.1}$$

where h_K is the diameter of element K and ρ_K is the diameter of the largest circle contained in K. The mesh parameter h is given by $h = \max(h_K)$ and the set of all interelement boundaries will be denoted by Γ_h. We will not assume $\mathcal{C}_h$ to be quasi-uniform.

The finite element subspaces of $P_k - P_l$ element are

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} = (v_1, v_2) \in H_0^1(\Omega)^2 : v_i|_K \in P_k(K),\ i = 1,2,\ K \in \mathcal{C}_h\}, \\ P_h &= \{p \in L_0^2(\Omega) : p|_K \in P_l(K),\ K \in \mathcal{C}_h\}, \end{aligned} \tag{2.2}$$

where P_s denotes the collection of all polynomials of degree not greater than s and $L_0^2(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions with zero mean value. Our notation is standard. The norms and seminorms in the Sobolev spaces $H^1(\Omega)^2$ are denoted by $\|\cdot\|_1$ and $|\cdot|_1$, respectively.

Given any regular triangulation $\mathcal{C}_h$, by a *macroelement* we now mean a connected set M of adjoining elements K from $\mathcal{C}_h$. Two macroelements M and $\bar{M}$ are said to be *equivalent* if there is a continuous one-to-one and onto mapping $F : M \to \bar{M}$ such that $F|_K$ is affine for each $K \subset M$. For a macroelement M we *define* the spaces $\mathbf{V}_{0,M}$ and P_M consistent with $\mathbf{V}_h$ and P_h:

$$\mathbf{V}_{0,M} = \{\, \mathbf{v} \in H_0^1(M)^2 : \mathbf{v}|_K \in P_k(K)^2,\ K \subset M\}, \tag{2.3}$$

$$P_M = \{p \in L_0^2(M) : p|_K \in P_l(K),\ K \subset M\}. \tag{2.4}$$

Further we *define*

$$N_M = \{p \in P_M : (\mathbf{v}, \nabla_h p)_M = 0,\ \mathbf{v} \in \mathbf{V}_{0,M}\}, \tag{2.5}$$

where $\nabla_h p$ is given by $\nabla p|_K$ on each $K \subset M$. The collection of edges of elements in the interior of M is denoted by Γ_M. The following seminorms defined in P_h turns out to be very useful for the analysis below:

$$|p|_h^2 = \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2, \qquad |[[p]]|_h^2 = \sum_{T \in \Gamma_h} h_T ([[p]], [[p]])_T. \tag{2.6}$$

In P_M we similarly define

$$|p|_M^2 = \sum_{K\subset M} h_K^2 \|\nabla p\|_{0,K}^2, \qquad |[[p]]|_M^2 = \sum_{T\in\Gamma_M} h_T([[p]],[[p]])_T. \tag{2.7}$$

Here, the collection of edges of elements in the interior of M is denoted by Γ_M, $\|\cdot\|_{0,K}$ is the L^2 norm on K, $(\cdot,\cdot)_T$ is the inner product in $L^2(T)$, h_T is the diameter of T, and $[[p]]_T$ is the jump in p along T.

The macroelement technique is based on the macroelement partitioning $\mathcal{M}_h$ satisfying the following conditions:

(M1) there is a fixed set of equivalence classes $\mathcal{D}_i, i = 1,\dots,q$, of macroelements such that each $M \in \mathcal{M}_h$ belongs to one of $\mathcal{D}_i$;

(M2) there is a positive integer L such that each $K \in \mathcal{C}_h$ is contained in at least one and not more than L macroelements of $\mathcal{M}_h$;

(M3) each $M \in \mathcal{D}_i$, $i = 1,\dots,q$, satisfies

(M3a) $p \in N_M$ implies that $|p|_M = 0$.

The usefulness of the macroelement concept and the above mesh-dependent norms is that it enables us to establish some weak stability estimates for the proof of (1.9).

Remark 2.1. We have modified the presentation of Sternberg [16, 17] to deal with the pressure stabilization and discontinuous pressure approximations.

Lemma 2.2. *Let $\mathcal{D}$ be a class of equivalent macroelements. Suppose that (M3a) is valid for every $M \in \mathcal{D}$. Then there is a constant $C > 0$ such that*

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_{0,M}} \frac{(\mathbf{v},\nabla_h p)_M}{|\mathbf{v}|_{1,M}} \geq C|p|_M, \quad p \in P_M \tag{2.8}$$

holds for all $M \in \mathcal{D}$.

Proof. For $M \in \mathcal{D}$, define a scaling invariant

$$\beta_M = \inf_{0\neq p\in P_M} \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_{0,M}} \frac{(\mathbf{v},\nabla_h p)_M}{|\mathbf{v}|_{1,M}|p|_M} \tag{2.9}$$

which is positive from the hypothesis. By virtue of the argument of Sternberg (cf. [15, 17]), the regularity condition (2.1) ensures that there is a constant C such that $\beta_M \geq C > 0$ for all $M \in \mathcal{D}$, which implies (2.8). □

Lemma 2.3. *Suppose that there is a macroelement partitioning $\mathcal{M}_h$ satisfying (M1), (M2), and (M3). Then the weak stability inequality*

$$\sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h} \frac{(\operatorname{div}\mathbf{v},p)}{\|\mathbf{v}\|_1} \geq C_1|p|_h - C_2|[[p]]|_h, \quad p \in P_h \tag{2.10}$$

is valid.

Proof. The local weak stability estimates (2.8) implies that for a given $p \in P_h$ and $M \in \mathcal{M}_h$, there is $\mathbf{v}_M \in \mathbf{V}_h$ with $\mathbf{v}_M = \mathbf{0}$ in $\Omega\backslash M$ such that

$$-(\mathbf{v}_M,\nabla_h p)_M \geq C|p|_M^2, \tag{2.11}$$

$$|\mathbf{v}_M|_1 = |\mathbf{v}_M|_{1,M} \leq |p|_M. \tag{2.12}$$

Then we have

$$(\operatorname{div}\mathbf{v}_M, p)_M = -(\mathbf{v}_M, \nabla_h p)_M + \sum_{T\in\Gamma_M} (\mathbf{v}_M\cdot\mathbf{n}, [[p]])_T, \tag{2.13}$$

$$(\operatorname{div}\mathbf{v}_M, p)_M \geq C|p|_M^2 - \left(\sum_{T\in\Gamma_M} h_T^{-1}\|\mathbf{v}_M\|_{0,T}^2\right)^{1/2} |[[p]]|_M. \tag{2.14}$$

Here $\mathbf{n}$ denotes a unit normal to T. From Lemma 2.4 we have the estimates

$$\sum_{T\in\Gamma_M} h_T^{-1}\|\mathbf{v}_M\|_{0,T}^2 \leq C\sum_{K\subset M} h_K^{-2}\|\mathbf{v}_M\|_{0,K}^2 \leq C'|\mathbf{v}|_{1,M}^2. \tag{2.15}$$

Combining (2.12), (2.14), and (2.15), we obtain

$$(\operatorname{div}\mathbf{v}_M, p)_M \geq C_1|p|_M^2 - C_2|p|_M|[[p]]|_M, \tag{2.16}$$

where $C_1 > 0$ and $C_2 > 0$ can be taken independent of M. Next let us define $\mathbf{v}\in\mathbf{V}_h$ through $\mathbf{v} = \sum_{M\in\mathcal{M}_h}\mathbf{v}_M$. Then the macroelement conditions (M1), (M2), (M3), and (2.16) give

$$\begin{aligned}
(\operatorname{div}\mathbf{v}, p) &= \sum_{M\in\mathcal{M}_h} (\operatorname{div}\mathbf{v}_M, p)_M \\
&\geq C_1\sum_{M\in\mathcal{M}_h}|p|_M^2 - C_2\sum_{M\in\mathcal{M}_h}|p|_M|[[p]]|_M \\
&\geq C_1|p|_h^2 - C_2\left(\sum_{M\in\mathcal{M}_h}|p|_M^2\right)^{1/2}\left(\sum_{M\in\mathcal{M}_h}|[[p]]|_M^2\right)^{1/2} \\
&\geq C_1|p|_h^2 - C_2(\sqrt{L}\,|p|_h)(\sqrt{L}|[[p]]|_h) \\
&= C_1|p|_h^2 - C_2L|p|_h|[[p]]|_h.
\end{aligned} \tag{2.17}$$

Since

$$\begin{aligned}
\|\mathbf{v}\|_1 \leq C|\mathbf{v}|_1 &\leq C\sum_{M\in\mathcal{M}_h}|\mathbf{v}_M|_{1,M} \leq C\sum_{M\in\mathcal{M}_h}|p|_M \\
&\leq CL\sum_{K\in\mathcal{C}_h} h_K\|\nabla p\|_{0,K} \leq CL|\Omega|^{1/2}|p|_h,
\end{aligned} \tag{2.18}$$

it follows from (2.17) that there are constants $C_1 > 0$ and $C_2 > 0$ satisfying (2.10). Here $|\Omega|$ denotes the measure of Ω. □

LEMMA 2.4. *Let M be a macroelement. Then we have for $\mathbf{u}\in\mathbf{V}_h$,*

$$\sum_{T\in\Gamma_M} h_T^{-1}\|\mathbf{u}\|_{0,T}^2 \leq C\sum_{K\subset M} h_K^{-2}\|\mathbf{u}\|_{0,K}^2, \tag{2.19}$$

and for $\mathbf{u}\in\mathbf{V}_{0,M}$,

$$\sum_{K\subset M} h_K^{-2}\|\mathbf{u}\|_{0,K}^2 \leq C|\mathbf{u}|_{1,M}^2, \tag{2.20}$$

where constants $C > 0$ depend only on the regularity constant σ of (2.1).

PROOF. According to the argument of [3, page 1045] it is not difficult to see that for $\mathbf{u} \in \mathbf{V}_h 0$,

$$\sum_{T \in \partial K} h_T^{-1} \|\mathbf{u}\|_{0,T}^2 \le C h_K^{-2} \|\mathbf{u}\|_{0,K}^2. \tag{2.21}$$

Then (2.19) follows immediately. Next, applying the argument of a proof of the inverse inequality for piecewise polynomials (cf. [13, page 195]), we can show that (2.20) holds for $\mathbf{u} \in \mathbf{V}_{0,M}$. □

LEMMA 2.5. *Suppose that either $k \ge 2$ in the definition (2.2) of $\mathbf{V}_h$ or $P_h \subset C(\Omega)$. Then there are two positive constants C_1 and C_2 such that*

$$\sup_{\mathbf{0} \ne \mathbf{v} \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}, p)}{\|\mathbf{v}\|_1} \ge C_1 \|p\|_0 - C_2 |p|_h, \quad p \in P_h. \tag{2.22}$$

PROOF. These are the cases (i) and (ii) of [8, Lemma 3.3]. See [8, pages 1685–1687] for the proof. □

LEMMA 2.6. *Under the assumption of Lemma 2.3 there are two positive constants C_1 and C_2 such that the weak stability inequality*

$$\sup_{\mathbf{0} \ne \mathbf{v} \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}, p)}{\|\mathbf{v}\|_1} \ge C_1 \|p\|_0 - C_2 |[[p]]|_h, \quad p \in P_h \tag{2.23}$$

holds.

PROOF. Equation (2.23) follows from (2.10) and (2.22). To be more precise, let C_1, C_2 and c_1, c_2 be the constants in (2.10) and (2.22), respectively. For $0 < t < 1$ we have

$$\begin{aligned} \sup_{\mathbf{0} \ne \mathbf{v} \in \mathbf{V}_h} \frac{(\operatorname{div} \mathbf{v}, p)}{\|\mathbf{v}\|_1} &\ge (1-t)(C_1 |p|_h - C_2 |[[p]]|_h) + t(c_1 \|p\|_0 - c_2 |p|_h) \\ &\ge t c_1 \|p\|_0 + ((1-t)C_1 - t c_2) |p|_h - (1-t) C_2 |[[p]]|_h. \end{aligned} \tag{2.24}$$

Then (2.23) follows provided $t < C_1 (C_1 + c_2)^{-1}$. □

We are ready to verify the stability condition (1.9) for the method (1.10). We do the case $\epsilon = 1$. The other case $\epsilon = -1$ is similar.

THEOREM 2.7. *Suppose that there is a macroelement partitioning $\mathcal{M}_h$ satisfying (M1), (M2), and (M3) for a regular triangulation $\mathcal{M}_h$ of $\Omega \subset \mathbb{R}^2$. Then given a stabilization parameter $\beta > 0$ the stability condition (1.9) for the method (1.10) is valid.*

PROOF. Let $(\mathbf{u}, p) \in \mathbf{V}_h \times P_h$. First, we note that

$$B(\mathbf{u}, p; \mathbf{u}, p) = \|\nabla \mathbf{u}\|_0^2 + \beta |[[p]]|_h^2 \ge C_1 \|\mathbf{u}\|_1^2 + \beta |[[p]]|_h^2. \tag{2.25}$$

Next, according to (2.23), there is $\mathbf{w} \in \mathbf{V}_h$ satisfying

$$(\operatorname{div} \mathbf{w}, p) \ge C_2 \|p\|_0^2 - C_3 \|p\|_0 |[[p]]|_h \tag{2.26}$$

and $\|\mathbf{w}\|_1 = \|p\|_0$. Then for $t_1 > 0$ and $t_2 > 0$,

$$\begin{aligned} B(\mathbf{u},p;-\mathbf{w},0) &= -(\nabla\mathbf{u},\nabla\mathbf{w}) + (\operatorname{div}\mathbf{w},p) \\ &\geq -\|\mathbf{u}\|_1\|p\|_0 + C_2\|p\|_0^2 - C_3\|p\|_0|[[p]]|_h \\ &\geq \left(C_2 - \frac{t_1}{2} - \frac{C_3 t_2}{2}\right)\|p\|_0^2 - \frac{\|\mathbf{u}\|_1^2}{2t_1} - \frac{C_3|[[p]]|_h^2}{2t_2}. \end{aligned} \tag{2.27}$$

Choosing t_1 and t_2 small enough, we have

$$B(\mathbf{u},p;-\mathbf{w},0) \geq C_4\|p\|_0^2 - C_5\|\mathbf{u}\|_1^2 - C_6|[[p]]|_h^2 \tag{2.28}$$

for some positive constants C_4, C_5, and C_6. Let us denote $(\mathbf{v},q) = (\mathbf{u}-\delta\mathbf{w},p)$. It follows from (2.25) and (2.28) that

$$\begin{aligned} B(\mathbf{u},p;\mathbf{v},q) &= B(\mathbf{u},p;\mathbf{u},p) + \delta B(\mathbf{u},p;-\mathbf{w},0) \\ &\geq \delta C_4\|p\|_0^2 + (C_1 - \delta C_5)\|\mathbf{u}\|_1^2 + (\beta - \delta C_6)\,|[[p]]|_h^2. \end{aligned} \tag{2.29}$$

Choosing $0 < \delta < \min\{C_1C_5^{-1}, \beta C_6^{-1}\}$ we obtain

$$B(\mathbf{u},p;\mathbf{v},q) \geq C_7\left(\|\mathbf{u}\|_1 + \|p\|_0\right)^2 \tag{2.30}$$

for some positive constant C_7. On the other hand, we have

$$\|\mathbf{v}\|_1 + \|q\|_0 \leq C_8\left(\|\mathbf{u}\|_1 + \|p\|_0\right) \tag{2.31}$$

for some positive constant C_8. Finally combining (2.30) and (2.31) we establish the stability condition (1.9) for the method (1.10). □

The error estimates are now obtained in the usual manner from the stability inequality (1.9) and from the following estimates (cf. [8, 9]):

$$\left(\sum_{K\in\mathcal{C}_h} h_K^2\|\nabla q\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T([[q]],[[q]])_T\right)^{1/2} \leq C\|q\|_0, \quad q\in P_h, \tag{2.32}$$

$$\begin{aligned} \inf_{q\in P_h}\left(\sum_{K\in\mathcal{C}_h} h_K^2\|\nabla(q-p)\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T([[q-p]],[[q-p]])_T\right)^{1/2} \\ \leq C\inf_{q\in P_h}\|q-p\|_0 \leq Ch^{l+1}|p|_{l+1}, \quad p\in H^{l+1}(\Omega), \end{aligned} \tag{2.33}$$

$$\inf_{\mathbf{v}\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}\|_1 \leq Ch^k|\mathbf{u}|_{k+1}, \quad \mathbf{u}\in H^{k+1}(\Omega)^2. \tag{2.34}$$

THEOREM 2.8. *Let the assumptions of Theorem 2.7 be valid. Assume further that the solution $(\mathbf{u},p)$ to (1.2) satisfies $\mathbf{u}\in H^{k+1}(\Omega)^2$ and $p\in H^{l+1}(\Omega)$. Then for $\beta > 0$, (1.10) has a unique solution $(\mathbf{u}_h,p_h)$ satisfying (1.5) and*

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|_0 \leq C\left(h^k|\mathbf{u}|_{k+1} + h^{l+1}|p|_{l+1}\right). \tag{2.35}$$

If in addition Ω is a convex polygon, then we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leq C\left(h^{k+1}|\mathbf{u}|_{k+1} + h^{l+2}|p|_{l+1}\right). \tag{2.36}$$

PROOF. We follow the argument of [8, page 1688]. Let $\tilde{\mathbf{u}} \in \mathbf{V}_h$ and $\tilde{p} \in P_h$ be the interpolants of $\mathbf{u}$ and p, respectively. The stability condition (1.9) of B implies the existence of $(\mathbf{v},q) \in \mathbf{V}_h \times P_h$ such that

$$\begin{aligned} \|\mathbf{v}\|_1 + \|q\|_0 &\le C, \\ \|\tilde{\mathbf{u}} - \mathbf{u}_h\|_1 + \|\tilde{p} - p_h\|_0 &\le B(\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p}; \mathbf{v}, q). \end{aligned} \tag{2.37}$$

Since

$$B(\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p}; \mathbf{v}, q) = B(\mathbf{u} - \tilde{\mathbf{u}}, p - \tilde{p}; \mathbf{v}, q), \tag{2.38}$$

$$\begin{aligned} B(\mathbf{u} - \tilde{\mathbf{u}}, p - \tilde{p}; \mathbf{v}, q) \le C &\left(\|\mathbf{u} - \tilde{\mathbf{u}}\|_1^2 + \|p - \tilde{p}\|_0^2 + \sum_{T \in \Gamma_h} h_T ([[p - \tilde{p}]], [[p - \tilde{p}]])_T \right)^{1/2} \\ &\cdot \left(\|\mathbf{v}\|_1^2 + \|q\|_0^2 + \sum_{T \in \Gamma_h} h_T ([[q]], [[q]])_T \right)^{1/2}, \end{aligned} \tag{2.39}$$

we get, from (2.32), (2.33), and (2.37),

$$\|\tilde{\mathbf{u}} - \mathbf{u}_h\|_1 + \|\tilde{p} - p_h\|_0 \le C(\|\mathbf{u} - \tilde{\mathbf{u}}\|_1 + \|p - \tilde{p}\|_0), \tag{2.40}$$

which gives (1.5) with the aid of the triangle inequality. Now (2.35) follows from (1.5), (2.33), and (2.34). Moreover, (2.36) follows from the Aubin-Nitsche argument using the a priori estimate [12],

$$\|\mathbf{u}\|_2 + \|p\|_1 \le C\|\mathbf{f}\|_0 \tag{2.41}$$

for a convex polygon. □

3. Higher order stable elements. In this section, we apply, essentially, Theorems 2.7 and 2.8 for the analysis of several higher order stable elements. We will verify the macroelement conditions (M1), (M2), and (M3) and the approximation properties (2.33) and (2.34) for each method to establish the error estimates (2.35) and (2.36). Our main concern is the verification of the condition (M3), since a construction of macroelement partitioning satisfying (M1) and (M2) is not difficult and the approximation properties (2.33) and (2.34) follow from the standard interpolation theory. For the $P_k^+ - P_{k-1}$ elements, $\mathbf{V}_h$ is enlarged using bubble functions on certain triangles. For the $P_k - P_k$ cross-grid elements or the $P_k - P_k$ iso-grid elements, the pressure triangulation $\mathcal{C}_h'$ or $\tilde{\mathcal{C}}_h$ is coarser than the velocity triangulation $\mathcal{C}_h$. But the results of Section 2 can be interpreted without difficulty.

We begin by recalling that the barycentric coordinates $\lambda_i = \lambda_i(\mathbf{x})$, $1 \le i \le 3$, of $\mathbf{x} = (x, y) \in \mathbb{R}^2$ with respect to the points $A_i = (x_i, y_i)$, $1 \le i \le 3$, which makes a triangle K, are the unique solution of the linear system

$$\sum_{i=1}^{3} \lambda_i A_i = \mathbf{x}, \quad \sum_{i=1}^{3} \lambda_i = 1. \tag{3.1}$$

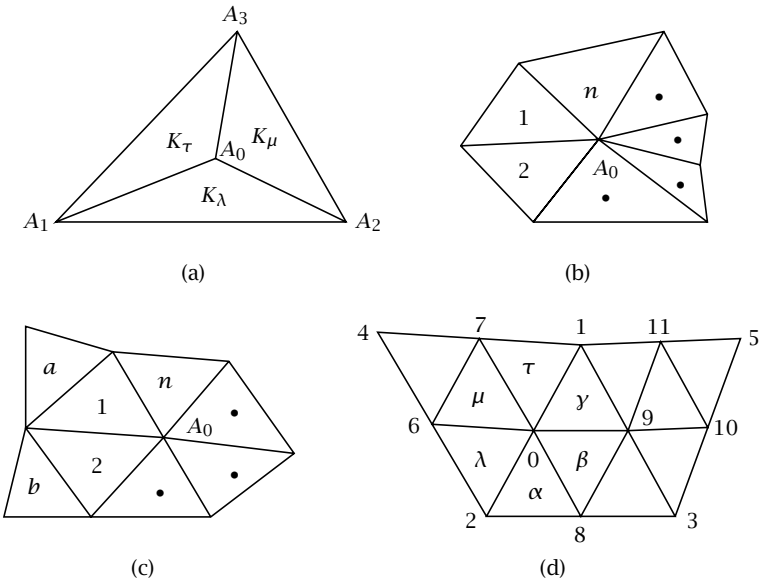

FIGURE 3.1. Examples of macroelement.

It follows that

$$\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} = \frac{1}{\epsilon|K|}\begin{bmatrix} y_2-y_3 & -(x_2-x_3)\\ -(y_1-y_3) & x_1-x_3\end{bmatrix}\begin{bmatrix} x-x_3\\ y-y_3\end{bmatrix}, \tag{3.2}$$

where $|K|$ denotes the measure of K and $\epsilon = \pm 1$ depending on the orientation of A_1, A_2, and A_3. Similar relations hold also for $\begin{bmatrix}\lambda_1\\ \lambda_3\end{bmatrix}$ and $\begin{bmatrix}\lambda_2\\ \lambda_3\end{bmatrix}$. Note that for any nonnegative integers i, j, k,

$$\int_K \lambda_1^i \lambda_2^j \lambda_3^k \, dx = \frac{2i!j!k!}{(i+j+k+2)!}|K|. \tag{3.3}$$

A few examples of macroelement are illustrated in Figure 3.1. For the macroelement in Figure 3.1a we interpret by $\lambda_0, \lambda_1, \lambda_2$ the barycentric coordinates on K_λ with respect to A_0, A_1, and A_2. The similar interpretation of notation will apply for the other figures.

3.1. The cross-grid $P_k - P_{k-1}$ elements, $k \geq 1$, using discontinuous pressures. In the cross-grid methods using discontinuous pressures the triangulation $\mathcal{C}_h$ is obtained from a triangulation $\mathcal{C}'_h$ by dividing each $K' \in \mathcal{C}'_h$ into three triangles inserting an interior vertex A_0 as in Figure 3.1a, where A_0 is not necessarily the center of gravity of K'. For $k \geq 1$ we define $\mathbf{V}_h$ by (2.2) and

$$P_h = \{p \in L_0^2(\Omega) : p|_K \in P_{k-1}(K),\ K \in \mathcal{C}_h\}. \tag{3.4}$$

LEMMA 3.1. *Let M be a macroelement consisting of three triangles aligned as in Figure 3.1a. Define $\mathbf{V}_{0,M}$ by (2.3) and*

$$P_M = \{p \in L_0^2(M) : p|_K \in P_{k-1}(K),\ K \subset M\} \quad \text{for } k \geq 2. \tag{3.5}$$

Then (M3a) is valid.

PROOF. Let $p \in N_M$ and write

$$p_x|_{K_\lambda} = \frac{g(\lambda_0,\lambda_1,\lambda_2)}{|K_\lambda|}. \tag{3.6}$$

Suppose that $p_x|_{K_\mu} = h(\mu_0,\mu_3,\mu_2)/|K_\mu|$. Here λ_i's and μ_j's are the barycentric coordinates of K_λ and K_μ, respectively. Choose $\mathbf{u} = (u_1, 0) \in \mathbf{V}_{h,M}$ such that

$$u_1 = \begin{cases} \lambda_0\lambda_2(g+h)(\lambda_0,\lambda_1,\lambda_2) & \text{in } K_\lambda, \\ \mu_0\mu_2(g+h)(\mu_0,\mu_3,\mu_2) & \text{in } K_\mu, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Then $(\mathbf{u}, \nabla_h p)_M = 0$ gives an equation

$$\left(\lambda_0\lambda_2(g+h)(\lambda_0,\lambda_1,\lambda_2), \frac{g(\lambda_0,\lambda_1,\lambda_2)}{|K_\lambda|}\right)_{K_\lambda} + \left(\mu_0\mu_2(g+h)(\mu_0,\mu_3,\mu_2), \frac{h(\mu_0,\mu_3,\mu_2)}{|K_\mu|}\right)_{K_\mu} = 0. \tag{3.8}$$

On the other hand, it is not difficult to see from (3.3) that

$$\frac{1}{K_\lambda}\int_{K_\lambda} \lambda_0\lambda_2(g^2+gh)(\lambda_0,\lambda_1,\lambda_2)\,dx = \frac{1}{K_\mu}\int_{K_\mu} \mu_0\mu_2(g^2+gh)(\mu_0,\mu_3,\mu_2)\,dx. \tag{3.9}$$

Combining (3.8) and (3.9) we get

$$\frac{1}{K_\mu}\int_{K_\mu} \mu_0\mu_2(g+h)^2(\mu_0,\mu_3,\mu_2)\,dx = 0 \tag{3.10}$$

which implies

$$(g+h)(\mu_0,\mu_3,\mu_2) = 0. \tag{3.11}$$

Then we have

$$p_x|_{K_\mu} = -\frac{g(\mu_0,\mu_3,\mu_2)}{|K_\mu|}. \tag{3.12}$$

By the same argument we obtain

$$p_x|_{K_\tau} = \frac{g(\tau_0,\tau_3,\tau_1)}{|K_\tau|}, \qquad p_x|_{K_\lambda} = -\frac{g(\lambda_0,\lambda_2,\lambda_1)}{|K_\lambda|}. \tag{3.13}$$

Thus g is a polynomial satisfying

$$\begin{aligned} g(\lambda_0,\lambda_1,\lambda_2) &= -g(\lambda_0,\lambda_2,\lambda_1), \\ g(\mu_0,\mu_2,\mu_3) &= -g(\mu_0,\mu_3,\mu_2), \\ g(\tau_0,\tau_3,\tau_1) &= -g(\tau_0,\tau_1,\tau_3). \end{aligned} \tag{3.14}$$

Let us consider the case $k = 4$ first before we turn to the general case $k \geq 1$. From (3.6), (3.14), and the assumption $k = 4$, we can write

$$p_x|_{K_\lambda} = \frac{a}{|K_\lambda|}(\lambda_1^2 - \lambda_2^2) + \frac{b}{|K_\lambda|}(\lambda_1\lambda_0 - \lambda_2\lambda_0) \tag{3.15}$$

with two parameters a and b in $\mathbb{R}$. Similarly, $p_y|_{K_\lambda}$ can be written as the right-hand side of (3.15) with the parameters a and b replaced by a' and b', respectively. Note also that p_x and p_y in K_μ or K_τ can be expressed using a,b and a',b', respectively. Then from (3.6), (3.15), and

$$(p_x)_y = (p_y)_x \tag{3.16}$$

which are valid on each $K \in \mathcal{C}_h$, we get

$$a(\lambda_1, y-\lambda_2, y) = a'(\lambda_1, x-\lambda_2, x), \qquad b(\lambda_1, y-\lambda_2, y) = b'(\lambda_1, x-\lambda_2, x). \tag{3.17}$$

Applying (3.2) and (3.17) in K_λ, K_μ, or K_τ, we find that (a,a') and (b,b') satisfies the homogeneous system in (s,t),

$$s(x_1-x_0)+t(y_1-y_0)=0, \qquad s(x_2-x_0)+t(y_2-y_0)=0, \tag{3.18}$$

of which the solution is trivial since the determinant of the coefficient matrix is equal to $|K_\lambda|/2 > 0$. This implies that $|p|_M = 0$ when $k = 4$. For the general case $k \geq 2$, we can write

$$p_x|_{K_\lambda} = \frac{1}{|K_\lambda|} \sum_{\substack{i+j+l=k-2 \\ i\geq j\geq 0,\ l\geq 0}} a_{i,j,l}(\lambda_1^i\lambda_2^j - \lambda_2^i\lambda_1^j)\lambda_0^l. \tag{3.19}$$

Similarly, $p_y|_{K_\lambda}$ can be written as the right-hand side of (3.19) with the parameters $a_{i,j,l}$ replaced by $a'_{i,j,l}$. Moreover p_x and p_y in K_μ or K_τ can be expressed using $a_{i,j,l}$ and $a'_{i,j,l}$ analogously. Then we find that $(a_{i,j,l}, a'_{i,j,l})$ satisfies (3.18) for each i,j,l. It follows that $a_{i,j,l} = a'_{i,j,k,l} = 0$ for each i,j,l and that $\nabla p|_K = 0$, for all $K \subset M$. This completes the proof. □

Thus we have a nonoverlapping macroelement partitioning, with one class of macroelements equivalent to $K' \in \mathcal{C}'_h$, which satisfies (M1), (M2), and (M3). A careful observation of the analysis of Section 2 also shows that for a nonoverlapping macroelement partitioning the coefficient of $\epsilon\beta$ in the approximation scheme (1.10) can be reduced to

$$\sum_{K'\in\mathcal{C}'_h} \sum_{T\in\Gamma_{K'}} h_T([[p]],[[q]])_T, \tag{3.20}$$

where $\Gamma_{K'}$ denotes the collection of edges of elements of $\mathcal{C}_h$ in the interior of K'. We will call the resulting scheme as the locally stabilized approximation scheme (cf. [11]). Since the approximation properties (2.33) and (2.34) are valid, we obtain the following result for the cross-grid $P_k - P_{k-1}$ elements, $k \geq 1$, using discontinuous pressures.

THEOREM 3.2. *Suppose that $\mathcal{C}_h$, which is obtained from $\mathcal{C}_h'$, is a regular triangulation of Ω. For $k \geq 1$, define $\mathbf{V}_h$ and P_h by (2.2) and (3.4), respectively. Then Theorem 2.8 is valid with $\beta > 0$ for the approximation scheme (1.10) or for the locally stabilized approximation scheme.*

REMARK 3.3. The case $k = 1$ can be considered as a special case of the scheme in [11].

3.2. The $P_k^+ - P_{k-1}$ elements, $k \geq 2$, using discontinuous pressures. The argument of Lemma 3.1 shows that in general a macroelement M of type (b) in Figure 3.1 consisting of n triangles with a common vertex A_0 in the interior of M satisfies (M3a) provided n is odd. When the index n of A_0 is even, we augment bubble functions on a triangle in M in order to verify (M3a). For a regular triangulation $\mathcal{C}_h$ of Ω, we can construct a macroelement partitioning $\mathcal{M}_h$, consisting of macroelements of type (b) and (c) in Figure 3.1. Let $\mathcal{E}$ be a set of triangles in $\mathcal{C}_h$ such that for each macroelement $M \in \mathcal{M}_h$ with an interior vertex of even index there is a triangle $K \in \mathcal{E}$. Then we have the following result for $P_k^+ - P_{k-1}$ element.

THEOREM 3.4. *Suppose that $\mathcal{C}_h$ is a regular triangulation of Ω. Define*

$$\mathbf{V}_h = \{\mathbf{v} \in H_0^1(\Omega)^2 : \mathbf{v}|_K \in P_k(K)^2,\ K \in \mathcal{C}_h;\ \mathbf{v}|_K \in [P_k(K) \oplus \lambda_1\lambda_2\lambda_3 P_{k-2}(K)]^2,\ K \in \mathcal{E}\},$$
$$P_h = \{p \in L_0^2(\Omega) : p|_K \in P_{k-1}(K), K \in \mathcal{C}_h\}. \tag{3.21}$$

Here $\lambda_1, \lambda_2, \lambda_3$ are the barycentric coordinates of the corresponding triangle K. Then Theorem 2.8 is valid for $k \geq 2$.

3.3. The cross-grid $P_k - P_k$ elements, $k \geq 1$. In the cross-grid $P_k - P_k$ elements, $k \geq 1$, the triangulation $\mathcal{C}_h$ for velocity is obtained from the triangulation $\tilde{\mathcal{C}}_h$ for pressure by dividing each $\tilde{K} \in \tilde{\mathcal{C}}_h$ into three triangles inserting an interior vertex A_0 as in Figure 3.1a, where A_0 is not necessarily the center of gravity of $\tilde{K}$.

For $k \geq 1$ we define $\mathbf{V}_h$ by (2.2) and

$$P_h = \{p \in L_0^2(\Omega) : p|_{\tilde{K}} \in P_k(\tilde{K}),\ \tilde{K} \in \tilde{\mathcal{C}}_h\}. \tag{3.22}$$

LEMMA 3.5. *Let M be a macroelement consisting of three triangles aligned as in Figure 3.1a. Define $\mathbf{V}_{0,M}$ by (2.3) and*

$$P_M = \{p \in P_k(M)\} \quad \text{for } k \geq 1. \tag{3.23}$$

Then (M3a) is valid.

PROOF. Let $p \in N_M$ and choose $\mathbf{u} \in \mathbf{V}_{0,M}$ such that

$$\mathbf{u} = \begin{cases} \lambda_0 \nabla p & \text{in } K_\lambda, \\ \mu_0 \nabla p & \text{in } K_\mu, \\ \tau_0 \nabla p & \text{in } K_\tau. \end{cases} \tag{3.24}$$

Then $(\mathbf{u}, \nabla_h p)_M = 0$ gives

$$(\lambda_0 \nabla p, \nabla p)_{K_\lambda} + (\mu_0 \nabla p, \nabla p)_{K_\mu} + (\tau_0 \nabla p, \nabla p)_{K_\tau} = 0 \tag{3.25}$$

which implies $\nabla p|_K = 0$ for all $K \subset M$ and thus $|p|_M = 0$. □

Thus we have a nonoverlapping partitioning, with one class of macroelements equivalent to (a) in Figure 3.1, satisfying (M1), (M2), and (M3). Since the approximation properties (2.33) and (2.34) are valid, we obtain the following result for the cross-grid $P_k - P_k$ elements, $k \geq 1$.

THEOREM 3.6. *Suppose that $\mathcal{C}_h$, which is obtained from $\tilde{\mathcal{C}}_h$, is a regular triangulation of Ω. For $k \geq 1$, define $\mathbf{V}_h$ and P_h by (2.2) and (3.22), respectively. Then Theorem 2.8 is valid with $\beta = 0$.*

3.4. The iso-grid $P_k - P_k$ elements, $k \geq 1$, using continuous pressures. In the $P_k - P_k$ iso-grid elements, $k \geq 1$, using continuous pressures, the triangulation $\mathcal{C}_h$ for velocity is obtained from a triangulation $\tilde{\mathcal{C}}_h$ for pressure by dividing each $\tilde{K} \in \tilde{\mathcal{C}}_h$ into four triangles inserting three vertices, one at each edge of $\tilde{K}$. Each of the inserted vertices is not necessarily a mid-point of the corresponding edge. For $k \geq 1$ we define $\mathbf{V}_h$ by (2.2) and

$$P_h = \{p \in L_0^2(\Omega) \cap C(\Omega) : p|_{\tilde{K}} \in P_k(\tilde{K}),\ \tilde{K} \in \tilde{\mathcal{C}}_h\}. \tag{3.26}$$

LEMMA 3.7. *Let M be a macroelement consisting of twelve triangles aligned as in Figure 3.1d. Define*

$$\begin{aligned} \mathbf{V}_{0,M} &= \{\mathbf{v} \in H_0^1(M)^2 : \mathbf{v}|_K \in P_k(K)^2,\ K \in \mathcal{C}_h \cap M\}, \\ P_M &= \{p \in C(M) : p|_{\tilde{K}} \in P_k(\tilde{K}),\ \tilde{K} \in \tilde{\mathcal{C}}_h \cap M\} \quad \text{for } k \geq 1. \end{aligned} \tag{3.27}$$

Then (M3a) is valid.

PROOF. Let $p \in N_M$ and define

$$g = \begin{cases} \tau_0 & \text{in } K_{017}, \\ \mu_0 & \text{in } K_{076}, \\ \lambda_0 & \text{in } K_{062}, \\ \alpha_0 & \text{in } K_{028}, \\ \beta_0 & \text{in } K_{089}, \\ \gamma_0 & \text{in } K_{091}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.28}$$

where K_{ijk} denotes the triangle with vertices A_i, A_j, and A_k. Let $\mathbf{e} = (e_1, e_2)$ be the unit vector in the direction of $\overrightarrow{A_1A_2}$, and choose $\mathbf{u} = (e_1 g p_{\mathbf{e}}, e_2 g p_{\mathbf{e}}) \in \mathbf{V}_{0,M}$. Note that $p_{\mathbf{e}} = \nabla p \cdot \mathbf{e}$ is continuous on $K_{123} \cup K_{142}$. Then $(\mathbf{u}, \nabla_h p)_M = 0$ gives

$$(\tau_0 p_{\mathbf{e}}, p_{\mathbf{e}})_{K_{017}} + (\mu_0 p_{\mathbf{e}}, p_{\mathbf{e}})_{K_{076}} + \cdots + (\gamma_0 p_{\mathbf{e}}, p_{\mathbf{e}})_{K_{091}} = 0 \tag{3.29}$$

which implies $p_{\mathbf{e}}|_{K_{017}} = 0, \ldots, p_{\mathbf{e}}|_{K_{091}} = 0$. The similar argument gives also $p_{\mathbf{d}}|_{K_{091}} = 0$, where $\mathbf{d}$ is the unit vector in the direction of $\overrightarrow{A_2A_4}$. Since $\mathbf{e}$ and $\mathbf{d}$ are linearly independent, we get $\nabla p|_{K_{091}} = 0$ and so $\nabla p|_{K_{123}} = 0$. Next consider $\mathbf{u} = (e_2 g p_{\mathbf{e}}^{\perp}, -e_1 g p_{\mathbf{e}}^{\perp}) \in \mathbf{V}_{0,M}$, where $p_{\mathbf{e}}^{\perp} \in P_{k-1}(\mathbb{R}^2)$ is the extension of $(e_2 p_x - e_1 p_y)|_{K_{142}}$. Then $(\mathbf{u}, \nabla_h p)_M = 0$ gives

$$(\tau_0 p_{\mathbf{e}}^{\perp}, p_{\mathbf{e}}^{\perp})_{K_{017}} + (\mu_0 p_{\mathbf{e}}^{\perp}, p_{\mathbf{e}}^{\perp})_{K_{076}} + (\lambda_0 p_{\mathbf{e}}^{\perp}, p_{\mathbf{e}}^{\perp})_{K_{062}} = 0 \tag{3.30}$$

which implies $p_{\mathbf{e}}^{\perp}|_{K_{062}} = 0$. Since (e_1, e_2) and $(e_2, -e_1)$ are orthogonal, $p_{\mathbf{e}}|_{K_{062}} = p_{\mathbf{e}}^{\perp}|_{K_{062}} = 0$ implies $\nabla p|_{K_{062}} = 0$ and so $\nabla p|_{K_{142}} = 0$. The analogous argument also gives $\nabla p|_{K_{135}} = 0$. It follows that $|p|_M = 0$. □

Now it is easy to construct a macroelement partitioning, with only one class of macroelements equivalent to (d) in Figure 3.1, satisfying (M1), (M2), and (M3). Since the approximation properties (2.33) and (2.34) are valid, we have the following result for the iso-grid $P_k - P_k$ elements, $k \geq 1$, using continuous pressures.

THEOREM 3.8. *Suppose that $\mathcal{C}_h$, which is obtained from $\tilde{\mathcal{C}}_h$, is a regular triangulation of Ω. For $k \geq 1$, define $\mathbf{V}_h$ and P_h by (2.2) and (3.26), respectively. Then Theorem 2.8 is valid with $\beta = 0$.*

3.5. The $P_k - P_{k-1}$ Taylor-Hood elements, $k \geq 2$

LEMMA 3.9. *Let M be a macroelement consisting of three triangles aligned as in Figure 3.1a. Define $\mathbf{V}_{0,M}$ by (2.3) and*

$$P_M = \{p \in C(M) : p|_K \in P_{k-1}(K),\ K \subset M\} \quad \text{for } k \geq 2. \tag{3.31}$$

Then (M3a) is valid.

PROOF. Let $\mathbf{e} = (e_1, e_2)$ be the unit vector in the direction of $\overrightarrow{A_0A_2}$. Choose $\mathbf{u} = (u_1, u_2) \in \mathbf{V}_{0,M}$ such that

$$u_1 = \begin{cases} e_1\lambda_0\lambda_2 p_{\mathbf{e}} & \text{in } K_\lambda, \\ e_1\mu_0\mu_2 p_{\mathbf{e}} & \text{in } K_\mu, \\ 0 & \text{in } K_\tau, \end{cases} \qquad u_2 = \begin{cases} e_2\lambda_0\lambda_2 p_{\mathbf{e}} & \text{in } K_\lambda, \\ e_2\mu_0\mu_2 p_{\mathbf{e}} & \text{in } K_\mu, \\ 0 & \text{in } K_\tau, \end{cases} \tag{3.32}$$

where $p_{\mathbf{e}} = \nabla p \cdot \mathbf{e}$. Let $p \in N_M$. Then $(\mathbf{u}, \nabla_h p)_M = 0$ gives

$$(\lambda_0\lambda_2 p_{\mathbf{e}}, p_{\mathbf{e}})_{K_\lambda} + (\mu_0\mu_2 p_{\mathbf{e}}, p_{\mathbf{e}})_{K_\mu} = 0 \tag{3.33}$$

which implies $p_{\mathbf{e}}|_{K_\lambda} = 0$ and $p_{\mathbf{e}}|_{K_\mu} = 0$. Similarly, we have $p_{\mathbf{d}}|_{K_\lambda} = 0$ and $p_{\mathbf{d}}|_{K_3} = 0$, where $\mathbf{d}$ is the unit vector in the direction of $\overrightarrow{A_0A_3}$. Since $\mathbf{e}$ and $\mathbf{d}$ are linearly independent, we have $\nabla p = 0$ in K_μ. By the same reasoning we also have $\nabla p = 0$ in K_λ or K_τ. It follows that $|p|_M = 0$. □

Now it is not difficult to construct a macroelement partitioning $\mathcal{M}_h$, consisting of macroelements of types (b) and (c) in Figure 3.1, satisfying (M1), (M2), and (M3). Since the approximation properties (2.33) and (2.34) are valid, we get the following result for $P_k - P_{k-1}$ Taylor-Hood elements, $k \geq 2$.

THEOREM 3.10. *Suppose that $\mathcal{C}_h$ is a regular triangulation of Ω. For $k \geq 2$, define $\mathbf{V}_h$ by (2.2) and*

$$P_h = \{p \in L_0^2(\Omega) \cap C(\Omega) : p|_K \in P_{k-1}(K),\ K \in \mathcal{C}_h\}. \tag{3.34}$$

Then Theorem 2.8 is valid with $\beta = 0$.

REMARK 3.11. It should be pointed out that some restrictions (cf. [5]) inherited from the result of Scott and Vogelius [14] are removed here. See [4] for a different proof.

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