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ON COMPLETE CONVERGENCE FOR L^p-MIXINGALES

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ABSTRACT. We provide in this paper sufficient conditions for the complete convergence for the partial sums and the random selected partial sums of **B**-valued L^p-mixingales.

Keywords and phrases. Complete convergence, p-smooth Banach space, L^p-mixingale.

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1. Introduction and results. McLeish [7] introduced first the concept of mixingales, a generalization of the concepts of mixing sequences and martingale differences, where the mixingale convergence theorems and the strong laws of large numbers have been proved. Furthermore, McLeish [6, 8] studied the invariance principles for mixingales. Yin [9] generalized McLeish's concept of mixingales to operator-valued mixingales, and proved the operator-valued mixingale convergence theorems. Hall and Heyde [2] also pointed out that mixingales include martingale differences, lacunary functions, linear processes, and uniformly mixing processes (also called Φ-mixing).

On the other hand, up till now, there have been an extensive literatures in complete convergence for independent and dependent random sequences (especially, martingale differences and various mixing sequences), see partially the references here. However, there are few papers reported on the complete convergence for mixingales; see, for example, Liang and Ren [5].

Preceding observations stir us to investigate the complete convergence for mixingales. In the present paper, we first generalize slightly McLeish's definition of mixingales to **B**-valued L^p-mixingales, and then give some general results about complete convergence for **B**-valued L^p-mixingales.

Next, we introduce some notations. Let $(\mathbf{B}, \|\cdot\|)$ be a Banach space. **B** is said to be q-smooth $(1 \le q \le 2)$ if there exists a constant $C_q > 0$ such that for every **B**-valued L^q-integrable martingale difference sequence $\{D_i; i \ge 1\}$,

$$E\left\|\sum_{i=1}^{n} D_i\right\|^q \le C_q \sum_{i=1}^{n} E\|D_i\|^q, \quad n \ge 1. \tag{1.1}$$

Let $\{X_n;\ n \ge 1\}$ be **B**-valued L^p-integrable $(1 \le p \le 2)$ random variables on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_n; -\infty < n < \infty\}$ be an increasing sequence of sub σ-fields of $\mathcal{F}$. Then $\{X_n, \mathcal{F}_n\}$ is called a L^p-mixingale if there exist sequences of nonnegative constants C_n and $\psi(n)$, where $\psi(m) \downarrow 0$ as $m \to \infty$, which satisfy the following properties:

(i) $\|E(X_n \mid \mathcal{F}_{n-m})\|_p \le \psi(m)C_n$;

(ii) $\|X_n - E(X_n \mid \mathcal{F}_{n+m})\|_p \le \psi(m+1)C_n$, for all $n \ge 1$ and $m \ge 0$, where $\|X\|_p = (E\|X\|^p)^{1/p}$.

Let $\{X_n; n \ge 1\}$ be **B**-valued random variables, and X_0 be a real nonnegative random variable. We call that $\{X_n\}$ is bounded in probability by X_0 (abbreviated $\{X_n\} \prec X_0$) if

$$P(\|X_n\| > t) \le P(X_0 > t) \quad \forall t > 0. \tag{1.2}$$

Given a positive function $l(x)$ defined on $(0, +\infty)$, we say that $l(x)$ is a slowly variable function as $x \to \infty$, if for all $c > 0$,

$$\lim_{x \to +\infty} \frac{l(cx)}{l(x)} = 1, \tag{1.3}$$

see also Laha and Rohatgi [4].

From now on, we use C to denote finite positive constants whose value may change from statement to statement. For real numbers x, y, $[x]$ denotes the largest integer $k \le x$, and $x \wedge y$ means $\min(x, y)$.

The following are the main results of this paper.

Theorem 1.1. *Let $1 \le t < q \le 2$, $0 < \delta < 1 \wedge 3(q/t - 1)$, $1 \le p \le 2$. **B** is a q-smooth Banach space. Suppose $\{X_n, \mathcal{F}_n\}$ is a **B**-valued L^p-mixingale, and X is a real nonnegative random variable satisfying $\{X_n\} \prec X$. Suppose $l(x)$ is an increasing slowly variable function as $x \to \infty$. If $E(X^{t+\delta} l(X^{t+\delta})) < \infty$ and there exist λ $(1 \le \lambda \le p)$ and $\beta > 0$ such that $t + (1-t)\lambda > 0$, $\beta < \min(\delta/2t,\ \delta/(t+qt),\ (q-t)/(t+qt))$, and*

$$\sum_{n=1}^{\infty} \psi^{\lambda}([n^{\beta}]) \max_{1 \le i \le n} C_i^{\lambda} < \infty, \tag{1.4}$$

then

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|S_n\| \ge n^{1/t}\varepsilon) < \infty \tag{1.5}$$

for $\varepsilon > 0$, where $S_n = \sum_{i=1}^{n} X_i$.

Theorem 1.2. *Let $1 \le t < q \le 2$, $0 < \delta < 1 \wedge 3(q/t - 1)$, and $1 \le p \le 2$. **B** is a q-smooth Banach space. Suppose $\{X_n, \mathcal{F}_n\}$ is a **B**-valued L^p-mixingale, and X is a real nonnegative random variable satisfying $\{X_n\} \prec X$. Suppose $l(x)$ is an increasing slowly variable function as $x \to \infty$. If $E(X^{t+\delta} l(X^{t+\delta}) \ln^+ X) < \infty$ and there are λ $(1 \le \lambda \le p)$ and $\beta > 0$ satisfying $\beta < \min(\delta/2t,\ \delta/(t+qt), (q-t)/(t+qt))$ and*

$$\sum_{j=1}^{\infty} j 2^{(1+2\lambda)j} \psi^{\lambda}([2^{\beta j}]) \max_{1 \le i \le 2^{j+1}} C_i^{\lambda} < \infty, \tag{1.6}$$

then

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P\left(\sup_{k \ge n} \left\| \frac{S_k}{k^{1/t}} \right\| \ge \varepsilon\right) < \infty \tag{1.7}$$

for $\varepsilon > 0$, where $S_n = \sum_{i=1}^{n} X_i$.

Based on Theorem 1.2, we can now obtain the analogue to random selected partial sums of L^p-mixingales.

Theorem 1.3. *Let $1 < t < q \le 2$, $0 < \delta < 1 \wedge 3(q/t - 1)$, $1 \le p \le 2$. $\mathbf{B}$ is a q-smooth Banach space. Suppose $\{X_n, \mathcal{F}_n\}$ is a $\mathbf{B}$-valued L^p-mixingale and X is a real nonnegative random variable satisfying $\{X_n\} \prec X$. Suppose $\{\nu_n; n \ge 1\}$ are random variables which only take positive integer values and are defined on the same probability space as $\{X_n\}$. Suppose $l(x)$ is an increasing slowly variable function as $x \to \infty$. If $E(X^{t+\delta} l(X^{t+\delta}) \ln^+ X) < \infty$, and there exist positive constants λ, β, and η such that $\beta < \min(\delta/2t, \delta/(t+qt), (q-t)/(t+qt))$,*

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P\left(\frac{\nu_n}{n} < \eta\right) < \infty, \tag{1.8}$$

$$\sum_{j=1}^{\infty} j 2^{(1+2\lambda)j} \psi^{\lambda}([2^{\beta j}]) \max_{1 \le i \le 2^{j+1}} C_i^{\lambda} < \infty, \tag{1.9}$$

then for $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P\left(\left\|\sum_{i=1}^{\nu_n} X_i\right\| \ge \nu_n^{1/t} \cdot \varepsilon\right) < \infty. \tag{1.10}$$

Remark 1.4. To our best knowledge, even if $\mathbf{B} = \mathbb{R}$ (the real numbers), the results here are new. Furthermore, conditions (1.4), (1.6), and (1.9) are reasonable. For this purpose, we now particularize the general situation as follows. Let $\mathbf{B} = \mathbb{R}$. In return, $q = 2$. Let $t = 1$, $p = 2$ and $\{X_n, \mathcal{F}_n\}$ be a L^2-mixingale (coinciding with mixingale of McLeish [7] or Hall and Heyde [2]). Consequently, $1 \wedge 3(q/t - 1) = 1$. Given $0 < \delta < 1$, then $\min(\delta/2t,\ \delta/(t+qt),\ (q-t)/(t+qt)) = \min(\delta/2,\ \delta/3,\ 1/3) = \delta/3$.

Suppose that $\{C_i;\ i \ge 1\}$ is bounded or $\sum_{i=1}^{\infty} C_i < \infty$ and that $\psi(m) = o(m^{-\theta})$ for some constant θ satisfying $\theta \cdot \delta > 3$, then condition (1.4) is satisfied with $\lambda = 1$ and each $\beta \in (1/\theta,\ \delta/3)$, as can be easily verified. Moreover, in addition to the above assumptions, suppose that $\psi(m) = o(m^{-\theta})$ for some constant θ satisfying $\theta \cdot \delta > 9$, then conditions (1.6) and (1.9) are satisfied with $\lambda = 1$ and each $\beta \in (3/\theta,\ \delta/3)$.

On the other hand, condition $\psi(m) = o(m^{-\theta})$ is implied with summability conditions such as $\sum_{m=1}^{\infty} \psi^{1/\theta}(m) < \infty$ (see also [6, 7]).

Remark 1.5. In general, if $\{C_i;\ i \ge 1\}$ is bounded or $\sum_{i=1}^{\infty} C_i^{\lambda} < \infty$ for some $1 \le \lambda \le p$ satisfying $t + (1-t)\lambda > 0$, and $\psi(m) = o(m^{-\theta})$ for some sufficiently large θ satisfying $(1+2\lambda)/\theta\lambda < \min(\delta/2t,\ \delta/(t+qt),\ (q-t)/(t+qt))$ then conditions (1.4), (1.6), and (1.9) are satisfied with the above λ and each $\beta \in ((1+2\lambda)/\theta\lambda,\ \min(\delta/2t,\ \delta/(t+qt),\ (q-t)/(t+qt)))$. Meanwhile, condition $\psi(m) = o(m^{-\theta})$ can also be implied with summability conditions such as $\sum_{m=1}^{\infty} \psi^{1/\theta}(m) < \infty$ (see also [6, 7]).

In any case, roughly speaking, conditions such as $\{C_i;\ i \ge 1\}$ is bounded or $\sum_{i=1}^{\infty} C_i^{\lambda} < \infty$ for some $1 \le \lambda \le p$ satisfying $t + (1-t)\lambda > 0$, plus a specific rate of convergence of $\psi(m)$ to 0, ensure conditions (1.4), (1.6), and (1.9).

Remark 1.6. Condition (1.8) is just one which is usually employed in literatures.

2. Proofs of the main results. For the sake of convenience, we begin with two lemmas, which will be needed below.

Lemma 2.1. *Suppose that $l(x)$ is a slowly variable function as $x \to \infty$, then we have*

(1) $\lim_{x\to\infty} l(x+u)/l(x) = 1,\ \forall u > 0$;
(2) $\lim_{k\to+\infty} \sup_{2^k \le x \le 2^{k+1}} l(x)/l(2^k) = 1$;
(3) $\lim_{x\to+\infty} x^{\delta} l(x) = +\infty,\ \lim_{x\to+\infty} x^{-\delta} l(x) = 0,\ \forall \delta > 0$;
(4) $C\cdot 2^{kr} l(\eta\cdot 2^k) \le \sum_{j=1}^{k} 2^{jr} l(\eta\cdot 2^j) \le C\cdot 2^{kr} l(\eta\cdot 2^k)$ *for every positive r, η and integer k;*
(5) $C\cdot 2^{kr} l(\eta\cdot 2^k) \le \sum_{j=k}^{\infty} 2^{jr} l(\eta\cdot 2^j) \le C\cdot 2^{kr} l(\eta\cdot 2^k)$, $C\cdot k2^{kr} l(2^k) \le \sum_{j=k}^{\infty} j2^{jr} l(2^j) \le C\cdot k2^{kr} l(2^k)$ *for every $r < 0,\ \eta > 0$ and integer k.*

We refer to Bai and Su [1] and Hu [3] for a proof of Lemma 2.1.

By applying integration by parts, it is easy to prove the following lemma.

Lemma 2.2. *Let X be a real random variable, then*

$$
\begin{aligned}
E\big(|X|^r I(|X| \le a)\big) &\le r\int_0^a t^{(r-1)} P(|X| > t)\,dt,\\
E\big(|X| I(|X| > a)\big) &= aP(|X| > a) + \int_a^{\infty} P(|X| > t)\,dt
\end{aligned}
\tag{2.1}
$$

for $r \ge 1,\ a > 0$, where and elsewhere $I(|X| \le a)$ means the indicator of $\{|X| \le a\}$.

Proof of Theorem 1.1. We write $\alpha = 1/t$. Notice first that

$$
\begin{aligned}
S_n = {} & \sum_{i=1}^{n} \Big(X_i - E\Big(X_i \mid \mathcal{F}_{i+[n^{\beta}]}\Big)\Big) + \sum_{i=1}^{n} \Big(E\Big(X_i \mid \mathcal{F}_{i+[n^{\beta}]}\Big) - E\Big(X_i \mid \mathcal{F}_{i+[n^{\beta}]-1}\Big)\Big)\\
& + \cdots + \sum_{i=1}^{n} \Big(E\Big(X_i \mid \mathcal{F}_{i-[n^{\beta}]+1}\Big) - E\Big(X_i \mid \mathcal{F}_{i-[n^{\beta}]}\Big)\Big) + \sum_{i=1}^{n} E\Big(X_i \mid \mathcal{F}_{i-[n^{\beta}]}\Big).
\end{aligned}
\tag{2.2}
$$

By denoting

$$
\begin{aligned}
\text{Part}\,1 &= \sum_{i=1}^{n} \Big(X_i - E\Big(X_i \mid \mathcal{F}_{i+[n^{\beta}]}\Big)\Big),\\
\text{Part}\,2 &= \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} \sum_{i=1}^{n} \big(E(X_i \mid \mathcal{F}_{i+l}) - E(X_i \mid \mathcal{F}_{i+l-1})\big),\\
\text{Part}\,3 &= \sum_{i=1}^{n} E\Big(X_i \mid \mathcal{F}_{i-[n^{\beta}]}\Big),
\end{aligned}
\tag{2.3}
$$

it is sufficient for us to prove

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|\text{Part}\,1\| \geq n^{\alpha}\varepsilon) < \infty, \tag{2.4}$$

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|\text{Part}\,2\| \geq n^{\alpha}\varepsilon) < \infty, \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|\text{Part}\,3\| \geq n^{\alpha}\varepsilon) < \infty. \tag{2.6}$$

By Chebyshev inequality, C_r-inequality, Lemma 2.1, and L^p-mixingale property we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|\text{Part}\,1\| \geq n^{\alpha}\varepsilon) &\leq C \cdot \sum_{n=1}^{\infty} l(n) n^{-(1+\alpha\lambda)} \left\{ \sum_{i=1}^{n} \left\| X_i - E\left(X_i \mid \mathcal{F}_{i+[n^{\beta}]}\right)\right\|_{\lambda} \right\}^{\lambda} \\ &\leq C \cdot \sum_{n=1}^{\infty} l(n) n^{\lambda-1-\alpha\lambda} \psi^{\lambda}([n^{\beta}]) \max_{1\leq i\leq n} C_i^{\lambda} \\ &\leq C \cdot \sum_{n=1}^{\infty} \psi^{\lambda}([n^{\beta}]) \max_{1\leq i\leq n} C_i^{\lambda} < \infty,\end{aligned} \tag{2.7}$$

which proves (2.4).

Similarly, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|\text{Part}\,3\| \geq n^{\alpha}\varepsilon) &\leq C \cdot \sum_{n=1}^{\infty} l(n) n^{-1-\alpha\lambda} \left\{ \sum_{i=1}^{n} \left\| E\left(X_i \mid \mathcal{F}_{i-[n^{\beta}]}\right)\right\|_{\lambda} \right\}^{\lambda} \\ &\leq C \cdot \sum_{n=1}^{\infty} l(n) n^{\lambda-1-\alpha\lambda} \psi^{\lambda}([n^{\beta}]) \max_{1\leq i\leq n} C_i^{\lambda} \\ &\leq C \cdot \sum_{n=1}^{\infty} \psi^{\lambda}([n^{\beta}]) \max_{1\leq i\leq n} C_i^{\lambda} < \infty,\end{aligned} \tag{2.8}$$

which is exactly (2.6).

To prove (2.5), let $Y_{i,n} = X_i I(\|X_i\| \leq n^{\alpha})$, $Z_{i,n} = X_i - Y_{i,n}$, $W_{l,i} = E(X_i \mid \mathcal{F}_{i+l}) - E(X_i \mid \mathcal{F}_{i+l-1})$, $U_{l,i} = E(Y_{i,n} \mid \mathcal{F}_{i+l}) - E(Y_{i,n} \mid \mathcal{F}_{i+l-1})$, $V_{l,i} = E(Z_{i,n} \mid \mathcal{F}_{i+l}) - E(Z_{i,n} \mid \mathcal{F}_{i+l-1})$, $n \geq 1$, $1 \leq i \leq n$, $-[n^{\beta}]+1 \leq l \leq [n^{\beta}]$. Clearly, $X_i = Y_{i,n} + Z_{i,n}$, $W_{l,i} = U_{l,i} + V_{l,i}$. For fixed l, $\{U_{l,i}, \mathcal{F}_{i+l}, 1 \leq i \leq n\}$ and $\{V_{l,i}, \mathcal{F}_{i+l}, 1 \leq i \leq n\}$ are martingale difference sequences. Hence,

$$\begin{aligned}\text{the left hand side (LHS) of (2.5)} &\leq \sum_{n=1}^{\infty} \frac{l(n)}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} P\left(\left\| \sum_{i=1}^{n} W_{l,i} \right\| \geq n^{\alpha-\beta} \cdot \frac{\varepsilon}{2} \right) \\ &\leq \sum_{n=1}^{\infty} \frac{l(n)}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} P\left(\left\| \sum_{i=1}^{n} U_{l,i} \right\| \geq n^{\alpha-\beta} \cdot \frac{\varepsilon}{4} \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{l(n)}{n} \sum_{l=-[n^{\beta}]+1}^{[n^{\beta}]} P\left(\left\| \sum_{i=1}^{n} V_{l,i} \right\| \geq n^{\alpha-\beta} \cdot \frac{\varepsilon}{4} \right) \\ &= \text{Part}\,4 + \text{Part}\,5.\end{aligned} \tag{2.9}$$

By Kolmogorov inequality and Lemma 2.2,

$$\begin{aligned}
\text{Part}\,4 &\le C\cdot\sum_{n=1}^{\infty}\frac{l(n)}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]} n^{(\beta-\alpha)q}\sum_{i=1}^{n}E\|Y_{i,n}\|^q\\
&= C\cdot\sum_{n=1}^{\infty}\frac{l(n)}{n}[n^\beta]\cdot n^{(\beta-\alpha)q}\sum_{i=1}^{n}\alpha q\cdot\int_0^n s^{\alpha q-1}P(\|X_i\|^t>s)\,ds\\
&\le C\cdot\sum_{n=1}^{\infty}l(n)n^{(\beta-\alpha)q+\beta}\int_0^n s^{\alpha q-1}P(X^t>s)\,ds\\
&\le C\cdot\sum_{j=1}^{\infty}2^{(1+\beta+\beta q-\alpha q)j}l(2^j)\int_0^{2^j}s^{\alpha q-1}P(X^t>s)\,ds.
\end{aligned}\tag{2.10}$$

Observing

$$\int_0^{2^j}s^{\alpha q-1}P(X^t>s)\,ds=\sum_{k=1}^{j}\int_{2^{k-1}}^{2^k}s^{\alpha q-1}P(X^t>s)\,ds+\int_0^1 s^{\alpha q-1}P(X^t>s)\,ds \tag{2.11}$$

$$\le C+\sum_{k=1}^{j}2^{\alpha qk}P(X^t>2^{k-1}), \tag{2.12}$$

we get

$$\text{Part}\,4\le C\cdot\sum_{j=1}^{\infty}2^{(1+\beta+\beta q-\alpha q)j}l(2^j)+C\cdot\sum_{j=1}^{\infty}2^{(1+\beta+\beta q-\alpha q)j}l(2^j)\sum_{k=1}^{j}2^{\alpha qk}P(X^t>2^{k-1}). \tag{2.13}$$

Since $1+\beta+\beta q-\alpha q<0$, from Lemma 2.1(5) we know that

$$\sum_{j=1}^{\infty}2^{(1+\beta+\beta q-\alpha q)j}l(2^j)<\infty. \tag{2.14}$$

Now from Lemma 2.1(5) again, it follows that

$$\begin{aligned}
\sum_{j=1}^{\infty}2^{(1+\beta+\beta q-\alpha q)j}l(2^j)\sum_{k=1}^{j}2^{\alpha qk}P(X^t>2^{k-1}) &\le C\cdot\sum_{k=1}^{\infty}2^{(1+\beta+\beta q)k}l(2^k)P(X^t>2^{k-1})\\
&\le C\cdot E\big(X^{t+(1+q)\beta t}l(X^{t+(1+q)\beta t})\big)\\
&\le C\cdot E\big(X^{t+\delta}l(X^{t+\delta})\big)<\infty.
\end{aligned}\tag{2.15}$$

By Kolmogorov inequality and Lemma 2.2 again, we get

$$\begin{aligned}
\text{Part}\,5 &\le C\cdot\sum_{n=1}^{\infty}\frac{l(n)}{n}\sum_{l=-[n^\beta]+1}^{[n^\beta]} n^{(\beta-\alpha)}\sum_{i=1}^{n}E\|Z_{l,i}\|\\
&\le C\cdot\sum_{n=1}^{\infty}\frac{l(n)}{n}n^\beta\cdot n^{\beta-\alpha}\cdot n\left[n^\alpha P(X^t>n)+\int_{n^\alpha}^{\infty}P(X>s)\,ds\right]\\
&\le C\cdot\sum_{n=1}^{\infty}n^{2\beta}l(n)P(X^t>n)+C\cdot\sum_{n=1}^{\infty}n^{2\beta-\alpha}l(n)\int_{n^\alpha}^{\infty}P(X>s)\,ds.
\end{aligned}\tag{2.16}$$

Keeping Lemma 2.1 in mind, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} n^{2\beta} l(n) P(X^t > n) &\le C \cdot \sum_{j=1}^{\infty} 2^{(2\beta+1)j} l(2^j) P(X^t > 2^{j-1}) \\ &\le C \cdot E(X^{t+2\beta t} l(X^{t+2\beta t})) \\ &\le C \cdot E(X^{t+\delta} l(X^{t+\delta})) < \infty,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\sum_{n=1}^{\infty} n^{2\beta-\alpha} l(n) \int_{n^\alpha}^{\infty} P(X > s)\, ds &\le C \cdot \sum_{j=1}^{\infty} 2^{(1+2\beta-\alpha)j} l(2^j) \int_{2^{\alpha j}}^{\infty} P(X > s) ds \\ &\le C \cdot \int_1^{\infty} E\left(\left(\frac{X}{s}\right)^{(1+2\beta)t} l\left(\left(\frac{X}{s}\right)^{(1+2\beta)t}\right)\right) ds \\ &\le C \cdot E(X^{t+\delta} l(X^{t+\delta})) < \infty.\end{aligned} \tag{2.18}$$

Hence, equation (2.5) follows from (2.9), (2.13), (2.14), (2.15), (2.16), (2.17), and (2.18). Theorem 1.1 is proved. □

Proof of Theorem 1.2. By Lemma 2.1, we know first that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{l(n)}{n} P\left(\sup_{k\ge n}\left\|\frac{S_k}{k^{1/t}}\right\| \ge \varepsilon\right) &\le C \cdot \sum_{j=1}^{\infty} l(2^j) P\left(\sup_{k\ge 2^j}\left\|\frac{S_k}{k^{1/t}}\right\| \ge \varepsilon\right) \\ &\le C \cdot \sum_{m=1}^{\infty} m l(2^m) P\left(\max_{2^m \le k < 2^{m+1}} \|S_k\| \ge 2^{m/t}\varepsilon\right).\end{aligned} \tag{2.19}$$

Hence, it is enough to show that

$$\sum_{j=1}^{\infty} j l(2^j) P\left(\max_{2^j \le n < 2^{j+1}} \|S_n\| \ge 2^{j/t}\varepsilon\right) < \infty \quad \forall \varepsilon > 0. \tag{2.20}$$

Observe that for $2^j \le n < 2^{j+1}$,

$$\begin{aligned}S_n &= \sum_{i=1}^{n} \left(X_i - E\left(X_i \mid \mathcal{F}_{i+[2^{\beta j}]}\right)\right) \\ &\quad + \sum_{l=-[2^{\beta j}]+1}^{[2^{\beta j}]} \sum_{i=1}^{n} \left(E(X_i \mid \mathcal{F}_{i+l}) - E(X_i \mid \mathcal{F}_{i+l-1})\right) + \sum_{i=1}^{n} E(X_i \mid \mathcal{F}_{i-[2^{\beta j}]}) \\ &= I_1 + II_2 + III_3,\end{aligned} \tag{2.21}$$

we only have to prove

$$\sum_{j=1}^{\infty} j l(2^j) P\left(\max_{2^j \le n < 2^{j+1}} \|I_1\| \ge 2^{j/t}\varepsilon\right) < \infty, \tag{2.22}$$

$$\sum_{j=1}^{\infty} jl(2^j)P\left(\max_{2^j \le n < 2^{j+1}} \|II_2\| \ge 2^{j/t}\varepsilon\right) < \infty, \tag{2.23}$$

$$\sum_{j=1}^{\infty} jl(2^j)P\left(\max_{2^j \le n < 2^{j+1}} \|III_3\| \ge 2^{j/t}\varepsilon\right) < \infty. \tag{2.24}$$

To this end, we write $\alpha = 1/t$. By Lemma 2.1 and (1.6), we have

$$\begin{aligned}
\text{LHS of (2.22)} &\le \sum_{j=1}^{\infty} jl(2^j) \sum_{2^j \le n < 2^{j+1}} P\left(\left\|\sum_{i=1}^{n}\left(X_i - E\left(X_i \mid \mathcal{F}_{i+[2^{\beta j}]}\right)\right)\right\| \ge 2^{(\alpha-1)j}\varepsilon\right) \\
&\le C \cdot \sum_{j=1}^{\infty} jl(2^j) \sum_{2^j \le n < 2^{j+1}} 2^{(1-\alpha)\lambda j}\left(\sum_{i=1}^{n}\left\|X_i - E\left(X_i \mid \mathcal{F}_{i+[2^{\beta j}]}\right)\right\|_{\lambda}\right)^{\lambda} \\
&\le C \cdot \sum_{j=1}^{\infty} jl(2^j) \cdot 2^j \cdot 2^{(1-\alpha)\lambda j} \cdot 2^{\lambda j} \cdot \psi^{\lambda}([2^{\beta j}]) \max_{1 \le i \le 2^{j+1}} C_i^{\lambda} \\
&\le C \cdot \sum_{j=1}^{\infty} j2^{(1+2\lambda)j}\psi^{\lambda}([2^{\beta j}]) \max_{1 \le i \le 2^{j+1}} C_i^{\lambda} < \infty.
\end{aligned} \tag{2.25}$$

Similarly, we can get

$$\begin{aligned}
\text{LHS of (2.24)} &\le C \cdot \sum_{j=1}^{\infty} jl(2^j) \cdot 2^j \cdot 2^{(\lambda-\alpha\lambda)j} \cdot 2^{\lambda j} \cdot \psi^{\lambda}([2^{\beta j}]) \max_{1 \le i \le 2^{j+1}} C_i^{\lambda} \\
&\le C \cdot \sum_{j=1}^{\infty} j2^{(1+2\lambda)j}\psi^{\lambda}([2^{\beta j}]) \max_{1 \le i \le 2^{j+1}} C_i^{\lambda} < \infty.
\end{aligned} \tag{2.26}$$

Now, all that remains is to prove (2.23). For this purpose, we denote $Y_{i,j} = X_i I(\|X_i\|^t \le 2^j)$, $Z_{i,j} = X_i - Y_{i,j}$, $W_{l,i} = E(X_i \mid \mathcal{F}_{i+l}) - E(X_i \mid \mathcal{F}_{i+l-1})$, $U_{i,l} = E(Y_{i,j} \mid \mathcal{F}_{i+l}) - E(Y_{i,j} \mid \mathcal{F}_{i+l-1})$, $V_{l,i} = E(Z_{i,j} \mid \mathcal{F}_{i+l}) - E(Z_{i,j} \mid \mathcal{F}_{i+l-1})$ for $2^j \le n < 2^{j+1}$, $1 \le i \le n$, $-[2^{\beta j}]+1 \le l \le [2^{\beta j}]$. Therefore,

$$\begin{aligned}
\text{LHS of (2.23)} &\le \sum_{j=1}^{\infty} jl(2^j) \sum_{l=-[2^{\beta j}]+1}^{[2^{\beta j}]} P\left(\max_{2^j \le n < 2^{j+1}}\left\|\sum_{i=1}^{n} W_{l,i}\right\| \ge 2^{(\alpha-\beta)j}\varepsilon\right) \\
&\le \sum_{j=1}^{\infty} jl(2^j) \sum_{l=-[2^{\beta j}]+1}^{[2^{\beta j}]} P\left(\max_{2^j \le n < 2^{j+1}}\left\|\sum_{i=1}^{n} U_{l,i}\right\| \ge 2^{(\alpha-\beta)j}\frac{\varepsilon}{2}\right) \\
&\quad + \sum_{j=1}^{\infty} jl(2^j) \sum_{l=-[2^{\beta j}]+1}^{[2^{\beta j}]} P\left(\max_{2^j \le n < 2^{j+1}}\left\|\sum_{i=1}^{n} V_{l,i}\right\| \ge 2^{(\alpha-\beta)j}\frac{\varepsilon}{2}\right) \\
&= IV_4 + V_5.
\end{aligned} \tag{2.27}$$

By Kolmogorov inequality and Lemma 2.2,

$$\begin{aligned}
VI_4 &\le C\cdot\sum_{j=1}^{\infty} jl(2^j)\sum_{l=-[2^{\beta j}]+1}^{[2^{\beta j}]} 2^{(\beta-\alpha)qj}\sum_{i=1}^{2^{j+1}} E\|U_{l,i}\|^q \\
&\le C\cdot\sum_{j=1}^{\infty} jl(2^j)\sum_{l=-[2^{\beta j}]+1}^{[2^{\beta j}]} 2^{(\beta-\alpha)qj}\sum_{i=1}^{2^{j+1}} \int_0^{2^j} s^{(\alpha q-1)}P(\|X_i\|^t > s)\,ds \\
&\le C\cdot\sum_{j=1}^{\infty} jl(2^j)2^{\beta j}\left[2^{(\beta-\alpha)qj}2^j + 2^{(\beta-\alpha)qj}2^j\sum_{k=1}^{j}\int_{2^{k-1}}^{2^k} s^{(\alpha q-1)}P(X^t > s)\,ds\right] \\
&\le C\cdot\sum_{j=1}^{\infty} jl(2^j)2^{(1+\beta+\beta q-\alpha q)j} \\
&\quad + C\cdot\sum_{j=1}^{\infty} jl(2^j)2^{(1+\beta+\beta q-\alpha q)j}\sum_{k=1}^{j} P(X^t > 2^{k-1})2^{\alpha qk}.
\end{aligned} \tag{2.28}$$

Since $1+\beta+\beta q-\alpha q < 0$, by Lemma 2.2(5) we know

$$\sum_{j=1}^{\infty} jl(2^j)2^{(1+\beta+\beta q-\alpha q)j} < \infty. \tag{2.29}$$

Furthermore, from Lemma 2.1, we get

$$\begin{aligned}
&\sum_{j=1}^{\infty} jl(2^j)2^{(1+\beta+\beta q-\alpha q)j}\sum_{k=1}^{j} 2^{\alpha qk}P(X^t > 2^{k-1}) \\
&\quad\le C\cdot\sum_{k=1}^{\infty} 2^{\alpha qk}P(X^t > 2^{k-1})\sum_{j=k}^{\infty} jl(2^j)2^{(1+\beta+\beta q-\alpha q)j} \\
&\quad\le C\cdot\sum_{k=1}^{\infty} 2^{(1+\beta+\beta q)k}l(2^k)P(X^t > 2^{k-1}) \\
&\qquad + C\cdot\sum_{j=1}^{\infty}(k-1)2^{(1+\beta+\beta q)(k-1)}l(2^{(k-1)})P(X^t > 2^{k-1}) \\
&\quad\le C\cdot E\left(X^{t+(\beta+\beta q)t}l(X^{t+(\beta+\beta q)t})\right) + C\cdot E\left(X^{t+(\beta+\beta q)t}l(X^{t+(\beta+\beta q)t})\ln^+ X\right) \\
&\quad\le C\cdot E\left(X^{t+\delta}l(X^{t+\delta})\right) + C\cdot E\left(X^{t+\delta}l(X^{t+\delta})\ln^+ X\right) < \infty,
\end{aligned} \tag{2.30}$$

since $(1+q)\beta t < \delta$. Consequently, $VI_4 < \infty$. Now all that remains is to prove $V_5 < \infty$.

In fact, by Chebyshev inequality and Lemma 2.2, we have

$$\begin{aligned}
V_5 &\le C\cdot\sum_{j=1}^{\infty} jl(2^j)2^{\beta j}2^{(\beta-\alpha)j}2^j\left[2^{\alpha j}P(X > 2^{\alpha j}) + \int_{2^{\alpha j}}^{\infty} P(X > s)ds\right] \\
&\le C\cdot\sum_{j=1}^{\infty} j2^{(1+2\beta)j}l(2^j)P(X^t > 2^j) + C\cdot\sum_{j=1}^{\infty} j2^{(1+2\beta-\alpha)j}l(2^j)\int_{2^{\alpha j}}^{\infty} P(X > s)\,ds.
\end{aligned} \tag{2.31}$$

Since $2\beta t < \delta$,

$$\begin{aligned}\sum_{j=1}^{\infty} j 2^{j(1+2\beta)} l(2^j) P(X^t > 2^j) &\le C \cdot E(X^{t+2\beta t} l(X^{t+2\beta t}) \ln^+ X) \\ &\le C \cdot E(X^{t+\delta} l(X^{t+\delta}) \ln^+ X) < \infty.\end{aligned} \tag{2.32}$$

At the same time, by Fubini's theorem,

$$\begin{aligned}\sum_{j=1}^{\infty} j 2^{(1+2\beta-\alpha)j} l(2^j) \int_{2^{\alpha j}}^{\infty} P(X > s)\, ds &= \sum_{j=1}^{\infty} j 2^{(1+2\beta)j} l(2^j) \int_1^{\infty} P(X^t > 2^j s)\, ds \\ &\le C \cdot \int_1^{\infty} E\left(\left(\frac{X}{s}\right)^{t+2\beta t} l\left(\left(\frac{X}{s}\right)^{t+2\beta t}\right) \ln^+ \left(\frac{X}{s}\right)\right) ds \\ &\le C \cdot E(X^{t+2\beta t} l(X^{t+2\beta t}) \ln^+ X) \int_1^{\infty} s^{-(t+2\beta t)}\, ds \\ &\le C \cdot E(X^{t+\delta} l(X^{t+\delta}) \ln^+ X) < \infty,\end{aligned} \tag{2.33}$$

which, together with (2.32), implies $V_5 < \infty$. This completes the proof. □

Proof of Theorem 1.3. Because

$$P\left(\left\|\sum_{i=1}^{\nu_n} X_i\right\| \ge \nu_n^{1/t} \varepsilon\right) \le P\left(\frac{\nu_n}{n} < \eta\right) + P\left(\sup_{k \ge \eta n} \left\|\sum_{j=1}^{k} \frac{X_j}{k^{1/t}}\right\| \ge \varepsilon\right), \tag{2.34}$$

keeping in mind (1.8), it is enough to show

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P\left(\sup_{k \ge \eta n} \left\|\frac{S_k}{k^{1/t}}\right\| \ge \varepsilon\right) < \infty, \tag{2.35}$$

where $S_k = \sum_{i=1}^{k} X_i$. Indeed Theorem 1.2 implies (2.35), hence Theorem 1.3 is proved. □

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