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STABILITY OF GENERALIZED ADDITIVE CAUCHY EQUATIONS

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ABSTRACT. A familiar functional equation $f(ax+b)=cf(x)$ will be solved in the class of functions $f:\mathbb{R}\to\mathbb{R}$. Applying this result we will investigate the Hyers-Ulam-Rassias stability problem of the generalized additive Cauchy equation

$$f(a_1x_1+\cdots+a_mx_m+x_0)=\sum_{i=1}^{m} b_i f(a_{i1}x_1+\cdots+a_{im}x_m)$$

in connection with the question of Rassias and Tabor.

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1. Introduction. The starting point of the study of stability of functional equations seems to be the famous talk of Ulam [21], in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a function $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$?

The case of approximately additive functions was solved by Hyers [6] under the assumption that G_1 and G_2 are Banach spaces. Later, the result of Hyers was generalized by Rassias [14] and other mathematicians (see [3, 4, 5, 7, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20]). The work of Rassias stimulated a number of mathematicians to investigate the stability problem of various functional equations. The terminology "Hyers-Ulam-Rassias stability" of the additive Cauchy equation was originated from these historical backgrounds. This terminology is also applied to the other functional equations.

For a given integer $m\ge 2$, let (a_{ij}) be a matrix in $\mathbb{C}^{m\times m}$ whose determinant ω is not zero. For $i\in\{1,\dots,m\}$, we denote by ω_i the determinant of the matrix that remains after all entries of the ith column in (a_{ij}) are replaced by 1.

In [13], the first author investigated the Hyers-Ulam-Rassias stability problem of a generalized additive Cauchy equation

$$f(a_1x_1+\cdots+a_mx_m+x_0)=\sum_{i=1}^{m} b_i f(a_{i1}x_1+\cdots+a_{im}x_m) \tag{1.1}$$

in the class of functions from a complex normed space E_1 into a complex Banach space E_2, where $a_1,\dots,a_m,b_1,\dots,b_m$ are complex numbers for which there exist some $j,k\in\{1,\dots,m\}$ with $a_j\ne 0$ and $b_k\ne 0$.

Let us define

$$r := \sum_{i=1}^{m} a_i \frac{\omega_i}{\omega}, \qquad B := \sum_{i=1}^{m} b_i. \tag{1.2}$$

Given $x_0 \in E_1$, we define a sequence $\{s_n(x)\}$ by

$$s_0(x) := x, \qquad s_1(x) := rx + x_0, \qquad s_{n+1}(x) := s_n(s_1(x)) \tag{1.3}$$

for any $x \in E_1$ and $n \in \mathbb{N}$. Assume that a function $\varphi : E_1^m \to [0,\infty)$ satisfies

$$\Phi(x_1,\dots,x_m) := \sum_{i=0}^{\infty} \frac{1}{|B|^{i+1}} \varphi\left(\frac{\omega_1}{\omega} s_i(x_1),\dots,\frac{\omega_m}{\omega} s_i(x_m)\right) < \infty, \tag{1.4}$$

$$\Phi(s_n(x),\dots,s_n(x)) = o(|B|^n) \quad \text{as } n \longrightarrow \infty \tag{1.5}$$

for all $x, x_1, \dots, x_m \in E_1$.

The following is one of the main theorems in [13].

Theorem 1.1. *Let E_1 and E_2 be a complex normed space and a complex Banach space, respectively. Assume that a function $f : E_1 \to E_2$ satisfies the inequality*

$$\left\| f(a_1x_1 + \cdots + a_mx_m + x_0) - \sum_{i=1}^{m} b_i f(a_{i1}x_1 + \cdots + a_{im}x_m) \right\| \le \varphi(x_1,\dots,x_m) \tag{1.6}$$

for all $x_1,\dots,x_m \in E_1$ and for a fixed $x_0 \in E_1$. If $|r|,\ |B| \notin \{0,1\}$, then there exists a unique function $F : E_1 \to E_2$ such that

$$F(rx + x_0) = BF(x), \qquad \|f(x) - F(x)\| \le \Phi(x,\dots,x) \tag{1.7}$$

for all $x \in E_1$.

The following familiar functional equation

$$f(ax + b) = \mathrm{cf}(x) \tag{1.8}$$

frequently appear in solving various functional equations (especially, in solving the generalized additive Cauchy equations). But to the best of our knowledge no author solved this equation. In Section 2, we solve the functional equation (1.8) in the class of functions $f : \mathbb{R} \to \mathbb{R}$. And, in Section 3, we use the result of Section 2 to study the Hyers-Ulam-Rassias stability of the generalized additive Cauchy equation (1.1). The question of Rassias and Tabor concerning the stability of the generalized additive Cauchy equations is studied in Section 4.

2. General solution of (1.8). Aczél and Kuczma, [1], solved a familiar functional equation

$$f(kx) = k^y f(x), \tag{2.1}$$

where y is a real constant and $k \neq 1$ is a positive constant. Indeed, they proved the following theorem.

Theorem 2.1. *The general solution $f : (0,\infty) \to \mathbb{R}$ of (2.1) is given by*

$$f(x) = x^{y} p(\log_k x), \tag{2.2}$$

where $p : \mathbb{R} \to \mathbb{R}$ is an arbitrary periodic function of period 1.

By using the idea of Aczél and Kuczma, we solve (1.8).

Theorem 2.2. *Let $a > 0$ $(a \neq 1)$, b and $c > 0$ be real constants. The general solution $f : \mathbb{R} \to \mathbb{R}$ of the functional equation (1.8) is given by*

$$f(x) = \begin{cases} c^{\log_a(x+b/(a-1))} p_1\left[\log_a\left(x+\dfrac{b}{a-1}\right)\right] & \left(x > -\dfrac{b}{a-1}\right), \\ d & \left(x = -\dfrac{b}{a-1}\right), \\ c^{\log_a(-x-b/(a-1))} p_2\left[\log_a\left(-x-\dfrac{b}{a-1}\right)\right] & \left(x < -\dfrac{b}{a-1}\right), \end{cases} \tag{2.3}$$

where $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ are arbitrary periodic functions of period 1. If $c \neq 1$, $d = 0$ and if $c = 1$, d is an arbitrary real number.

Proof. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ is given by the formula (2.3). Obviously, $x > -b/(a-1)$, $x = -b/(a-1)$, and $x < -b/(a-1)$ imply $ax+b > -b/(a-1)$, $ax+b = -b/(a-1)$, and $ax+b < -b/(a-1)$, respectively. The following facts

$$\begin{aligned} f(ax+b) &= c^{\log_a(ax+b+b/(a-1))} p_1\left[\log_a\left(ax+b+\frac{b}{a-1}\right)\right] \\ &= c^{\log_a(x+b/(a-1))+1} p_1\left[\log_a\left(x+\frac{b}{a-1}\right)+1\right] \\ &= cf(x) \end{aligned} \tag{2.4}$$

for all $x > -b/(a-1)$, and

$$\begin{aligned} f(ax+b) &= c^{\log_a(-ax-b-b/(a-1))} p_2\left[\log_a\left(-ax-b-\frac{b}{a-1}\right)\right] \\ &= c^{\log_a(-x-b/(a-1))+1} p_2\left[\log_a\left(-x-\frac{b}{a-1}\right)+1\right] \\ &= cf(x) \end{aligned} \tag{2.5}$$

for any $x < -b/(a-1)$, together with

$$f(ax+b) = f\left(-\frac{b}{a-1}\right) = cf(x) \tag{2.6}$$

for $x = -b/(a-1)$, imply that f is a solution of (1.8).

Now, assume that $f : \mathbb{R} \to \mathbb{R}$ is a solution of equation (1.8). Let us define functions $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ by

$$p_1(t) := c^{-t} f\left(a^t - \frac{b}{a-1}\right), \qquad p_2(t) := c^{-t} f\left(-a^t - \frac{b}{a-1}\right). \tag{2.7}$$

Then,

$$\begin{aligned} p_1(t+1) &= c^{-t-1} f\left(a^{t+1} - \frac{b}{a-1}\right) = c^{-t-1} f\left[a\left(a^t - \frac{b}{a-1}\right) + b\right] \\ &= c^{-t} f\left(a^t - \frac{b}{a-1}\right) = p_1(t), \\ p_2(t+1) &= c^{-t-1} f\left(-a^{t+1} - \frac{b}{a-1}\right) = c^{-t-1} f\left[a\left(-a^t - \frac{b}{a-1}\right) + b\right] \\ &= c^{-t} f\left(-a^t - \frac{b}{a-1}\right) = p_2(t) \end{aligned} \tag{2.8}$$

which mean that p_1 and p_2 are periodic functions of period 1.

If we put $t = \log_a(x + b/(a-1))$ and $t = \log_a(-x - b/(a-1))$ in the first and second definition of (2.7), respectively, then we see that (2.3) is true. □

3. Hyers-Ulam-Rassias stability of (1.1). Throughout this section, let (a_{ij}) be a matrix in $\mathbb{R}^{m\times m}$ whose determinant ω is nonzero, where m is an integer greater than 1. We denote by ω_i the determinant of the matrix that remains after all entries of the ith column in (a_{ij}) are replaced by 1. Let a_i and b_i, $i \in \{1,\dots,m\}$, be real numbers for which there exist some $j,k \in \{1,\dots,m\}$ with $a_j \neq 0$ and $b_k \neq 0$. Let a function $\varphi : E_1^m \to [0,\infty)$ satisfy the conditions (1.5) and (3.1), instead of (1.4), for all $x \in E_1$:

$$\Phi(x,\dots,x) := \sum_{i=0}^{\infty} \frac{1}{|B|^{i+1}} \varphi\left(\frac{\omega_1}{\omega} s_i(x),\dots,\frac{\omega_m}{\omega} s_i(x)\right) < \infty. \tag{3.1}$$

The following lemma is a reduced version of Theorem 1.1 which will be applied to the case of real normed spaces. We may slightly modify the proof of Theorem 1.1 to prove the following lemma.

Lemma 3.1. *Let E_1 be a real normed space and E_2 a real Banach space. If a function $f : E_1 \to E_2$ satisfies inequality (1.6) for all $x_1,\dots,x_m \in E_1$ and for some $x_0 \in E_1$ and $|B| > 0$, then there exists a unique function $F : E_1 \to E_2$ such that*

$$F(rx + x_0) = BF(x), \qquad \|f(x) - F(x)\| \le \Phi(x,\dots,x) \tag{3.2}$$

for each $x \in E_1$.

Combining Theorem 2.2 and Lemma 3.1, we obtain the following theorem.

Theorem 3.2. *Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality (1.6) for all $x_1,\dots,x_m \in \mathbb{R}$ and for some $x_0 \in \mathbb{R}$. Let r and B be given by the formulae in (1.2). If $r > 0$ $(r \neq 1)$ and $B > 0$, then there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ such that*

$$\|f(x) - F(x)\| \le \Phi(x,\dots,x) \tag{3.3}$$

for all $x \in \mathbb{R}$. Furthermore, the function F may be represented by

$$F(x) = \begin{cases} B^{\log_r(x+x_0/(r-1))} p_1\left[\log_r\left(x + \dfrac{x_0}{r-1}\right)\right] & \left(x > -\dfrac{x_0}{r-1}\right), \\ d & \left(x = -\dfrac{x_0}{r-1}\right), \\ B^{\log_r(-x-x_0/(r-1))} p_2\left[\log_r\left(-x - \dfrac{x_0}{r-1}\right)\right] & \left(x < -\dfrac{x_0}{r-1}\right), \end{cases} \tag{3.4}$$

where $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ are periodic functions of period 1. If $B \neq 1$, $d = 0$ and if $B = 1$, d is an arbitrary real number.

4. Question of Rassias and Tabor. Rassias and Tabor [18] asked whether the generalized Cauchy functional equation

$$f(a_1x + a_2y + v) = b_1 f(x) + b_2 f(y) + w, \tag{4.1}$$

where $a_1a_2b_1b_2 \neq 0$, is stable in the sense of Hyers, Ulam, and Rassias. For the case when $a_1 = b_1$, $a_2 = b_2$, and $v = w = 0$ in (4.1), Badea [2] proved the Hyers-Ulam-Rassias stability. Furthermore, the stability problem for the case of $v \neq 0$ in (4.1) was investigated in [13].

We will give a partial answer to the question of Rassias and Tabor which is indeed a corollary to Theorem 3.2.

Theorem 4.1. *Given real numbers $a, b > 0$, let a_1, a_2, b_1, b_2 be real constants with $a_1 + a_2 > 0$, $a_1 + a_2 \neq 1$, $b_1 + b_2 > \max\{(a_1+a_2)^a, (a_1+a_2)^b\}$, and $b_1 + b_2 \neq 1$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality*

$$\left| f(a_1x + a_2y) - b_1 f(x) - b_2 f(y) - w \right| \le \theta\left(|x|^a + |y|^b\right) \tag{4.2}$$

for some $\theta \ge 0$, for a fixed $w \in \mathbb{R}$ and for all $x, y \in \mathbb{R}$, then there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ with

$$\left| f(x) - F(x) - \frac{w}{1 - b_1 - b_2} \right| \le \frac{\theta|x|^a}{b_1 + b_2 - (a_1+a_2)^a} + \frac{\theta|x|^b}{b_1 + b_2 - (a_1+a_2)^b} \tag{4.3}$$

for any $x \in \mathbb{R}$, where

$$F(x) = \begin{cases} (b_1+b_2)^{\log_{a_1+a_2} x} p_1(\log_{a_1+a_2} x) & (x > 0), \\ 0 & (x = 0), \\ (b_1+b_2)^{\log_{a_1+a_2}(-x)} p_2(\log_{a_1+a_2}(-x)) & (x < 0), \end{cases} \tag{4.4}$$

with periodic functions $p_1, p_2 : \mathbb{R} \to \mathbb{R}$ of period 1.

Proof. With $g(x) = f(x) - w/(1 - b_1 - b_2)$, inequality (4.2) yields

$$\left| g(a_1x + a_2y) - b_1 g(x) - b_2 g(y) \right| \le \theta(|x|^a + |y|^b) \tag{4.5}$$

for all $x, y \in \mathbb{R}$. For this case, we have $a_{11} = 1$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = 1$, and so $\omega = \omega_1 = \omega_2 = 1$. If we put $\varphi(x,y) = \theta(|x|^a + |y|^b)$, then $\varphi : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ satisfies conditions (1.5) and (3.1). Hence, Lemma 3.1 says that there exists a unique function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F((a_1+a_2)x) = (b_1+b_2)F(x), \tag{4.6}$$

$$\begin{aligned} |g(x) - F(x)| &\le \Phi(x,x) \\ &= \sum_{i=0}^{\infty} \frac{\theta}{(b_1+b_2)^{i+1}}\Big(|(a_1+a_2)^i x|^a + |(a_1+a_2)^i x|^b\Big) \\ &= \frac{\theta|x|^a}{b_1+b_2-(a_1+a_2)^a} + \frac{\theta|x|^b}{b_1+b_2-(a_1+a_2)^b} \end{aligned} \tag{4.7}$$

for all $x \in \mathbb{R}$.

Theorem 2.2, together with (4.6), implies (4.4). □

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