Internat. J. Math. & Math. Sci.
Vol. 24, No. 12 (2000) 821–823
S0161171200002702

ON A FOUR-GENERATOR COXETER GROUP

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(Received 4 December 1999)

ABSTRACT. We study one of the 4-generator Coxeter groups and show that it is SQ-universal (SQU). We also study some other properties of the group.

Keywords and phrases. Group presentation, Reidemeister-Schreier process, Coxeter group, SQ-universality.

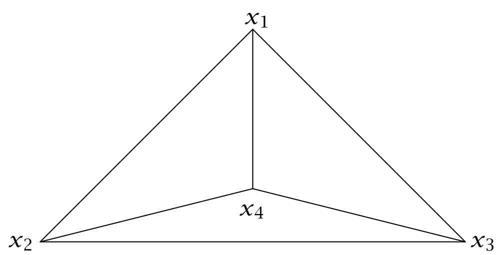
2000 Mathematics Subject Classification. Primary 20F05.

1. Introduction.

We consider the Coxeter group P given by the presentation

$$P = \langle x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2 = x_3^2 = x_4^2 = (x_1x_2)^3 = (x_2x_3)^3 = (x_1x_3)^3 = (x_1,x_4)^3 = (x_3x_4)^3 = (x_2x_3)^3 = e\rangle. \tag{1.1}$$

The Coxeter graph of this group is clearly just a combinatorial tetrahedron:



We observe that each face is the graph of the Euclidean triangle group $\triangle(3,3,3)$ which is an affine Weyl group and this contains a nilpotent subgroup of finite index. The group P is infinite and it will be interesting to see its largeness by answering whether it is SQ-universal or not.

2. SQ-universality.

We let S_3 be the symmetric group of degree 3. Thus

$$S_3 = \langle y_1, y_2 \mid y_1^2 = y_2^2 = (y_1y_2)^3 = e\rangle. \tag{2.1}$$

We consider the map $\theta : P \to S_3$ defined by

$$\theta(x_1) = y_1, \qquad \theta(x_2) = y_2, \qquad \theta(x_3) = \theta(x_4) = y_1y_2y_1. \tag{2.2}$$

It is easy to see that θ is an epimorphism and $P/\ker\theta \cong S_3$. A Schreier transversal for S_3 in P is $\{e, x_1, x_2, x_1x_2, x_2x_1, x_1x_2x_1\}$. A straightforward application of the

Reidemeister-Schreier process gives the following presentation for $\ker\theta$:

$$\ker\theta = \langle a,b,c,d \mid (ad)^3 = (bc)^3 = (abcd)^3 = e\rangle. \tag{2.3}$$

Letting $a = d^{-1}$ and $b = c^{-1}$, we see that $\ker\theta$ is mapped homomorphically onto the free group of rank 2, F_2. Hence $\ker\theta$ is SQU. Since the index of $\ker\theta$ in P is finite (6), we get that P is also SQU [4].

3. The growth series. Let (P,X) be a Coxeter system and let $Y \subseteq X$. We denote the subgroup of P, generated by Y, by P_Y. Then (W_Y,Y) is also a Coxeter system. In Bourbaki [2, Section 1 of Chapter 4], Exercise 26 gives the following formula for computing the growth series of P (word growth in the sense of Milner and Gromov):

$$\sum_{Y\subseteq x} \frac{(-1)^{|y|}}{P_Y(t)} = \begin{cases} \dfrac{t^m}{P(t)} & \text{if } P \text{ is finite,} \\ 0 & \text{if } P \text{ is infinite.} \end{cases} \tag{3.1}$$

In the formula, $G(t)$ is the growth series of G, m is the length of the unique element of P of maximal length.

We use (3.1) to compute $P(t)$. We compute $P(t)$ in steps corresponding to the cardinality of Y:

$|Y| = 0$ is the trivial subgroup with growth series $y_0 = 1$.

$|Y| = 1$ four cyclic subgroups of order 2 with growth series $y_1 = 1+t$.

$|Y| = 2$ six dihedral subgroups of order 6 with growth series $y_2 = (1+t)(1+t+t^2)$.

$|Y| = 3$ four affine subgroups with growth series given by $1/y_0 - 3/y_1 + 3/y_2 - 1/y_3 = 0$, that is, $y_3 = (1+t+t^2)/(1-t)^2$.

$|Y| = 4$ the whole group with growth $y_4(t) = P(t)$ given by $1/y_0 - 4/y_1 + 6/y_2 - 4/y_3 + 1/y_4 = 0$, that is, $y_4 = (1+t)(1+t+t^2)/(1-t)(1-t-3t^2)$.

The growth coefficients $\{c_n\}$ are given by the linear recurrence $c_0 = 1,\ c_1 = 4,\ c_2 = 12,\ c_3 = 30,\ c_n = 2c_{n-1} + 2c_{n-2} - 3c_{n-3},\ n \geq 4$ (see [3]). We observe from the growth series y_4 that zeros of the denominator are not on the unit circle. This implies that P has no nilpotent subgroup of finite index—in accordance with the fact that P is SQU.

It is possible to show that the group P and the Geisking group $G = \langle x,y \mid x^2y^2 = xy\rangle$ are isometric and hence y_4 is also the growth series of G (see [3]). In [1], it appears that the two Coxeter groups T_n and S_n are also isometric and so have the same growth series.

4. The commutator subgroup. Using the Reidemeister-Schreier process, we get the following presentation for P':

$$P' = \langle x,y,z \mid x^3 = y^3 = z^3 = (xy)^3 = (xz)^3 = \left(yz^{-1}\right)^3 = e\rangle. \tag{4.1}$$

We use P' to show that P is SQU in a different method. Let K be the normal closure of the elements $xy^{-1}, xz^{-1}, yz^{-1}$ in P'. The group K has index 3 in P'. Using the Reidemeister-Schreier process, we get the following presentation for K:

$$\begin{aligned} K = \langle u_1,u_2,u_3,v_1,v_2,v_3 \mid v_1^2 &= v_2^2 = v_3^2 = u_1u_2u_3 \\ &= u_1u_3u_2 = v_1v_2v_3 = u_1v_2u_3v_1u_2u_3 = e\rangle. \end{aligned} \tag{4.2}$$

Letting $u_3 = v_3 = e$, we see that K is mapped homomorphically onto $Z * Z_3$. Since $Z * Z_3$ is SQU (see [4]), therefore K is SQU. Since K is of finite index in P' and P' is of finite index in P, we get that P is SQU.

Acknowledgement. The author thanks King Fahd University of Petroleum and Minerals for supporting him in his research.

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