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ABOUT THE EXISTENCE OF THE THERMODYNAMIC LIMIT FOR SOME DETERMINISTIC SEQUENCES OF THE UNIT CIRCLE

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Abstract. We show that in the set $\Omega = \mathbb{R}_+ \times (1,+\infty) \subset \mathbb{R}^2_+$, endowed with the usual Lebesgue measure, for almost all $(h,\lambda) \in \Omega$ the limit $\lim_{n\to+\infty}(1/n)\ln|h(\lambda^n - \lambda^{-n}) \bmod [-\frac{1}{2},\frac{1}{2})|$ exists and is equal to zero. The result is related to a characterization of relaxation to equilibrium in mixing automorphisms of the two-torus. It is nothing but a curiosity, but maybe you will find it nice.

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1. Introduction. In the analysis of relaxation to equilibrium of mixing automorphisms of the two-torus [1, 2, 3] one encounters the following problem. Suppose that the one-torus is parameterized by the unit interval $[-\frac{1}{2},\frac{1}{2})$ and for appropriate constants $h > 0$ and $\lambda > 1$ consider the real sequence

$$x_n = h(\lambda^n - \lambda^{-n}) \bmod \left[-\frac{1}{2},\frac{1}{2}\right) \quad \forall n \in \mathbb{N}. \tag{1.1}$$

A significant definition of an exponential "relaxation rate" can be given if the so-called "thermodynamic" limit [3],

$$\lim_{n\to+\infty} -\frac{1}{n}\ln|x_n| \tag{1.2}$$

exists and is equal to zero. Existence of (1.2) is clearly not obvious, since the x_n's typically wander through the whole interval $[-\frac{1}{2},\frac{1}{2})$ but every so often they visit a small neighborhood of zero, where the logarithm is singular. Actually, not even if one replaces the ordinary limit in (1.2) with a supremum limit the finiteness of the result is assured.

This note is devoted to a measure theoretical discussion of the previous problem. One can show that existence to zero of limit (1.2) occurs almost surely, for almost any choice of the parameters h and λ, with respect to a measure suitably defined.

2. Results. Our goal is to prove the statement below.

Theorem 2.1. *Consider the set $\Omega = \mathbb{R}_+ \times (1,+\infty) \subset \mathbb{R}^2_+$ endowed with the usual Lebesgue measure μ. Then, for μ almost all $(h,\lambda) \in \Omega$ there holds*

$$\lim_{n\to+\infty} \frac{1}{n}\ln\left|h(\lambda^n - \lambda^{-n}) \bmod \left[-\frac{1}{2},\frac{1}{2}\right)\right| = 0. \tag{2.1}$$

This result can be easily deduced by means of standard arguments of measure theory once the following main theorem is proved.

Theorem 2.2. *Let $h > 0$ and $Q \in \mathbb{N}$, $Q > 1$, some fixed constants. Consider the set G of all $\lambda \in [1,Q]$ for which a (possibly λ-dependent) real sequence $(a_n)_{n\in\mathbb{N}}$ and an integer $n' \in \mathbb{N}$ exist such that*

(a) $a_n > 0 \ \forall n > n'$;
(b) $a_n \le |h(\lambda^n - \lambda^{-n}) \bmod [-\frac{1}{2},\frac{1}{2})| \ \forall n > n'$;
(c) $\lim_{n\to+\infty}(1/n)\ln a_n = 0$.

Then, if μ denotes the Lebesgue measure on $\mathbb{R}$:

(1) *the set $G \subseteq [1,Q]$ is actually nonempty;*
(2) *G is μ-measurable and its measure holds $\mu(G) = Q-1$.*

As a consequence, the set $B = [1,Q]\backslash G$, where conditions (a), (b), and (c) are not simultaneously satisfied, is also μ-measurable and of vanishing measure.

We firstly prove the result by considering values of λ in the interval $[1+\eta,Q]$, with η small positive number arbitrarily fixed ($\eta < 1/2$). We therefore look for the subset G_η of $\lambda \in [1+\eta,Q]$, where hypotheses (a), (b), and (c) are satisfied by a suitable choice of the sequence $(a_n)_{n\in\mathbb{N}}$ and of the integer $n' \in \mathbb{N}$. The basic idea of the proof is that the μ-measure of G_η turns out to be $Q-1-\eta$ even if we confine ourselves to choose the sequence $(a_n)_{n\in\mathbb{N}}$ in the form

$$a_n = \frac{1}{n^2} \quad \forall n \in \mathbb{N}, \tag{2.2}$$

which certainly fulfills requirements (a) and (c), and enable us to deal with the *only* condition (b) on λ.

Let us then take $a_n = 1/n^2$ for all $n \in \mathbb{N}$ and an arbitrarily given value of $n \in \mathbb{N}$. Before tackling the real proof, we need some definitions.

Definition 2.3. We introduce the set $B_n \subseteq [1+\eta,Q]$

$$B_n = \left\{\lambda \in [1+\eta,Q] : a_n > \left| h(\lambda^n - \lambda^{-n}) \bmod \left[-\frac{1}{2},\frac{1}{2}\right)\right|\right\}, \tag{2.3}$$

that is, the set of $\lambda \in [1+\eta,Q]$, where the condition $a_n \le |h(\lambda^n - \lambda^{-n}) \bmod [-\frac{1}{2},\frac{1}{2})|$ is not satisfied for the assigned $n \in \mathbb{N}$.

Notice that B_n is a finite union of intervals because the function $\Phi_n(\lambda) = \lambda^n - \lambda^{-n}$ is strictly increasing in $[+1,+\infty)$ at fixed n. In fact

$$\Phi_n'(\lambda) = n(\lambda^{n-1} + \lambda^{-(n+1)}) > 0 \quad \forall \lambda \in [1,+\infty). \tag{2.4}$$

Consequently, B_n is μ-measurable as a finite union of bounded intervals.

Definition 2.4. We further introduce the set $\hat{B}_{n'} \subseteq [1+\eta,Q]$, $n' \in \mathbb{N}$, given by

$$\hat{B}_{n'} = \left\{\lambda \in [1+\eta,Q] : a_n > \left| h(\lambda^n - \lambda^{-n}) \bmod \left[-\frac{1}{2},\frac{1}{2}\right)\right|, \ n > n'\right\} = \bigcup_{n>n'} B_n \tag{2.5}$$

which is obviously μ-measurable as a countable union of μ-measurable sets.

Definition 2.5. We finally introduce the "bad" set $B_\eta \subseteq [1+\eta, Q]$

$$B_\eta = \bigcap_{n'=1}^{\infty} \hat{B}_{n'}, \tag{2.6}$$

where condition (b) is not satisfied—with this particular choice of the sequence $(a_n)_{n\in\mathbb{N}}$. B_η is also a μ-measurable set, as a countable intersection of μ-measurable sets.

An immediate consequence of the previous definitions is that $[1+\eta, Q] \setminus G_\eta = B_\eta$. Our goal is to prove that $\mu(B_\eta) = 0$. To this end, since for all $n' \in \mathbb{N}$, $B_\eta \subseteq \hat{B}_{n'}$ by definition, it is enough to show that

$$\lim_{n'\to+\infty} \mu(\hat{B}_{n'}) = 0. \tag{2.7}$$

Therefore, we can confine ourselves to consider values of $n' \in \mathbb{N}$ large enough, and owing to Definition 2.4, we can also assume values of $n \in \mathbb{N}$ greater that n'. More precisely, we impose the following technical requirements on the size of n' and n. We take $n > n' \in \mathbb{N}$ such that:

(i) $a_{n'} = 1/n'^2 < \eta \Rightarrow a_n < \eta \ \forall n > n'$.
(ii) $a_n - h/(1+\eta)^n = 1/n^2 - h/(1+\eta)^n > 0 \ \forall n > n'$.
(iii) $h[(1+\eta)^n - (1+\eta)^{-n}] > 3/2$ and $h[Q^n - Q^{-n}] > 5/2 \ \forall n > n'$.

Under the previous conditions we can state the following lemmas.

Lemma 2.6. *The μ-measure of B_n, n as above, admits the upper bound*

$$\mu(B_n) \le 2\varepsilon_n \left(\frac{1}{h}\right)^{1/n} [h(Q^n - Q^{-n})]^{1/n}, \tag{2.8}$$

where $\varepsilon_n = a_n + h/(1+\eta)^n > 0$.

Proof. We firstly notice that $1 + a_n < 1+\eta$ by (i); on the other hand, since $\eta < 1/2$ by hypothesis, (i) implies $a_n < 1/2$, so that all the intervals $(p - a_n, p + a_n)$, $p = 2, \dots, \lfloor h(Q^n - Q^{-n}) \rfloor + 1$ are disjoint.

By using (iii) and denoted with I_n the integer set $\{2, 3, \dots, \lfloor h(Q^n - Q^{-n}) \rfloor + 1\}$, we deduce

$$\begin{aligned} B_n &\subseteq \left\{ \lambda \in [1+\eta, Q] : h(\lambda^n - \lambda^{-n}) \in \bigcup_{p=2}^{\lfloor h(Q^n - Q^{-n})\rfloor + 1} (p - a_n, p + a_n) \right\} \\ &= \{\lambda \in [1+\eta, Q] : p - a_n < h(\lambda^n - \lambda^{-n}) < p + a_n,\ p \in I_n\} \\ &= \{\lambda \in [1+\eta, Q] : p - (a_n - h\lambda^{-n}) < h\lambda^n < p + a_n + h\lambda^{-n},\ p \in I_n\}. \end{aligned} \tag{2.9}$$

Now it is clear that for all $\lambda \in [1+\eta, Q]$,

$$a_n - \frac{h}{(1+\eta)^n} \le a_n - h\lambda^{-n} < a_n + h\lambda^{-n} \le a_n + \frac{h}{(1+\eta)^n} \tag{2.10}$$

and by (ii),

$$a_n - \frac{h}{(1+\eta)^n} > 0 \tag{2.11}$$

from which we obtain

$$0 < a_n - \frac{h}{(1+\eta)^n} \le a_n - h\lambda^{-n} < a_n + h\lambda^{-n} \le a_n + \frac{h}{(1+\eta)^n} \quad \forall \lambda \in [1+\eta, Q]. \tag{2.12}$$

By enlarging each covering interval in (2.9), we are then led to the inclusion

$$B_n \subseteq \left\{\lambda \in [1+\eta, Q] : p - \left(a_n + \frac{h}{(1+\eta)^n}\right) < h\lambda^n < p + a_n + \frac{h}{(1+\eta)^n},\ p \in I_n\right\} \tag{2.13}$$

and recalling the definition of ε_n,

$$\begin{aligned} B_n &\subseteq \left\{\lambda \in [1+\eta, Q] : \left(\frac{p-\varepsilon_n}{h}\right)^{1/n} < \lambda < \left(\frac{p+\varepsilon_n}{h}\right)^{1/n},\ p \in I_n\right\} \\ &\subseteq \bigcup_{p=2}^{\lfloor h(Q^n - Q^{-n})\rfloor + 1} \left(\left(\frac{p-\varepsilon_n}{h}\right)^{1/n}, \left(\frac{p+\varepsilon_n}{h}\right)^{1/n}\right), \end{aligned} \tag{2.14}$$

the final set being μ-measurable as a finite union of intervals. Whence

$$\mu(B_n) \le \sum_{p=2}^{\lfloor h(Q^n - Q^{-n})\rfloor + 1} \left(\frac{1}{h}\right)^{1/n} \left[(p+\varepsilon_n)^{1/n} - (p-\varepsilon_n)^{1/n}\right]. \tag{2.15}$$

Moreover, for all $p = 2, 3, \ldots, \lfloor h(Q^n - Q^{-n})\rfloor + 1$ Lagrange mean value theorem implies the equalities below

$$(p+\varepsilon_n)^{1/n} - (p-\varepsilon_n)^{1/n} = \frac{1}{n}(p+\xi_p)^{(1/n)-1} 2\varepsilon_n \tag{2.16}$$

for some $\xi_p \in (-\varepsilon_n, \varepsilon_n)$, and since

$$(p+\xi_p)^{(1/n)-1} = \frac{1}{(p+\xi_p)^{1-(1/n)}} \le \frac{1}{(p-1)^{1-(1/n)}} \tag{2.17}$$

we conclude that

$$\begin{aligned} \mu(B_n) &\le \sum_{p=2}^{\lfloor h(Q^n - Q^{-n})\rfloor + 1} \left(\frac{1}{h}\right)^{1/n} \frac{1}{n} \frac{1}{(p-1)^{1-(1/n)}} 2\varepsilon_n \\ &= \frac{2\varepsilon_n}{n}\left(\frac{1}{h}\right)^{1/n} \sum_{p=1}^{\lfloor h(Q^n - Q^{-n})\rfloor} \frac{1}{p^{1-(1/n)}}. \end{aligned} \tag{2.18}$$

As $(p^{1-(1/n)})^{-1}$ is a decreasing function of p, the following upper estimate holds

$$\begin{aligned} \sum_{p=1}^{\lfloor h(Q^n - Q^{-n})\rfloor} \frac{1}{p^{1-(1/n)}} &\le \int_0^{\lfloor h(Q^n - Q^{-n})\rfloor} p^{(1/n)-1} dp = \left[np^{1/n}\right]_0^{\lfloor h(Q^n - Q^{-n})\rfloor} \\ &= n\lfloor h(Q^n - Q^{-n})\rfloor^{1/n} \end{aligned} \tag{2.19}$$

and finally $\mu(B_n) \le 2\varepsilon_n (1/h)^{1/n} [h(Q^n - Q^{-n})]^{1/n}$, which completes the proof. □

Lemma 2.7. *If $n' > 0$ (satisfying (i), (ii), and (iii)) is sufficiently large, the μ-measure of $\hat{B}_{n'}$ is bounded by*

$$\mu(\hat{B}_{n'}) \leq 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n \tag{2.20}$$

for some small $\varepsilon > 0$.

Proof. Because of the identity $\hat{B}_{n'} = \cup_{n>n'} B_n$ and using Lemma 2.6, we have the following estimate

$$\mu(\hat{B}_{n'}) \leq \sum_{n=n'+1}^{\infty} \mu(B_n) \leq 2 \sum_{n=n'+1}^{\infty} \varepsilon_n \left(\frac{1}{h}\right)^{1/n} \left[h(Q^n - Q^{-n})\right]^{1/n}. \tag{2.21}$$

Notice that for all $h > 0$, and $Q \in \mathbb{N}$, $Q > 1$

$$\lim_{n\to+\infty} \left(\frac{1}{h}\right)^{1/n} \left[h(Q^n - Q^{-n})\right]^{1/n} = Q \tag{2.22}$$

so that for some $\varepsilon > 0$, $\varepsilon \ll Q$, and n' sufficiently large there holds

$$Q - \varepsilon < \left(\frac{1}{h}\right)^{1/n} \left[h(Q^n - Q^{-n})\right]^{1/n} < Q + \varepsilon \quad \forall n \in \mathbb{N},\ n > n'. \tag{2.23}$$

Whence for n' as above

$$\mu(\hat{B}_{n'}) \leq 2 \sum_{n=n'+1}^{\infty} \varepsilon_n (Q+\varepsilon) = 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n \tag{2.24}$$

which is finite, owing to

$$\sum_{n=1}^{\infty} \varepsilon_n = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{h}{(1+\eta)^n}\right) = \frac{\pi^2}{6} + \frac{h}{\eta}. \tag{2.25}$$

□

Lemma 2.8. *The measure of B_η is zero*

$$\mu(B_\eta) = 0. \tag{2.26}$$

Proof. Since for all $n' \in \mathbb{N}$ we have that $B_\eta \subseteq \hat{B}_{n'}$, in particular this will be true for all $n' \in \mathbb{N}$ large enough to satisfy the requirements of the previous lemmas. Thus

$$\mu(B_\eta) \leq \mu(\hat{B}_{n'}) \leq 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n \tag{2.27}$$

and therefore

$$\mu(B_\eta) \leq \lim_{n'\to+\infty} 2(Q+\varepsilon) \sum_{n=n'+1}^{\infty} \varepsilon_n, \tag{2.28}$$

where the limit is obviously zero, because of $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. By the nonnegativity of measure we have the result. □

Proof of Theorem 2.2. As a consequence of Lemma 2.8, the "good" set $G_\eta = [1+\eta, Q]\backslash B_\eta$ of λ-values in $[1+\eta, Q]$ satisfying condition (b) for the particular choice of $(a_n)_{n\in\mathbb{N}}$, $a_n = 1/n^2$, is of course μ-measurable and with Lebesgue measure

$$\mu(G_\eta) = Q - 1 - \eta - \mu(B_\eta) = Q - 1 - \eta. \tag{2.29}$$

If we now consider an *arbitrary* choice of the sequence $(a_n)_{n\in\mathbb{N}}$, compatible again with conditions (a) and (c), the previous set G_η will maybe "grow" by a subset $\tilde{G}_\eta \subseteq [1+\eta, Q]\backslash G_\eta$:

$$G_\eta \longrightarrow G_\eta^o = G_\eta \cup \tilde{G}_\eta. \tag{2.30}$$

But as $\mu([1+\eta, Q]\backslash G_\eta) = 0$ it follows that $\tilde{G}_\eta$ is also μ-measurable and of vanishing μ-measure. Hence we finally conclude that the *full* set G_η^o, corresponding to arbitrary (a)- and (c)-conditioned sequences $(a_n)_{n\in\mathbb{N}}$, is μ-measurable with measure

$$\mu(G_\eta^o) = Q - 1 - \eta \tag{2.31}$$

and that the corresponding *full* set $B_\eta^o = [1+\eta, Q]\backslash G_\eta^o$ of λ values where condition (b) is not fulfilled for *any* (a)- and (c)-conditioned sequence $(a_n)_{n\in\mathbb{N}}$ is in turn μ-measurable with vanishing μ-measure:

$$\mu(B_\eta^o) = Q - 1 - \eta - \mu(G_\eta^o) = 0. \tag{2.32}$$

So far we have proved that B_η is a set of vanishing measure in any closed interval $[1+\eta, Q]$ with $\eta > 0$. Consider now B, G in $[1, Q]$, that is, according to the previous notation

$$B = B_0, \qquad G = G_0. \tag{2.33}$$

We firstly notice that B and G are both μ-measurable because

$$G = \bigcup_{n'=1}^{\infty} \bigcap_{n>n'} S_n, \tag{2.34}$$

where S_n is the finite union of *subintervals* in $[1, Q]$ (dependent on n), and $B = [1, Q]\backslash G$. Then take

$$B = \big(B \cap [1, 1+\eta)\big) \cup \big(B \cap [1+\eta, Q]\big) \tag{2.35}$$

union of disjoint sets, for some fixed $\eta \in (0, \frac{1}{2})$. The μ-measurable set $B \cap [1+\eta, Q]$ is the "bad" set in $[1+\eta, Q]$, so that by Lemma 2.8,

$$\mu\big(B \cap [1+\eta, Q]\big) = 0. \tag{2.36}$$

As for the μ-measurable set $B \cap [1, 1+\eta)$, we have the identity

$$\begin{aligned} B \cap [1, 1+\eta) &= B \cap \left(\{1\} \cup \bigcup_{k=1}^{\infty} \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)\right) \\ &= B \cap \{1\} \cup \bigcup_{k=1}^{\infty} B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right) \end{aligned} \tag{2.37}$$

union of disjoint sets. But the μ-measurable set

$$B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right), \quad k \in \mathbb{N}\setminus\{0\}, \tag{2.38}$$

satisfies

$$B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right) \subseteq B \cap \left[1 + \frac{\eta}{k+1}, Q\right] \tag{2.39}$$

and since $1/2 > \eta/(k+1) > 0$ for any given $k \in \mathbb{N}\setminus\{0\}$, we obtain

$$\mu\left(B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)\right) = 0. \tag{2.40}$$

On the other hand, there trivially holds $\mu(B \cap \{1\}) = 0$, so that

$$\mu(B \cap [1, 1+\eta)) = \mu(B \cap \{1\}) + \sum_{k=1}^{\infty} \mu\left(B \cap \left[1 + \frac{\eta}{k+1}, 1 + \frac{\eta}{k}\right)\right) = 0. \tag{2.41}$$

Whence, finally,

$$\mu(B) = \mu(B \cap [1, 1+\eta)) + \mu(B \cap [1+\eta, Q)) = 0, \tag{2.42}$$

that is, $\mu(B) = 0$ and $\mu(G) = Q - 1$. The proof is complete. □

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