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$\aleph_1$-DIRECTED INVERSE SYSTEMS OF CONTINUOUS IMAGES OF ARCS

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Abstract. The main purpose of this paper is to prove that if $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a usual $\aleph_1$-directed inverse system of continuous images of arcs with monotone bonding mappings, then $X = \lim \mathbf{X}$ is a continuous image of an arc (Theorem 2.4). Some applications of this statement are also given.

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1. Introduction. By *an approximate inverse system* in this paper we will mean an approximate inverse system in the sense of Mardešić [11]. See the appendix, Definition 5.1.

An *arc* is a continuum with precisely two nonseparating points.

A space X is said to be a continuous image of an arc if there exists an arc L and a continuous surjection $f : L \to X$.

Inverse sequence of continuous images of arcs with monotone bonding mappings was studied in [16], where was proved that the limit of a usual inverse sequence of continuous images of arcs and monotone bonding mappings is the continuous image of an arc.

The well-ordered and σ-directed usual inverse system of continuous images of arcs with monotone bonding mappings were studied in [7].

In this paper, we shall study the $\aleph_1$-directed inverse systems of continuous images of arcs. The main theorem of this paper states that if $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a usual $\aleph_1$-directed inverse system of continuous images of arcs and monotone bonding mappings, then $X = \lim \mathbf{X}$ is the continuous image of an arc (Theorem 2.4). Using Theorem 2.10 we shall obtain a characterization of continuous image of an arc by its images Y with $w(Y) = \aleph_1$ (Theorem 3.3). Finally, we shall use this characterization to produce some new theorems concerning the approximate inverse system of continuous images of arcs (Theorems 4.3 and 4.6).

2. $\aleph_1$-directed inverse systems. We start with some lemmas.

Lemma 2.1. *Let $\mathcal{M} = \{M_\mu : \mu \in M\}$ be an $\aleph_1$-directed (partially ordered by inclusion) family of compact metric subspaces M_μ of a space X. Then $N = \bigcup\{M_\mu : \mu \in M\}$ is a compact metrizable subspace of X.*

Proof. Suppose that $w(N) \geq \aleph_1$. By virtue of [5] (or [17, Theorem 1.1] if X is a T_3-space), for $\lambda = \aleph_1$, there exists a subspace N_λ of N such that $\mathrm{card}(N_\lambda) \leq \aleph_1$ and

$w(N_\lambda) \geq \aleph_1$. For each $x \in N_\lambda$ there exists a $M_\mu(x) \in \mathcal{M}$ such that $x \in M_\mu(x)$. The family $\mathcal{M}_1 = \{M_\mu(x) : x \in N_\lambda\}$ has the cardinality $\leq \aleph_1$. By the $\aleph_1$-directedness of $\mathcal{M}$ there exists a $M_\nu \in \mathcal{M}$ such that $M_\nu \supseteq M_\mu(x)$ for each $x \in N_\lambda$. This means that $N_\lambda \subseteq M_\nu$. We infer that $w(N_\lambda) \leq \aleph_0$ since M_ν is a metric subspace of X. This contradicts the assumption $w(N_\lambda) \geq \aleph_1$. Hence, $w(N) \leq \aleph_0$. There exists a countable dense subset $Z = \{z_n : N \in \mathbb{N}\}$ of N. For each z_n there is a $M_\mu(n) \in \mathcal{M}$ such that $z_n \in M_\mu(n)$. It is clear that $W = \bigcup\{M_\mu(n) : n \in \mathbb{N}\}$ is dense in N. By virtue of the $\aleph_1$-directedness of $\mathcal{M}$ there exists a $M_\nu \in \mathcal{M}$ such that $M_\nu \supseteq M_\mu(n)$ for each $M_\mu(n)$. We infer that $M_\mu \supseteq W$ and, consequently, M_μ is dense in N. From the compactness of M_μ it follows that $N = M_\mu$. Hence, N is a compact metrizable subspace of X. □

Let τ be an infinite cardinal. We say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *τ-directed* if for each $b \subseteq A$ with $\operatorname{card}(B) \leq \operatorname{card}(A)$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. We say that $\mathbf{X}$ is *σ-directed* if $\mathbf{X}$ is $\aleph_0$-directed.

Lemma 2.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate τ-directed inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a compact space with $w(Y) \leq \tau$ and let Z be a closed subspace of X. For each surjective mapping $f : Z \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : p_b(Z) \to Y$ such that $f = g_b \circ (p_b \mid Z)$. Moreover, if f is a monotone surjection, then g_b is a monotone surjection.*

Proof. Let $Z_a = p_a(Z)$ and $q_a = p_a \mid Z$, $a \in A$. The proof is broken into several steps.

Step 1. For each normal covering $\mathcal{U}$ of Z there exists an $a(\mathcal{U}) \in A$ such that for each $b \geq a(\mathcal{U})$ there is a normal covering $\mathcal{U}_b$ of Z_b such that $q_b^{-1}(\mathcal{U}_b)$ is a refinement of $\mathcal{U}$.

For each normal covering $\mathcal{U}$ of Z there exists a normal covering $\mathcal{V}$ of X such that $\mathcal{V} \mid Z = \{V \bigcap Z : V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}$ [1, Theorem 15.14]. By virtue of (B1) (see [13, Theorems 2.8 and 4.2]), [11, Lemma 4] there exists an $a(\mathcal{U}) \in A$ such that for each $b \geq a(\mathcal{U})$ there is a normal cover $\mathcal{V}_b$ of X_b with $p_b^{-1}(\mathcal{V}_b) \prec \mathcal{V}$. Consider the covering $\mathcal{U}_b = \mathcal{V}_b \mid Z_b$. It is clear that $q_b^{-1}(\mathcal{U}_b)$ is a refinement of $\mathcal{U}$.

Step 2. For each family of normal coverings $\mathcal{U}$ of Z with $\operatorname{card}(\mathcal{U}) \leq \tau$ there exists an $a \in A$ such that for each $b \geq a$ there is a family of normal coverings $\mathcal{U}_b$ of Z_b, $\operatorname{card}(\mathcal{U}_b) \leq \tau$ such that $q_b^{-1}(\mathcal{U}_b)$ is a refinement of $\mathcal{U}$.

This follows from Step 1 and from the τ-directedness of A.

Step 3. For each basis $\mathcal{B} = \{\mathcal{U}_n : n < \tau\}$ of normal coverings of Y there exists an $a \in A$ such that for each $b \geq a$ there is a family of normal coverings $\mathcal{U}_{nb}$ of Z_b such that $q_b^{-1}(\mathcal{U}_{nb})$ is a refinement of $\mathcal{U}_n$, $\mathcal{U}_n \in \mathcal{B}$.

Now, each basis of normal coverings of Y has the cardinality $\leq \tau$ (Lemma 5.2). Apply Step 2.

Step 4. If Z_b is as in Step 3, then for each $x_b \in Z_b$ the set $f(q_b^{-1}(x_b))$ is degenerate.

Suppose that there exists a pair u, v of distinct points of Y such that $u, v \in f(q_b^{-1}(x_b))$. Then there exists a pair x, y of distinct points of $q_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let U, V be a pair of disjoint open sets of Y such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, Y \setminus \{u, v\}\}$ of Y. There exists a normal covering $\mathcal{V}_n \in \mathcal{V}$

such that $\mathcal{V}_n \prec \{U, V, X \setminus \{u, v\}\}$. We infer that there is a normal covering $\mathcal{V}_{nb}$ of Z_b such that $q_b^{-1}(\mathcal{V}_{nb}) \prec f^{-1}(\mathcal{V}_n)$. It follows that $q_b(x) \neq q_b(y)$ since x and y lie in the disjoint members of the covering $f^{-1}(\mathcal{V}_n)$. This is impossible since $x, y \in q_b^{-1}(x_b)$. Thus, $f(q_b^{-1}(x_b))$ is degenerate.

Step 5. There exists a mapping $g_b : Z_b \to Y$.

Now we define g_b by $g_b(x_b) = f(q_b^{-1}(x_b))$. It is clear that $g_b q_b = f$.

Let us prove that g_b is continuous. Let U be open in Y. Then $g_b^{-1}(U)$ is open since $(q_b)^{-1}(g_b^{-1}(U)) = f^{-1}(U)$ is open and q_b is quotient (as a closed mapping).

Step 6. If the mapping f is monotone, then g_b is monotone.

This follows from the relation $g_b q_b = f$. □

Lemma 2.3. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a usual $\aleph_1$-directed inverse system of compact metric spaces X_a and surjective bonding mappings. Then $X = \lim \mathbf{X}$ is a compact metrizable space.*

Proof. Let us prove that if $f : X \to Y$ is a continuous surjection, then Y is a metrizable space. We shall use transfinite induction on $w(Y)$. If $w(Y) \leq \aleph_1$, then there exists an $a \in A$ and a surjective mapping $g_a : X_a \to Y$ such that $f = g_a p_a$ (Lemma 2.2). It follows that Y is a metrizable space. Suppose that this is true for each space Y with $w(Y) < \aleph_\beta$. Let us prove that this is true for the spaces Y with $w(Y) = \aleph_\beta$. By virtue of Theorem 5.9 there exists a well-ordered inverse system $\mathbf{Y} = \{Y_\mu, \rho_{\mu\nu}, M\}$ such that $Y = \lim \mathbf{Y}$ and $w(Y_\mu) < \aleph_\beta$. Moreover, by inductive hypothesis it follows that $w(Y_\mu) = \aleph_0$ since there exists a mapping $\rho_\mu f : X \to Y_\mu$. By [17, Theorem 2.2] it follows that $w(Y) \leq \aleph_1$. Now, there exists an $a \in A$ and a surjective mapping $g_a : X_a \to Y$ such that $f = g_a p_a$. This means that Y is a metrizable space. Finally, we shall prove that $w(X) = \aleph_0$. By virtue of Theorem 5.9 there exists a well-ordered inverse system $\mathbf{Y} = \{Y_\mu, \rho_{\mu\nu}, M\}$ such that $X = \lim \mathbf{Y}$ and $w(Y_\mu) < \aleph_\beta$. Moreover, $w(Y_\mu) = \aleph_0$. By [17, Theorem 2.2] it follows that $w(Y) \leq \aleph_1$. There exists an $a \in A$ and a surjective mapping $g_a : X_a \to Y$ such that $f = g_a p_a$. This means that X is a metrizable space. □

Let X be a non-degenerate locally connected continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property of containing no separating point of itself. A cyclic element of locally connected continuum is again locally connected continuum. We let

$$\mathbf{L}_X = \{Y \subset X : Y \text{ is a non-degenerate cyclic element of } X\}. \tag{2.1}$$

For other details see [2, 16].

Now we shall prove the main theorem of this paper.

Theorem 2.4. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a usual $\aleph_1$-directed inverse system of continuous images of arcs and monotone bonding mappings. Then $X = \lim \mathbf{X}$ is the continuous image of an arc.*

Proof. By [2, Theorem 1] it suffices to prove that each cyclic element Z of X is a continuous image of arc. From [16, Theorem 2.7] it follows that there exists a usual inverse system $(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$ of cyclic elements of X_γ such that $Z = \lim \operatorname{inv}(Z_\gamma, g_{\gamma\gamma'}, \Gamma)$. It is clear that each Z_γ is a continuous image of an arc. Let x, y, and z be distinct points of Z. There exists an $a \in A$ such that for each $b \geq a$ $g_b(x)$, $g_b(y)$, $g_b(y)$ are different points of Z_b. Moreover there exists a minimal metrizable T-set T_b which contains $g_b(x)$, $g_b(y)$ and $g_b(y)$. For each $b \geq a$ the family $\{p_{bc}(T_c) : c \geq b\}$ satisfies the assumptions of Lemma 2.1. This means that $N_b = \bigcup\{g_{bc}(T_c) : c \geq b\}$ is a compact metric subspace of Z_b. Moreover, $g_{bc}(N_c) = N_b$. We have a usual inverse system $\mathbf{N} = \{N_b, g_{bc} \mid N_c, a \leq b \leq c\}$ which satisfies the assumptions of Lemma 2.3. It follows that $N = \lim \mathbf{N}$ is compact and metrizable. Moreover, by [7, Lemma 2.15] it follows that N is a T-set in X which contains x, y, and z. From [16, Theorem 4.4] it follows that Z is a continuous image of an arc. □

Remark 2.5. Let us observe that Theorem 2.4 is not true for σ-directed inverse system. This is shown by an example of Nikiel [14, Example 4.3]. The sufficient and necessary condition for the limit of σ-directed inverse system to be the continuous image of an arc are given in [7, Theorem 2.21].

Lemma 2.6. *Let B be an infinite subset of directed set A. There exists a directed subset $F_\infty(B)$ of A such that $B \subseteq F_\infty(B)$ and $\operatorname{card}(F_\infty(B)) = \operatorname{card}(B)$.*

Proof. Let ν be any finite subset of A. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup\{\delta(\nu) : \nu \in B\}$, where ν is a finite subset of B. Put

$$F_{n+1} = F_1(F_n(B)), \qquad F_\infty(B) = \bigcup\{F_n(B) : n \in \mathbb{N}\}. \tag{2.2}$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \cdots \subseteq F_n(B) \subseteq \cdots. \tag{2.3}$$

The set $F_\infty(B)$ is directed since each finite subset ν of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$. From $\operatorname{card}(B) \geq \aleph_0$, it follows $\operatorname{card}(\{\delta(\nu) : \nu \in B\}) \leq \operatorname{card}(B)\aleph_0$. We infer that $\operatorname{card}(F_1(B)) \leq \operatorname{card}(B)\aleph_0$. Similarly, $\operatorname{card}(F_n(B)) \leq \operatorname{card}(B)\aleph_0$. This means that $\operatorname{card}(F_\infty(B)) \leq \operatorname{card}(B)\aleph_0$. Thus

$$\operatorname{card}\left(F_\infty(B)\right) \leq \operatorname{card}(B)\aleph_0. \tag{2.4}$$

We infer that $\operatorname{card}(F_\infty(B)) = \operatorname{card}(B)$. □

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a usual inverse system of compact spaces and let $\tau < \operatorname{card}(A)$ be an infinite cardinal. Consider the set A_τ of all $F_\infty(B)$, $B \subseteq A$, $\operatorname{card}(B) = \tau$, ordered by inclusion. It is clear that A_τ is τ-directed. Each element α of A_τ is some $F_\infty(B)$. We define X_α as the limit of $\{X_a, p_{ab}, F_\infty(B)\}$. Let $\alpha = F_\infty(B)$ and $\beta = F_\infty(C)$. If $\alpha \subseteq \beta$, then there exists the natural projection $q_{\alpha\beta} : X_\beta = \lim\{X_a, p_{ab}, F_\infty(C)\} \to X_\alpha = \lim\{X_a, p_{ab}, F_\infty(B)\}$. It is clear that $q_{\alpha\gamma} = q_{\alpha\beta}q_{\beta\gamma}$ if $\alpha \leq \beta \leq \gamma$. It follows that $\{X_\alpha, q_{\alpha\beta}, A_{\omega_1}\}$ is a usual inverse system.

Theorem 2.7. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a usual inverse system of compact spaces with limit X. For each infinite cardinal $\tau < \operatorname{card}(A)$ there exists a τ-directed usual inverse system $\{X_\alpha, q_{\alpha\beta}, A_\tau\}$ such that X is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, A_\tau\}$.*

Proof. The proof is the same as the proof of [16, Theorem 9.4]. □

COROLLARY 2.8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a usual inverse system of compact spaces with limit* X. *If* $\aleph_1 < \operatorname{card}(A)$, *then there exists an* $\aleph_1$*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, A_{\aleph_1}\}$ *such that* X *is homeomorphic to* $\lim\{X_\alpha, q_{\alpha\beta}, A_{\aleph_1}\}$.

COROLLARY 2.9. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a usual inverse system of compact spaces with limit* X. *If* $\aleph_0 < \operatorname{card}(A)$, *then there exists a* σ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, A_{\aleph_0}\}$ *such that* X *is homeomorphic to* $\lim\{X_\alpha, q_{\alpha\beta}, A_{\aleph_0}\}$.

At the end of this section, we shall prove the following theorem.

THEOREM 2.10. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a usual inverse system of continuous images of arcs with limit* X *and* $\operatorname{cf}(\operatorname{card}(A)) \neq \aleph_1$. *If the mappings* p_{ab} *are monotone surjections, then the following are equivalent:*

(a) X *is the continuous image of an arc.*

(b) *Each proper subsystem* $\mathbf{Y} = \{X_b, p_{bc}, B\}$ *of* $\mathbf{X}$ *with* $\operatorname{card}(B) = \aleph_1$ *has the limit* Y *which is the continuous image of an arc.*

PROOF. (a)⇒(b). Obvious since there exists the natural projection $\mathrm{f}_q : \lim \mathbf{X} \to \lim \mathbf{Y}$.

(b)⇒(a). Inverse system $\{X_\alpha, q_{\alpha\beta}, A_{\aleph_1}\}$ is $\aleph_1$-directed. Apply Theorem 2.4. □

3. Characterizing spaces by images of the weight less than or equal to $\aleph_1$.

In this section, we shall characterize a compact locally connected space by its images of the weight $\leq \aleph_1$.

A continuum X is said to be *hereditarily locally connected* if each subcontinuum of X is locally connected.

We start with the following theorem.

THEOREM 3.1. *Let* X *be a compact space. The following are equivalent:*

(a) X *is locally connected.*

(b) *If* $f : X \to Y$ *is a surjection and* Y *is a metric space, then* Y *is locally connected.*

PROOF. (a)⇒(b). If X is locally connected, then from [19, Theorem 1.6, page 70] it follows that Y is locally connected.

(b)⇒(a). By Theorem 5.3 there exists a σ-directed usual inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. By (b) each X_a is locally connected. Finally, by Theorem 5.6 we infer that X is locally connected. □

THEOREM 3.2. *Let* X *be a locally connected continuum. The following are equivalent:*

(a) X *is hereditarily locally connected.*

(b) *If* Z *is a subcontinuum of* X *and if* $f : Z \to Y$ *is a surjection onto a metric space* Y, *then* Y *is locally connected.*

PROOF. (a)⇒(b). If X is hereditarily locally connected, then each subcontinuum $Z \subseteq X$ is locally connected. By [19, Theorem 1.6, page 70] it follows that Y is locally connected.

(b)⇒(a). By Theorem 5.4 there exists a σ-directed usual inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective monotone bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. Now, each X_a is hereditarily locally connected.

Namely, for each subcontinuum K of X_a we have a subcontinuum $Z = p_a^{-1}(K)$ and a mapping $p_a \mid Z : p_a \mid Z : Z \to K$. By (b) K is locally connected. Hence, each X_a is hereditarily locally connected. From Theorem 5.7 it follows that X is hereditarily locally connected. □

From Theorem 2.4 it follows the following characterization of continuous images of arcs.

Theorem 3.3. *Let X be a locally connected continuum X with $\operatorname{cf}(w(X)) \neq \omega_1$. The following are equivalent:*

(a) *X is a continuous image of an arc.*

(b) *If $f : X \to Y$ is a continuous surjection and $w(Y) = \aleph_1$, then Y is a continuous image of an arc.*

Proof. (a)⇒(b). Obvious.

(b)⇒(a). By [3, Theorem 2.3.23] the space X is embeddable in $I^{w(X)}$. Since $I^{w(X)}$ is an inverse limits of finite cube I^n, we infer that there exists a usual inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric spaces X_a such that X is homeomorphic to $\lim\mathbf{X}$. It is clear that there exists no a subset B which is cofinal in A and $\operatorname{card}(B) = \aleph_1$. This means that there exists an $\aleph_1$-directed usual inverse system $\{X_\alpha, q_{\alpha\beta}, A_{\aleph_1}\}$ such that X is homeomorphic to $\lim\{X_\alpha, q_{\alpha\beta}, A_{\aleph_1}\}$ and $w(X_a) \leq \aleph_1$. By (b) each X_a is the continuous image of an arc. The system $\{X_\alpha, q_{\alpha\beta}, A_{\aleph_1}\}$ satisfies the conditions of Theorem 2.10. Hence, X is the continuous image of an arc. □

Theorem 3.4. *Let X be a locally connected nonmetrizable continuum such that $\operatorname{cf}(w(Z)) \neq \omega_1$ for each cyclic element Z of X. The following statements are equivalent:*

(a) *X is a continuous image of an arc.*

(b) *If Z is a non-degenerate cyclic element of X or $Z = X$, $f : Z \to Y$ is a surjection onto a space Y with $w(Y) = \omega_1$, then Y is a continuous image of an arc.*

Proof. (a)⇒(b). Let Z be a cyclic element of X. By [2, Theorem 1] Z is a continuous image of an arc. It is clear that if $f : Z \to Y$ is a surjection, then Y is a Y is a continuous image of an arc.

(b)⇒(a). Repeat the proof of the implication (b)⇒(a) of Theorem 3.3 replacing X by Z. □

Corollary 3.5. *Let X be a locally connected nonmetrizable continuum of regular weight $w(Z) > \omega_1$ for each cyclic element Z of X. The following statements are equivalent:*

(a) *X is a continuous image of an arc.*

(b) *If Z is a non-degenerate cyclic element of X or $Z = X$, $f : Z \to Y$ is a surjection onto a space Y with $w(Y) = \omega_1$, then Y is a continuous image of an arc.*

4. $\aleph_1$-directed approximate inverse systems. In this section, we will apply theorems from the last section to obtain some theorems for approximate limits. At the begin we shall prove the following theorem.

Theorem 4.1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed approximate inverse system of locally connected compact spaces. Then $X = \lim\mathbf{X}$ is locally connected.*

Proof. By Theorem 3.1 it suffices to prove that if $f : X \to Y$ is a surjection and Y is a metric continuum, then Y is locally connected. From Lemma 2.2 (for $Z = X$) it follows that there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : X_b \to Y$. We infer that Y is locally connected since X_a is locally connected. Hence, X is locally connected. □

Theorem 4.2. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed approximate inverse system of hereditarily locally connected continua. Then $X = \lim \mathbf{X}$ is hereditarily locally connected.*

Proof. By Theorem 3.2 it suffices to prove that if $f : X \to Y$ is a surjection and Y is a metric continuum, then Y is locally connected. By virtue of Theorem 4.1 it follows that X is locally connected. From Lemma 2.2 it follows that there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : p_b(Z) \to Y$. Now, $p_b(Z)$ is locally connected since X_b is hereditarily locally connected. We infer that Y is locally connected since $p_b(Z)$ is locally connected. Thus, X is hereditarily locally connected. □

Theorem 4.3. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continuous images of arcs X_a with the limit X. If $\mathbf{X}$ is $\aleph_1$-directed and $\operatorname{cf}(w(X)) \neq \omega_1$, then X is a continuous image of an arc.*

Proof. By virtue of Theorem 4.1 X is locally connected. From Theorem 3.3 it follows that it suffices to prove that if $f : X \to Y$ is a continuous surjection and $w(Y) = \aleph_1$, then Y is a continuous image of an arc. Now, using Lemma 2.2 we infer that there exists a $b \in A$ and a surjective mapping $g_b : X_b \to Y$. It follows that Y is a continuous image of an arc since X_b is a continuous image of an arc. □

Remark 4.4. Let us observe that the bonding mappings in Theorem 4.3 are not assumed to be monotone.

Corollary 4.5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continuous images of arcs X_a with the limit X. If $\mathbf{X}$ is $\aleph_1$-directed and if $w(X)$ is a regular cardinal $> \omega_1$, then X is a continuous image of an arc.*

Theorem 4.6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continuous images of arcs X_a with the limit X. If $\mathbf{X}$ is $\aleph_1$-directed and $\operatorname{cf}(w(Z)) \neq \omega_1$ for each cyclic element Z of X, then X is a continuous image of an arc.*

Proof. The proof is broken into several steps.

Step 1. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continua and let Y be a subcontinuum of $X = \lim \mathbf{X}$. If $r(p_a(Y)) \leq k$ for all $a \in A$ and some fixed integer $k \geq 0$, then $r(Y) \leq k$.

Let C, D be a pair of subcontinua of Y. First, we prove that for each pair K, L of different components of $C \bigcap D$ there is an $a \in A$ such that $p_b(K)$, $p_b(L)$ are subsets of different components of $p_b(C) \bigcap p_b(D)$, for each $b \geq a$. Suppose, on the contrary, that there is a set B, cofinal in A, such that $p_b(K)$ and $p_b(L)$, $b \in B$, lie in the same component C_b of $p_b(C) \bigcap p_b(D)$. We have a net $\{h_b(C_b) : b \in B\}$ which has a nonempty $\operatorname{Li}\{h_b(C_b) : b \in B\} = K \bigcup L$. By virtue of [8, Lemma 2.10] $\operatorname{Ls}\{h_b(C_b) : b \in B\}$ is a nonempty subcontinuum of $C \bigcap D$ which contains $K \bigcup L$. This is impossible since

K and L are all the components of $C \bigcap D$. Second, we prove that $C \bigcap D$ has $\leq k+1$ components. Suppose that $C \bigcap D$ has $\geq k+2$ components $K_1, \ldots, K_{k+2}, \ldots$. Consider the family $\{K_1, \ldots, K_{k+1}, K_{k+2}\}$. For each pair K_i, K_j there exists an $a_{ij} \in A$ such that $p_b(K_i)$, $p_b(K_j)$ lie in different components of $p_b(C) \bigcap p_b(D)$ for each $b \geq a_{ij}$. From the directedness of A it follows that there is an index $b \in B$, $b \geq a_{ij}$, $i, j = 1, \ldots, k+2$. We infer that $p_b(C) \bigcap p_b(D)$ has $\geq k+2$ components.This is impossible and the proof of Step 1 is completed.

Step 2. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continua and let Y be a subcontinuum of $X = \lim \mathbf{X}$. If $p_a(Y)$ is hereditarily unicoherent for all $a \in A$, then Y is hereditarily unicoherent.

Step 3. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of locally connected continua X_a with monotone bonding mappings and let Z be a non-degenerate cyclic element of $X = \lim \mathbf{X}$. There exists an a_0 such that $\mathbf{L}_b(Z_b) \neq \emptyset$.

Suppose that $\mathbf{L}_{Z_b} = \emptyset$ for each $b \in B$ (i.e., all Z_b are dendron), where B is cofinal in A. By Step 2, Z is hereditarily unicoherent. This means that Z is a dendron since it is locally connected. Thus, $L_Z = \emptyset$. This contradicts the assumption that $Z \in \mathbf{L}_X$.

Step 4. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of locally connected continua X_a with monotone bonding mappings and let Z be a cyclic element of $X = \lim \mathbf{X}$. If Y_a is a non-degenerate cyclic element of $Z_a = p_a(Z)$, then $p_a^{-1}(Y_a) \subseteq Z$ and Y_a is a cyclic element of X_a.

By virtue of [8, Corollary 5.6] the projections p_a are monotone. By virtue of [16, Lemma 2.3] there exists a cyclic element W of X such that $p_a^{-1}(Y_a) \subseteq W$. This means that $\operatorname{card}(Z \bigcap W) \geq 2$. We infer that $Z = W$. Similarly, if W_a is a cyclic element in X_a containing Y_a, then $p_a^{-1}(W_a) \subseteq Z$. Thus, $W_a \subseteq Z_a$. Hence, Y_a is a cyclic element of X_a.

Step 5. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of continuous images of arcs X_a with monotone bonding mappings and let Z be a cyclic element of $X = \lim \mathbf{X}$. Then each $Z_a = p_a(Z)$ is a continuous image of an arc.

It suffices to prove that each cyclic element W_a of Z_a is a continuous image of an arc. If W_a is degenerate, then it is a continuous image of an arc. If W_a is non-degenerate, then by Step 4, W_a is a cyclic element of X_a. From [2, Theorem 1] it follows that W_a is a continuous image of an arc. Hence, Z_a is a continuous image of an arc.

Step 6. Finally, let us prove Theorem 4.6. Let Z be any cyclic element of X, $f : Z \to Y$ a surjection and $w(Y) = \aleph_1$. From Lemma 2.2 it follows that there exists an $a \in A$ and $g_a : Z_a \to Y$. We infer that Y is the continuous image of an arc since Z_a is the continuous image of an arc (Step 5). By Corollary 3.5 we infer that X is the continuous image of an arc. □

We close this section with two questions.

In connection with [16, Theorem 5.1] it is naturally to ask the following question.

Question 4.7. Let $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of continuous images of arcs with monotone bonding mappings. Is it true that $X = \lim \mathbf{X}$ is the continuous image of an arc?

Comparing Theorems 2.4 and 4.3 it is naturally to ask the following question.

Question 4.8. Suppose that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an $\aleph_1$-directed approximate inverse system of continuous image of arcs and monotone bonding mappings. Does it follow that the approximate inverse limit space $X = \lim \mathbf{X}$ is a continuous image of an arc?

5. Appendix. $\mathrm{Cov}(X)$ is the set of all normal coverings of a topological space X. For other details see [1].

Definition 5.1. An approximate inverse system is a collection $\mathbf{X} = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, X_a, $a \in A$, is a topological space and $p_{ab} : X_b \to X_a$, $a \leq b$, X_a, $a \leq b$, are mappings such that $p_{aa} = \mathrm{id}$ and the following condition (A2) is satisfied:

(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is an index $b \geq a$ such that $(p_{ac}p_{cd}, p_{ad}) \prec \mathcal{U}$, whenever $a \leq b \leq c \leq d$.

Other basic notions, including approximate mapping and the limit of an approximate inverse system are defined as in [11, 13].

An inverse system in the sense [3, page 135] we call a *usual inverse system*.

An approximate inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be *commutative* [13, Definition 1.4] provided it satisfies the commutativity condition:

$$p_{ab}p_{bc} = p_{ac}, \quad \text{for } a < b < c. \tag{5.1}$$

Let us observe that if $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a commutative approximate inverse system of Tychonoff spaces, then the limit of $\mathbf{X}$ in the usual sense and the approximate limit coincide [13, Remark 1.15].

A basis of (open) normal coverings of a space X is a collection $\mathcal{C}$ of normal coverings such that every normal covering $\mathcal{U} \in \mathrm{Cov}(X)$ admits a refinement $\mathcal{V} \in \mathcal{C}$. In this case, we write $\mathcal{V} \prec \mathcal{U}$. We denote by $\mathrm{cw}(X)$ (*covering weight*) the minimal cardinal of a basis of a normal coverings of X [12, page 181].

Lemma 5.2 (see [12, Example 2.2])**.** *If X is a compact Hausdorff space, then* $\mathrm{cw}(X) = w(X)$.

Theorem 5.3. *Let X be a compact spaces. There exists a σ-directed usual inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Proof. Apply [10, pages 152 and 164] and Corollary 2.9. □

Theorem 5.4. *If X is a locally connected compact space, then there exists a usual σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric locally connected compact space, each p_{ab} is a monotone surjection and X is homeomorphic to* $\lim \mathbf{X}$. *Conversely, the inverse limit of a such system is always a locally connected compact space.*

Proof. Apply [10, Theorem 2] and Corollary 2.9. □

Theorem 5.5 (see [4, Theorem 5])**.** *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a usual σ-directed inverse system of locally connected continua X_a. Then $X = \lim \mathbf{X}$ is locally connected.*

Theorem 5.6 (see [6, Theorem 5]). *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a usual σ-directed inverse system of locally connected locally compact spaces X_a and perfect mappings p_{ab}. Then $X = \lim \mathbf{X}$ is locally connected.*

Theorem 5.7 (see [4, Corollary 3]). *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of hereditarily locally connected continua X_a. Then $X = \lim \mathbf{X}$ is hereditarily locally connected.*

Theorem 5.8 (see [15, Corollary 2.9]). *If X is a hereditarily locally connected continuum, then there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metrizable hereditarily locally connected continuum, each p_{ab} is a monotone surjection and X is homeomorphic to $\lim \mathbf{X}$.*

In the next theorem, we assume that $w(X)$ is an aleph, i.e., $w(X) = \aleph_{\tau(X)}$ and that $\omega_{\tau(X)}$ is the initial ordinal number belonging to $\aleph_{\tau(X)} = w(X)$ [9, page 279].

Theorem 5.9 (see [9, Theorem 3]). *Every nonmetrizable (Hausdorff) compact space X is homeomorphic with the inverse limit of an inverse system $\{X_\beta, p_{\beta\beta'}\}$, where β ranges through all the ordinals $\beta < \omega_{\tau(X)}$, while X_β are (Hausdorff) compact spaces satisfying*

(1) $\dim X_\beta \leq \dim X$.

(2) $w(X_\beta) < w(X)$.

Moreover,

(3) $w(X_\beta) \leq \operatorname{card}(\beta) = \operatorname{card}(\{\alpha : \alpha < \beta\})$, $\omega_0 \leq \beta < \omega_{\tau(X)}$.

If β is a limit ordinal, then

(4)
$$X_\beta = \lim \{X_\alpha,\ p_{\alpha\alpha'}\}, \quad \alpha < \beta, \tag{5.2}$$

$p_{\alpha\beta} : X\beta \to X_\alpha$ being the corresponding projections.

Corollary 5.10. *Every nonmetrizable compact locally connected space X is homeomorphic to the inverse limit of a transfinite inverse sequence $\mathbf{X} = \{X_a p_{ab}, w(X)\}$, where p_{ab} are monotone surjection, X_a are compact locally connected spaces such that $w(X_a) < w(X)$ and $w(X_a) \leq \operatorname{card}(a)$ provided $\omega_0 \leq a < w(X)$.*

If c is limit and $0 < c < w(X)$, then $X_c = \lim\{(X_a, p_{ab}, c)\}$ and the corresponding maps $p_{ab} : p_{ab} : X_b \to X_a$ are the corresponding projections for $a < c$.

Proof. Let $(X_\alpha, f_\beta^\alpha, w(X))$ be the inverse system from Theorem 5.9. Applying the monotone-light factorization [18] to the natural projections $f_\alpha : X \to X_\alpha$, we get compact spaces X_a, monotone surjection $m_a : X \to X_a$ and light surjections $l_a : X_a \to X_\alpha$ such that $f_\alpha = l_a \circ m_a$, $a < w(X)$. By [10, Lemma 8] there exist monotone surjections $p_{ab} : X_a \to X_b$ such that $p_{ab} \circ m_a = m_b$, $a \leq b < w(X)$. It follows that $\mathbf{X} = \{X_a, p_{ab}, w(X)\}$ is a transfinite inverse sequence such that X is homeomorphic to $\lim \mathbf{X}$. It is obvious that each X_a is locally connected. Moreover, by [10, Theorem 1] it follows that $w(X_a) = w(X_\alpha)$. $\square$

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