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A NOTE ON RAGHAVAN-REILLY'S PAIRWISE PARACOMPACTNESS

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ABSTRACT. The bitopological unstability of RR-pairwise paracompactness in presence of pairwise Hausdorff separation axiom is caused by a bitopological property which is much weaker and more local than RR-pairwise paracompactness. We slightly generalize some Michael's constructions and characterize RR-pairwise paracompactness in terms of bitopological θ-regularity, and some other weaker modifications of pairwise paracompactness.

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1. Preliminaries. Recall that a topological space X is (countably) *θ-regular* [3] (see [5] for countable version), if every filter base with a θ-cluster point has a cluster point. A topological space X is said to be *a-paracompact* [2] if every open cover of X has a locally finite (not necessarily open) refinement. A family is called *σ-locally finite* if it consists of countably many locally finite subfamilies (this notion has nothing common with the topology that also may be denoted by σ). Let Φ be a family of subsets of X. We denote by Φ^F the collection of all finite unions of members of Φ. A family Φ is called *directed* if Φ^F is a refinement of Φ.

Let X be a set with three topologies τ, σ, and ρ. We say that X is $(\tau-\sigma)$ *(semi)paracompact with respect to* ρ if every τ-open cover of X has a σ-open refinement which is (σ-) locally finite with respect to the topology ρ (see [7]). In [4], there was defined the following generalization of θ-regularity. We say that X is (*countably*) (τ,σ,ρ)-*θ-regular* if for every (countable) τ-open cover Ω of X and each $x \in X$ there exists a σ-open neighborhood U of x such that $\mathrm{cl}_\rho U$ can be covered by a finite subfamily of Ω. Note that in [4] the (countably) (τ,σ,ρ)-θ-regular spaces are also characterized in terms of filter bases and convergence. However, the covering characterization will be more useful for our purposes. Similarly as θ-regularity, there are numbers of simple examples of (τ,σ,ρ)-θ-regular spaces, including various modifications of regularity, compactness, local compactness or paracompactness and we leave them to the reader. For example, a space $(\tau-\rho)$ paracompact with respect to σ is (τ,σ,ρ)-θ-regular. In particular, if $\tau=\sigma=\rho$, we obtain an alternative definition of (countable) θ-regularity (see also [5]).

In [7], Raghavan and Reilly investigated various modifications of pairwise paracompactness. Let X be a bitopological space with the topologies τ and σ. Then the space X is called *RR-pairwise (semi)paracompact* if X is $(\tau-\tau)$ (semi)paracompact with respect to σ and $(\sigma - \sigma)$ (semi)paracompact with respect to τ . We say that X is FHP-*pairwise* (semi)paracompact if X is $(\tau - \sigma)$ (semi)paracompact with respect to σ and $(\sigma - \tau)$ (semi)paracompact with respect to τ. The bitopological space X is said to be *α-pairwise paracompact* if X is $(\tau - (\tau \vee \sigma))$ paracompact with respect to σ and $(\sigma - (\tau \vee \sigma))$ paracompact with respect to τ. We say that the space X is β-pairwise paracompact if X is $(\tau - (\tau \vee \sigma))$ paracompact with respect to τ and $(\sigma - (\tau \vee \sigma))$ paracompact with respect to σ. Finally, the bitopological space X is said to be *bi-paracompact* if both of its topologies are paracompact. Most of the previous concepts were originally introduced by Raghavan and Reilly, with one exception—the concept of FHP-pairwise paracompactness is due to Fletcher, Hoyle III and Patty. The notion of a-paracompactness we can naturally extend to bitopological spaces. We simply say that a bitopological space X is *bi-a-paracompact* if both its topologies are a-paracompact. In [4], there are studied several variants of pairwise θ-regularity. For this paper, the following one will be sufficient. A bitopological space X is called *β-pairwise (countably) θ-regular* if X is (countably) (τ, σ, τ)-θ-regular and (countably) (σ, τ, σ)-θ-regular.

2. Results. Let us start with the following proposition which is essentially contained in the proof of Boyte's [1] generalization of well-known Michael's theorem (see [1, Theorem 9]). In fact, the proof is based on a modification of Michael's technique of an expansion of a locally finite cover to an open locally finite cover. One of the nice properties of θ-regularity is that it can fully substitute regularity in Michael's proof. Since we will use the proposition as our starting point, we will repeat the proof in a brief, sketch form.

Proposition 2.1. *Let X be a topological space. Then X is paracompact if and only if X is θ-regular and a-paracompact.*

Sketch of the proof. Let X be θ-regular and a-paracompact. Let Ω be an open cover of X. Since X is a-paracompact, Ω has a locally finite (not necessarily open) refinement, say Γ. Then there exists an open cover Φ of X, such that every element of Φ meets only finitely many members of Γ. From θ-regularity of X it follows that there is an open cover Ψ of X such that the closures of members of Ψ can be covered by finitely many elements of the cover Φ. Hence, for every $S \in \Psi$, the set $\operatorname{cl} S$ meets only finitely many members of Γ. Let Ξ be a locally finite refinement of ψ. We put

$$U(G) = X \setminus \bigcup \{\operatorname{cl} F \mid F \in \Xi,\ G \cap \operatorname{cl} F = \varnothing\} \tag{2.1}$$

for every $G \in \Gamma$. By a standard process due to Michael [6] one can easily check that $\{U(G) \mid G \in \Gamma\}$ is an open locally finite expansion of Γ. For every $G \in \Gamma$ there is some $V(G) \in \Omega$ with $G \subseteq V(G)$. Let $W(G) = U(G) \cap V(G)$. The cover $\{W(G) \mid G \in \Gamma\}$ is the desired open locally finite refinement of Ω. So X is paracompact. Conversely, a paracompact space obviously is θ-regular and a-paracompact. $\square$

A considerable problem of bitopological spaces is the bitopological unstability with respect to presence of the pairwise Hausdorff separation axiom. If a pairwise Hausdorff bitopological space satisfies some covering properties, for example FHP-pairwise paracompactness, RR-pairwise paracompactness or α-pairwise paracompactness, its two topologies may collapse and revert to the unitopological setting. The following theorem shows that the critical covering property causing the collapse of the bitopological setting is much weaker and more local than some variants of pairwise paracompactness.

Theorem 2.2. *Let X be a bitopological space with the topologies τ and σ. Then X is β-pairwise θ-regular and pairwise Hausdorff if and only if $\tau = \sigma$ and the topological space is regular and T_1.*

Proof. If X is regular T_1 topological space, obviously it is θ-regular and Hausdorff. If we formally consider X as bitopological space we immediately obtain the desired conclusion. Conversely, suppose that X is β-pairwise θ-regular and pairwise Hausdorff. Let $U \in \tau$ and $x \in U$. We show that there exists $W \in \sigma$ such that $x \in W$ and $\mathrm{cl}_\tau W \subseteq U$. Since X is pairwise Hausdorff, for every $y \in X \setminus U$ there exist $U_y \in \tau$, $V_y \in \sigma$ such that $x \in V_y$, $y \in U_y$ and $U_y \cap V_y = \emptyset$. Then $\Omega = \{U\} \cup \{U_y \mid y \in X \setminus U\}$ is a τ-open cover of X, which implies that there exists $V \in \sigma$ and $y_1, y_2, \dots, y_k \in X \setminus U$ such that $\mathrm{cl}_\tau V \subseteq U \cup (\bigcup_{i=1}^k U_{y_i})$. Let $W = V \cap (\bigcap_{i=1}^k V_{y_i})$. Clearly, $x \in W$ and it follows that $\mathrm{cl}_\tau W \subseteq U \cup (\bigcup_{i=1}^k U_{y_i})$. On the other hand, for every $i \in \{1, 2, \dots, k\}$ we have $\mathrm{cl}_\tau W \cap U_{y_i} = \emptyset$. Hence, $W \subseteq \mathrm{cl}_\tau W \subseteq U$. Then $U \in \sigma$, which implies that $\tau \subseteq \sigma$. From the symmetry it also follows $\sigma \subseteq \tau$, which consequently gives $\tau = \sigma$. Obviously, this topology is regular and T_1. □

Since RR-pairwise paracompact spaces and FHP-pairwise paracompact spaces are β-pairwise θ-regular [4], the bitopological unstability of RR-pairwise paracompact or FHP-paracompact spaces now follows as a corollary.

Lemma 2.3. *Let X be a β-pairwise θ-regular bitopological space with the topologies τ and σ. Let Φ be a collection of subsets of X. Then Φ is locally finite with respect to τ if and only if it is locally finite with respect to σ.*

Proof. Suppose that Φ is locally finite with respect to one of the topologies, say τ. There exists a τ-open cover Ω of X such that any element of Ω meets only finitely many members of Φ. Let $x \in X$. Since X is β-pairwise θ-regular, there exists a σ-open neighborhood U of x, such that $\mathrm{cl}_\tau U$ can be covered by finitely many elements of Ω. Hence U meets only finitely many members of Φ, so Φ is locally finite also with respect to σ. □

Theorem 2.4. *Let X be a bitopological space. The following statements are equivalent:*

(i) *X is RR-pairwise paracompact.*
(ii) *X is β-pairwise θ-regular and α-pairwise paracompact.*
(iii) *X is β-pairwise θ-regular and β-pairwise paracompact.*
(iv) *X is β-pairwise θ-regular and bi-a-paracompact.*
(v) *X is β-pairwise θ-regular and bi-paracompact.*

Proof. The implications (i)⇒(ii) and (iii)⇒(iv) are clear. Since in a β-pairwise θ-regular space both topologies are θ-regular [4], from Proposition 2.1 it follows (iv)⇒(v) Finally, from Lemma 2.3 it follows (ii)⇒(iii) and (v)⇒(i). □

From the condition (v) it follows that a RR-pairwise paracompact space can be considered as a space with two relatively independent paracompact topologies only connected together by much local, regularity-like bitopological property, β-pairwise θ-regularity, which is responsible for the pairwise paracompact effect. For instance, if both topologies are compact β-pairwise θ-regularity is satisfied automatically. Since FHP-pairwise paracompactness implies β-pairwise θ-regularity and α-pairwise paracompactness, from (ii) we immediately obtain the following corollary.

Corollary 2.5. *Every* FHP-*pairwise paracompact bitopological space is RR-pairwise paracompact.*

Example 2.6. Consider $\mathbb{N} = \{1,2,\dots\}$ topologized by $\tau = \{U \mid u \subseteq \mathbb{N},$ if $1 \in U$ then $\mathbb{N} \setminus U$ is finite$\}$ and $\sigma = \{V \mid V \subseteq \mathbb{N},$ if $2 \in V$ then $\mathbb{N} \setminus V$ is finite$\}$. Both the topologies τ and σ are compact, which implies that $\mathbb{N}$ is RR-pairwise paracompact. But $\Omega = \{\{2\}, \mathbb{N} \setminus \{2\}\}$ is a τ-open cover of $\mathbb{N}$ which has no σ-open refinement. Therefore, $\mathbb{N}$ is not FHP-pairwise paracompact. Observe, that both τ and σ are T_1 and $\mathbb{N}$ is β-pairwise θ-regular but not pairwise Hausdorff.

Remark 2.7. From the previous example it follows that RR-pairwise paracompactness is strictly weaker and more general than FHP-pairwise paracompactness. However, modifying slightly the definition of FHP-pairwise paracompactness, we can obtain a FHP-like characterization of RR-pairwise paracompactness.

Theorem 2.8. *A bitopological space X with the topologies τ and σ is RR-pairwise paracompact if and only if every directed τ-open cover has a σ-open refinement locally finite with respect to σ and every directed σ-open cover has a τ-open refinement locally finite with respect to τ.*

Proof. Suppose that X satisfies the conditions stated in the theorem. Then X is (τ,σ,σ)-θ-regular and (σ,τ,τ)-θ-regular. Let Ω be a τ-open cover of X. There is a σ-open directed cover Φ of X such that the family $\operatorname{cl}_\sigma \Phi$ refines Ω^F. Similarly, there exists a τ-open directed cover Ψ of X such that $\operatorname{cl}_\tau \Psi$ refines Φ. And finally, there is some σ-open cover Ξ of X such that the family $\operatorname{cl}_\sigma \Xi$ refines Ψ. Let $x \in X$. There exist some $V \in \Xi$, $W \in \Psi$, $U \in \Phi$ and $U_1, U_2, \dots, U_k \in \Omega$ such that $x \in V \subseteq \operatorname{cl}_\sigma V \subseteq W \subseteq \operatorname{cl}_\tau W \subseteq U \subseteq \operatorname{cl}_\sigma U \subseteq \bigcup_{i=1}^k U_i$. Then $\operatorname{cl}_\tau V \subseteq \bigcup_{i=1}^k U_i$, which implies that X is (τ,σ,τ)-θ-regular. Reversing the role of τ and σ we obtain that X is also (σ,τ,σ)-θ-regular, which implies that X is β-pairwise θ-regular. But X obviously is α-pairwise paracompact, so X is RR-pairwise paracompact by Theorem 2.4. Conversely, if X is RR-pairwise paracompact, then X is β-pairwise θ-regular. Hence every directed cover of X which is open in one topology has a refinement open in the other topology. Since both topologies are paracompact by Theorem 2.4, one can easily verify that the condition stated in the theorem is satisfied. $\square$

We close this paper by a theorem which completes the characterization of RR-pairwise paracompactness given by Theorem 2.4. For the proof we refer the reader to [4]. Remark that one can also replace RR by FHP and the following theorem will remain true [4].

Theorem 2.9. *Let X be a bitopological space. Then X is RR-pairwise paracompact if and only if X is β-pairwise countably θ-regular and RR-pairwise semiparacompact.*

Proof. See [4, Corollary 4]. □

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