Internat. J. Math. & Math. Sci.
Vol. 24, No. 6 (2000) 423–432
S0161171200002477

OBSERVABILITY AND UNIQUENESS THEOREM FOR A COUPLED HYPERBOLIC SYSTEM

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(Received 23 December 1998)

ABSTRACT. We deal with the inverse inequality for a coupled hyperbolic system with dissipation. The inverse inequality is an indispensable inequality that appears in the Hilbert Uniqueness Method (HUM), to establish equivalence of norms which guarantees uniqueness and boundary exact controllability results. The term observability is due to the mathematician Ho (1986) who used it in his works relating it to the inverse inequality. We obtain the inverse inequality by the Lagrange multiplier method under certain conditions.

Keywords and phrases. Observability, inverse inequality, uniqueness theorem, unique continuation.

2000 Mathematics Subject Classification. Primary 35L05, 35L65.

1. Introduction. Several approaches are known concerning the principle of unique continuation. One of them consists in the classical principle of identity for analytical function, that is, holomorphic (analytic) functions which are defined in some region can frequently be extended to holomorphic functions in some larger region. These extensions are uniquely determined by the given functions (cf. [6, 7, 10]). Another extension process, introduced by Holgren, is based on the resolution of the homogeneous boundary value problem for the wave equation, that is, given $\{f(x), f_1(x)\}$ in $\mathfrak{D}(\Omega)\times\mathfrak{D}(\Omega)$, let us consider the homogeneous boundary value problem:

$$\begin{aligned} &u_{tt}-\Delta u = 0, \quad \text{in } Q=\Omega\times]0,T[,\\ &u(x,0)=f(x), \qquad u_t(x,0)=f_1(x), \text{in } \Omega,\\ &u=0, \quad \text{on } \Sigma=\Gamma\times]0,T[. \end{aligned} \tag{1.1}$$

It is well known that in this case (1.1) has a strong solution and

$$\frac{\partial u}{\partial \nu}\in L^2(\Sigma). \tag{1.2}$$

Then we define in $\mathfrak{D}(\Omega)\times\mathfrak{D}(\Omega)$ the quadratic form:

$$\left|\left|\{f,f_1\}\right|\right|_{\mathfrak{D}(\Omega)\times\mathfrak{D}(\Omega)} = \left[\int_{\Sigma_0}\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma\, dt\right]^{1/2}, \tag{1.3}$$

where Σ_0 is a part of the lateral boundary Σ of the cylinder Q, with positive measure. The quadratic form (1.3) is a seminorm in $\mathfrak{D}(\Omega)\times\mathfrak{D}(\Omega)$. We have the following result: if u is a solution of (1.1) with $\{f,f_1\}\in\mathfrak{D}(\Omega)\times\mathfrak{D}(\Omega)$, and $(\partial u/\partial\nu)=0$ on Σ_0, then this implies that u is zero in Q, i.e., the seminorm (1.3) is, indeed, a norm in $\mathfrak{D}(\Omega)\times\mathfrak{D}(\Omega)$. This is true due to the Holgren's theorem (cf. [3, 8]).

Our purpose in this paper is to establish a result of unique continuation in the direction of the Holgren's theorem based on a result of Ruiz [11]. We present this approach as follows.

Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. In the cylinder $Q = \Omega\times]0,T[$ we consider the initial boundary value problem for the wave equation (1.1). We are interested in the following result: there exists an open subset $\mathfrak{D}\subset\Omega$ such that there exists a time $T>0$ and a constant $C>0$, so that

$$\int_\Omega \Big[|u_t|^2+|\nabla u|^2\Big]dx \le C\Big[\int_0^T\int_{\mathfrak{D}}|u_t|^2\,dx\,dt+\|u\|^2_{L_2(\Omega\times(0,T))}\Big], \quad \forall\{f,f_1\}\in H_0^1(\Omega)\times L^2(\Omega). \tag{1.4}$$

Lions (cf. [9]) proved that this estimate holds if Ω is of class C^2, and $\mathfrak{D}$ is a neighborhood of the part of the boundary $\Gamma_0=\Gamma_0(x^0)=\{x\in\Gamma;(x-x^0,\nu(x))>0\}$, where x^0 is some point of $\mathbb{R}^n$, and $\nu(x)$ is the unit outer normal.

Bardos, Lebeau and Rauch [2] showed that, when Ω is of class C^∞, inequality (1.4) holds if $\mathfrak{D}$ satisfies some "geometric control property," that is: there exists some $T>0$ such that every ray of geometric optics intersects the set $\mathfrak{D}\times(0,T)$.

The estimate (1.4) implies the following unique continuation result: if $v\in H^1(\Omega\times]0,T[)$ is solution of

$$\begin{aligned} &v_{tt}-\Delta v+b(x)v=0, \quad \text{in } \Omega\times]0,T[,\\ &v=0, \quad \text{on } \Sigma=\Gamma\times]0,T[,\\ &v=0, \quad \text{in } \mathfrak{D}\times]0,T[, \end{aligned} \tag{1.5}$$

where $b\in L^\infty(\mathfrak{D}\times]0,T[)$, then

$$v\equiv 0, \quad \text{in } \Omega\times]0,T[, \tag{1.6}$$

see Zuazua [12].

2. Problem formulation. In the cylinder $\Omega\times]0,T[$ we consider the following initial boundary value problem for the coupled hyperbolic system with dissipation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x_i}\Big(A\frac{\partial u}{\partial x_i}\Big)+B(x)\frac{\partial v}{\partial t}=0, \tag{2.1a}$$

$$\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x_i}\Big(A\frac{\partial v}{\partial x_i}\Big)-B(x)\frac{\partial u}{\partial t}=0, \tag{2.1b}$$

$$\begin{cases} u(x,0)=f(x), & v(x,0)=g(x),\\ u_t(x,0)=f_1(x), & v_t(x,0)=g_1(x), \end{cases} \quad \text{in } \Omega, \tag{2.1c}$$

$$u=0, \qquad v=0, \quad \text{on } \Sigma=\Gamma\times]0,T[, \tag{2.1d}$$

where $u=(u^1,\dots,u^m),(v^1,\dots,v^m)$, $A=A^*$ and $B(x)=B^*(x)$ are square matrices of order m, and we are using the Einstein summation convention.

We assume that

$$\sum_{i=1}^{n} A\zeta_i \cdot \zeta_i \geq a_0 \sum_{i=1}^{n} |\zeta_i|^2, \quad a_0 > 0, \text{ where } \zeta_i = (\zeta_i^1, \ldots, \zeta_i^m) \in \mathbb{R}^n \tag{2.2}$$

and $B(x)\eta \cdot \eta \geq 0$, $x \in \Omega$, $\eta \in \mathbb{R}^m$. We denote by $\mathcal{H}$ the real Hilbert space of quadruples $\{u, v, u_1, v_1\}$ of m-compound vector-functions such that

$$u, v \in [H^1(\Omega)]^m, \qquad u|_\Gamma = v|_\Gamma = 0, \qquad u_1, v_1 \in [L^2(\Omega)]^m. \tag{2.3}$$

The inner product in $\mathcal{H}$ is defined by the formula

$$\langle \{u, v, u_1, v_1\}, \{f, g, f_1, g_1\} \rangle_{\mathcal{H}} = \int_\Omega \left(A \frac{\partial u}{\partial x_i} \cdot \frac{\partial f}{\partial x_i} + A \frac{\partial v}{\partial x_i} \cdot \frac{\partial g}{\partial x_i} + u_1 \cdot f_1 + v_1 \cdot g_1 \right) dx. \tag{2.4}$$

We defined an unbounded operator $\mathcal{A}$ in $\mathcal{H}$ given by:

$$\begin{aligned} D(\mathcal{A}) &= \left\{ \{u, v, u_1, v_1\} \in \mathcal{H};\ u, v \in [H^2(\Omega) \cap H_0^1(\Omega)]^m,\ u_1, v_1 \in [H_0^1(\Omega)]^m \right\}, \\ \mathcal{A}\{u, v, u_1, v_1\} &= \left[u_1, v_1, \frac{\partial}{\partial x_i}\left(A \frac{\partial u}{\partial x_i} \right) - B v_1, \frac{\partial}{\partial x_i}\left(A \frac{\partial v}{\partial x_i} \right) + B u_1 \right] \\ &\qquad \text{for } \{u, v, u_1, v_1\} \in \mathfrak{D}(\mathcal{A}). \end{aligned} \tag{2.5}$$

In standard way, we can check that the domain $\mathfrak{D}(\mathcal{A}^*)$ of the adjoint operator is given by:

$$\begin{aligned} D(\mathcal{A}^*) &= \left\{ \{f, g, f_1, g_1\} \in \mathcal{H};\ f, g \in [H^2(\Omega) \cap H_0^1(\Omega)]^m,\ f_1, g_1 \in [H_0^1(\Omega)]^m \right\}, \\ \mathcal{A}^*\{f, g, f_1, g_1\} &= -\left[f_1, g_1, \frac{\partial}{\partial x_i}\left(A \frac{\partial f}{\partial x_i} \right) - B g_1, \frac{\partial}{\partial x_i}\left(A \frac{\partial g}{\partial x_i} \right) + B f_1 \right] \\ &\qquad \text{for } \{f, g, f_1, g_1\} \in D(\mathcal{A}^*). \end{aligned} \tag{2.6}$$

Thus, the operator $\mathcal{A}$ is skew selfadjoint and from Stone's theorem it follows that it generates a 1-parameter group of unitary operators $\mathcal{U}(t)$ in $\mathcal{H}$. Moreover, $\mathcal{U}(t)$ is strongly continuous in t and $\mathcal{U}(t)\{f, g, f_1, g_1\}$ is strongly differentiable with respect to t for

$$\{f, g, f_1, g_1\} \in D(\mathcal{A}) \qquad \text{and} \qquad \frac{d}{dt}\mathcal{U}(t)\{f, g, f_1, g_1\} = \mathcal{A}\mathcal{U}(t)\{f, g, f_1, g_1\}. \tag{2.7}$$

Furthermore, $\mathcal{U}(t)$ takes $D(\mathcal{A})$ onto $D(\mathcal{A})$ are commutes with $\mathcal{A}$.

It follows that if $\{u, v, u_1, v_1\} = \mathcal{U}(t)\{f, g, f_1, g_1\}$, then u, v is strong solution to problem (2.1), with $u_t = u_1$, $v_t = v_1$, and possesses the following regularity:

$$\{u, v, u_t, v_t\} \in C^0([0, T]; D(\mathcal{A})) \cap C^1([0, T]; \mathcal{H}). \tag{2.8}$$

Now, we introduce the following notation:

$$\phi(u) = \sum_{i=1}^{n} A \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i}, \qquad E(t) = \int_\Omega \left(|u_t|^2 + |v_t|^2 + \phi(u) + \phi(v) \right) dx. \tag{2.9}$$

Since the operator $\mathcal{U}(t)$ is unitary, we have

$$E(t) \equiv \|\mathcal{U}(t)\{f,g,f_1,g_1\}\|_{\mathcal{H}}^2 = \|\{f,g,f_1,g_1\}\|_{\mathcal{H}}^2 \equiv E(0), \quad \forall t \in \mathbb{R}. \tag{2.10}$$

We also observe that, for $\mathcal{F} = \{f,g,f_1,g_1\} \in \mathcal{H}$, $\mathcal{U}(t)\mathcal{F}$ is a weak solution in $\mathcal{H}$ to the abstract Cauchy problem

$$\frac{d}{dt}\{u,v,u_1,v_1\} = \{u_1,v_1,Au,Av\} \tag{2.11}$$

in the following sense:

$$\int_0^T \left[\left\langle \mathcal{U}(t)\mathcal{F}, \frac{d\Phi}{dt}\right\rangle_{\mathcal{H}} + \langle \mathcal{U}(t)\mathcal{F}, A^*\Phi\rangle_{\mathcal{H}}\right] dt = -\langle \mathcal{F}, \Phi(0)\rangle_{\mathcal{H}}, \tag{2.12}$$

for every $\Phi \in L^2(0,T;D(A^*))$, $\Phi_t \in L^2(0,T;\mathcal{H})$, $\Phi(T) = 0$.

3. Observability and uniqueness theorem. In this section, we obtain the inverse inequality by using the Lagrange multiplier method with certain conditions over an arbitrary neighborhood $\mathfrak{D} \subset \Omega$.

It is easy to verify that the unperturbed wave equation of (2.1a) and (2.1b) is variational and that the expression

$$\mathcal{L}\left(u_t, \frac{\partial u}{\partial x_i}\right) = \frac{1}{2}\sum_{i=1}^{n} A\frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_i} - \frac{1}{2}|u_t|^2 \tag{3.1}$$

is its Lagrangian.

The invariance of the elementary action $\mathcal{L}dx_1 \cdots dx_n\,dt$ under the 1-parameter group of dilations in all variables with infinitesimal operator

$$t\frac{\partial}{\partial t} + (x_i - x_i^0)\frac{\partial}{\partial x_i} - \frac{n-1}{2}u^q\frac{\partial}{\partial u^q} \tag{3.2}$$

leads to a conservation law (cf. [1, E. Noether's theorem, page 88]). Now let $\{f,g,f_1,g_1\} \in D(\mathcal{A})$ and let $u(x,t), v(x,y)$ be a solution of problem (2.1).

After integration by parts, over the cylinder $\Omega \times]0,T[$, that conservation law we then obtain the identity:

$$\begin{aligned} TE(T) &+ \int_\Omega \left[2(\nabla\varphi,\nabla)u \cdot u_t + 2(\nabla\varphi,\nabla)v \cdot v_t + C(x)u \cdot u_t + C(x)v \cdot v_t\right] dx \Big|_{t=0}^{t=T} \\ &= \int_0^T \int_\Gamma \frac{\partial v}{\partial \nu}(\phi(u) + \phi(v))\,d\Gamma\,dt \\ &\quad + \int_0^T \int_\Omega \Big[2(\nabla\varphi,\nabla)v \cdot Bu_t - 2(\nabla\varphi,\nabla)u \cdot Bv_t + Cv \cdot Bu_t \\ &\qquad - CBv_t \cdot u + \frac{1}{2}\Delta C(Au \cdot u + Av \cdot v) + \mathcal{K}(u) + \mathcal{K}(v)\Big]\,dx\,dt, \end{aligned} \tag{3.3}$$

where the pair (u,v) is a solution of (2.1); $\varphi(x), C(x)$ are smooth functions in Ω, and

$$\mathcal{K}(u) = \phi(u)(\Delta\varphi + 1 - C(x)) + |u_t|^2(1 + C(x) - \Delta\phi) - 2\varphi_{x_i x_j} A u_{x_j} \cdot u_{x_i}. \tag{3.4}$$

We observe that identities of the kind in (3.3) are common in [4, 5]. We have that

$$\begin{aligned} |2(\nabla\varphi,\nabla)v\cdot Bu_t| &\le \varepsilon B(\nabla\varphi,\nabla)v\cdot(\nabla\varphi,\nabla)v+\frac{1}{\varepsilon}Bu_t\cdot u_t,\\ |2(\nabla\varphi,\nabla)u\cdot Bv_t| &\le \varepsilon B(\nabla\varphi,\nabla)u\cdot(\nabla\varphi,\nabla)u+\frac{1}{\varepsilon}Bv_t\cdot v_t,\\ Cv\cdot Bu_t-CBv_t\cdot u &= 2Cv\cdot Bu_t-\frac{\partial}{\partial t}(Cv\cdot Bu)\\ &\le -\frac{\partial}{\partial t}(Cv\cdot Bu)+\varepsilon|C|Bv\cdot v+\frac{1}{\varepsilon}|C|Bu_t\cdot u_t. \end{aligned} \tag{3.5}$$

From these inequalities we obtain

$$\begin{aligned} &\int_0^T\int_\Omega\Big[2(\nabla\varphi,\nabla)v\cdot Bu_t-2(\nabla\varphi,\nabla)u\cdot Bv_t+Cv\cdot Bu_t-CBv_t\cdot u\Big]\,dx\,dt\\ &\quad\le\int_\Omega Cv\cdot Bu\,dx\Big|_{t=T}^{t=0}+\frac{1}{\varepsilon}\int_0^T\int_\Omega\Big[(1+|C|)Bu_t\cdot u_t+Bv_t\cdot v_t\Big]\,dx\,dt\\ &\qquad+\varepsilon C_0(B,C(x),\varphi,a_0,\Omega)\int_0^T\int_\Omega(\phi(u)+\phi(v))\,dx\,dt. \end{aligned} \tag{3.6}$$

From the identity

$$\int_0^T\int_\Omega Bv_t\cdot v_t\,dx\,dt=\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt-\int_\Omega(u_t\cdot v_t+Au_{x_j}\cdot v_{x_j})\,dx\Big|_{t=0}^{t=T}, \tag{3.7}$$

it follows that

$$\int_0^T\int_\Omega Bv_t\cdot v_t\,dx\,dt\le E(0)+\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt. \tag{3.8}$$

So, using this inequality, from (3.6) we obtain

$$\begin{aligned} &\int_0^T\int_\Omega\Big[2(\nabla\varphi,\nabla)v\cdot Bu_t-2(\nabla\varphi,\nabla)u\cdot Bv_t+Cv\cdot Bu_t-CBv_t\cdot u\Big]\,dx\,dt\\ &\quad\le\frac{1}{\varepsilon}\int_0^T\int_\Omega\Big[(2+|C|)Bu_t\cdot u_t\Big]\,dx\,dt+\Big(C_0+\frac{1}{\varepsilon}\Big)E(0)+\varepsilon C_0TE(0),\\ &\Big|\int_\Omega\Big[2(\nabla\varphi,\nabla)u\cdot u_t+2(\nabla\varphi,\nabla)v\cdot v_t+C(x)u\cdot u_t+C(x)v\cdot v_t\Big]\,dx\Big|_{t=0}^{t=T}\Big|\\ &\quad\le C_1(|\nabla\varphi|,C(x),a_0,\Omega)E(0). \end{aligned} \tag{3.9}$$

We then arrive at the following inequality:

$$\begin{aligned} &TE(0)+\varepsilon C_0TE(0)-C_1E(0)-\Big(c_0+\frac{1}{\varepsilon}\Big)E(0)\\ &\quad\le\int_0^T\int_\Gamma\frac{\partial\varphi}{\partial\nu}(\phi(u)+\phi(v))\,d\Gamma+\frac{1}{\varepsilon}\int_0^T\int_\Omega[(2+|C|)Bu_t\cdot v_t]\,dx\,dt\\ &\qquad+\int_0^T\int_\Omega\Big[\frac{1}{2}\Delta C(Au\cdot u+Av\cdot v)+\mathcal{K}(u)+\mathcal{K}(v)\Big]\,dx\,dt. \end{aligned} \tag{3.10}$$

Now for an arbitrary point $x^0\in\mathbb{R}^n$ we find:

$$\Gamma_0=\{x\in\Gamma;(x-x^0,\nu(x))\ge 0\},\quad \Gamma_1=\Gamma\backslash\Gamma_0. \tag{3.11}$$

Let $\mathcal{D} \subset \Omega$ be an arbitrary neighborhood of Γ_0. There exists a function $\varphi(x)$ such that
(i) $(\partial\varphi/\partial\nu) \le 0$ on Γ,
(ii) $\varphi_0(x) \ge \mu > 0$, in $\Omega\backslash\mathcal{D}$, where $\varphi_0(x) = \inf(\partial^2\varphi/\partial_{x_i}\partial_{x_j})\eta_i\eta_j$.
Setting $C(x) = \Delta\varphi - 2\varphi_0(x) + \mu$, we have

$$\begin{aligned}
\mathcal{K}(u) &= \phi(u)(\Delta\phi + 1 - C(x)) + |u_t|^2(1 + C(x) - \Delta\varphi) - 2\varphi_{x_i x_j} A u_{x_i} u_{x_i} \\
&\le \phi(u)(\Delta\varphi + 1 - C(x) - 2\varphi_0)\phi(u) + |u_t|^2(1 + C(x) - \Delta\varphi) \\
&= (1-\mu)\phi(u) + (1 - 2\varphi_0(x) + \mu)|u_t|^2,
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \int_\Omega (\mathcal{K}(u) + \mathcal{K}(v))\,dx\,dt \\
&\quad\le \int_0^T \int_\Omega [(1-\mu)(\phi(u) + \phi(v)) + (1 + \mu - 2\varphi_0(x))\left(|u_t|^2 + |v_t|^2\right)\Big]\,dx\,dt \\
&\quad\le (1-\mu)\int_0^T \int_{\Omega\backslash\mathcal{D}} (|u_t|^2 + |v_t|^2 + \phi(u) + \phi(v))\,dx\,dt \\
&\qquad + \int_0^T \int_{\mathcal{D}} \Big[(1-\mu)(\phi(u) + \phi(v)) + (1 + \mu - 2\varphi_0)\left(|u_t|^2 + |v_t|^2\right)\Big]\,dx\,dt \\
&\quad= (1-\mu)\int_0^T \int_\Omega E(t)\,dt + \int_0^T \int_{\mathcal{D}} 2(\mu - \varphi_0)\left(|u_t|^2 + |v_t|^2\right)dx\,dt \\
&\quad\le (1-\mu)TE(0) + 2(\mu + \sup|\varphi_0|)\int_0^T \int_{\mathcal{D}} \left(|u_t|^2 + |v_t|^2\right)dx\,dt.
\end{aligned} \tag{3.12}$$

We assume that

$$B(x)u \cdot u \ge b_0|u|^2, \quad \text{in } \mathcal{D}, \tag{3.13}$$

then

$$\begin{aligned}
\int_0^T \int_{\mathcal{D}} |v_t|^2\,dx\,dt &\le \frac{1}{b_0}\int_0^T \int_{\mathcal{D}} B(x)v_t \cdot v_t\,dx\,dt \\
&\le \frac{1}{b_0}\int_0^T \int_\Omega B(x)v_t \cdot v_t\,dx\,dt \\
&\le \frac{1}{b_0}E_0 + \frac{1}{b_0}\int_0^T \int_\Omega B(x)u_t \cdot u_t\,dx\,dt,
\end{aligned} \tag{3.14}$$

and similarly,

$$\int_0^T \int_{\mathcal{D}} |u_t|^2\,dx\,dt \le \frac{1}{b_o}\int_0^T \int_\Omega B(x)u_t \cdot u_t\,dx\,dt. \tag{3.15}$$

From these inequalities we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega (\mathcal{K}(u) + \mathcal{K}(v))\,dx\,dt \\
&\qquad\le (1-\mu)TE(0) + \frac{2}{b_0}(\mu + \sup|\varphi_0|)E(0) \\
&\qquad\quad + \frac{4}{b_0}(\mu + \sup|\varphi_0|)\int_0^T \int_\Omega Bu_t \cdot u_t\,dx\,dt.
\end{aligned} \tag{3.16}$$

Inequalities (3.10) and (3.16) lead to

$$\begin{aligned} &T(\mu-\varepsilon C_0)E(0)-\Big(C_0+C_1+\frac{1}{\varepsilon}+\frac{2}{b_0}(\mu+\sup|\varphi_0|)\Big)E(0)\\ &\quad\le\Big[\frac{1}{\varepsilon}(2+\sup|C|)+\frac{4}{b_0}(\mu+\sup|\varphi_0|)\Big]\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt \\ &\quad+\int_0^T\int_\Omega\frac{1}{2}\Delta C(Au\cdot u+Av\cdot v)\,dx\,dt. \end{aligned} \tag{3.17}$$

In the general case, we can choose function $\varphi(x)$ in the following way:

$$\varphi(x)=\frac{1}{2}|x-x^0|^2,\quad x\in\Omega\backslash\mathfrak{D}, \tag{3.18}$$

and extend it in $\mathfrak{D}$ satisfying the property (i). Then,

$$\Delta C=\Delta(\Delta\varphi-2\varphi_0(x)+\mu)\equiv 0,\quad \text{in } \Omega\backslash\mathfrak{D}. \tag{3.19}$$

Thus, from (3.17), we arrive at the inequality:

$$\begin{aligned} &T(\mu-\varepsilon C_0)E(0)+\Big(C_0+C_1+\frac{1}{\varepsilon}+\frac{2}{b_0}(\mu+\sup|\varphi_0|)\Big)E(0)\\ &\quad\le\Big[\frac{1}{\varepsilon}(2+\sup|C|)+\frac{4}{b_0}(\mu+\sup|\varphi_0|)\Big]\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt \\ &\quad+C_2(\varphi,A)\int_0^T\Big(\|u\|^2_{L^2(\mathfrak{D})}+\|v\|^2_{L^2(\mathfrak{D})}\Big)\,dt. \end{aligned} \tag{3.20}$$

Now we can assume that there exists a function $\varphi(x)$ which satisfy (i), (ii), and
(iii) $\Delta^2\varphi-2\Delta\varphi_0(x)\le 0$, in Ω.
(We can construct some examples of these functions under certain assumptions on Ω and $\mathfrak{D}$.) In this case, from (3.17), it follows

$$\begin{aligned} &T(\mu-\varepsilon C_0)E(0)-\Big(C_0+C_1+\frac{1}{\varepsilon}+\frac{2}{b_0}(\mu+\sup|\varphi_0|)\Big)E(0)\\ &\quad\le\Big[\frac{1}{\varepsilon}\Big(2+\sup|C|+\frac{4}{b_0}(\mu+\sup|\varphi_0|)\Big)\Big]\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt. \end{aligned} \tag{3.21}$$

We choose ε such that $\mu>\varepsilon C_0$ (we can set $\varepsilon=(\mu/2C_0)$), and

$$T>\frac{2C_0}{\mu}+\frac{2C_1}{\mu}+\frac{4C_0}{\mu^2}+\frac{4}{b_0}\Big(1+\frac{1}{\mu}\sup|\varphi_0|\Big)\equiv T_0. \tag{3.22}$$

Then (3.20) and (3.21) can be rewritten in the following way:

$$\frac{\mu}{2}(T-T_0)E(0)\le\widetilde{C}\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt+C_2(\varphi,A)\int_0^T\Big(\|u\|^2_{L^2(\mathfrak{D})}\|v\|^2_{L^2(\mathfrak{D})}\Big)\,dt, \tag{3.23}$$

$$\frac{\mu}{2}(T-T_0)E(0)\le\widetilde{C}\int_0^T\int_\Omega Bu_t\cdot u_t\,dx\,dt, \tag{3.24}$$

where $\widetilde{C}=\widetilde{C}(\varphi,b_0,a_0,\Omega)$.

4. Conclusion. We arrive at the following assertion: in the cylinder $\Omega\times]0,T[$, we consider the initial boundary value problem for the following coupled hyperbolic system with dissipation:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x_i}\left(A\frac{\partial u}{\partial x_i}\right)+\mathscr{X}_{\mathfrak{D}}(x)B(x)\frac{\partial v}{\partial t}=0,\\
&\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x_i}\left(A\frac{\partial v}{\partial x_i}\right)-\mathscr{X}_{\mathfrak{D}}(x)B(x)\frac{\partial u}{\partial t}=0,\\
&\begin{cases} u(x,0)=f(x), & v(x,0)=g(x),\\ u_t(x,0)=f_1(x), & v_t(x,0)=g_1(x),\end{cases}\quad \text{in } \Omega,\\
&u=v=0,\quad \text{on } \Sigma=\Gamma\times]0,T[.
\end{aligned}
\tag{4.1}
$$

Here $A=A^*$, $B(x)=B^*(x)$, $B(x)\eta\cdot\eta\ge b_0|\eta|^2, b_0>0$; $A\zeta_i\cdot\zeta_i\ge a_0|\zeta_i|^2$, $a_0>0$, $B\in L^\infty(\mathfrak{D})$, and

$$
\mathscr{X}_{\mathfrak{D}}(x)=\begin{cases}1, & x\in\mathfrak{D},\\ 0, & x\notin\mathfrak{D},\end{cases}
\tag{4.2}
$$

where $\mathfrak{D}$ is an arbitrary neighborhood of the boundary (or a part of the boundary). We have that:

(1) If $\mathfrak{D}$ is an arbitrary neighborhood of $\Gamma_0=\{x\in\Gamma;(x-x^0,\nu(x))\ge 0\}$, then there exists $T_0>0$ such that, for $T>T_0$, the following inequality holds:

$$
\begin{aligned}
(T-T_0)&\int_\Omega\left[|u_t|^2+|v_t|^2+\phi(u)+\phi(v)\right]dx\\
&\le C_1\int_0^T\int_{\mathfrak{D}}|u_t|^2\,dx\,dt+C_2\int_0^T\left(\|u\|^2_{L^2(\mathfrak{D})}+\|v\|^2_{L^2(\mathfrak{D})}\right)dt.
\end{aligned}
\tag{4.3}
$$

(2) If $\mathfrak{D}$ and Ω are such that there exists a function $\varphi(t)$ with properties (i), (ii), and (iii), then, for $T>T_0$, we have

$$
(T-T_0)\int_\Omega\left[|u_t|^2+|v_t|^2+\phi(u)+\phi(v)\right]dx\le C\int_0^T\int_{\mathfrak{D}}|u_t|^2\,dx\,dt.
\tag{4.4}
$$

Remark 4.1. Let us consider the coupled system:

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x_i}\left(A\frac{\partial u}{\partial x_i}\right)+G(x)u+\mathscr{X}\mathfrak{D}(x)B(x)\frac{\partial v}{\partial t}=0,\\
&\frac{\partial^2 v}{\partial t^2}-\frac{\partial}{\partial x_i}\left(A\frac{\partial v}{\partial x_i}\right)+G(x)v-\mathscr{X}\mathfrak{D}(x)B(x)\frac{\partial u}{\partial t}=0,
\end{aligned}
\tag{4.5}
$$

where A,B, and $\mathfrak{D}$ are defined above, $G=G^*$, with $G(x)\eta\cdot\eta\ge 0$, $x\in\Omega$; and support $(G)\subseteq\bar{\mathfrak{D}}$, $G\in W^{1,\infty}(\mathfrak{D})$.

These new terms considered add to the right-hand side of the main identity (3.3) (and the rest of the formulas) the following expression:

$$
\int_0^T\int_\Omega\Big[\varphi_{x_i}G_{x_i}u\cdot u+(\Delta\varphi+1-C)Gu\cdot u+\varphi_{x_i}G_{x_i}v\cdot v+(\Delta\varphi+1-C)Gv\cdot v\Big]dx\,dt.
\tag{4.6}
$$

(In this case, $\widetilde{E}(t) = \int_\Omega [|u|^2 + |v|^2 + \phi(u) + \phi(v) + Gu \cdot u + Gv \cdot v]\,dx$.) After the substitution $C = \Delta\varphi - 2\varphi_0 + \mu$, this expression takes the form

$$(1-\mu)\int_0^T \int_\Omega (Gu\cdot u + Gv\cdot v)\,dx\,dt + \int_0^T \int_\Omega \Big[\varphi_{x_i} G_{x_i} u\cdot u + 2\varphi_0 Gu\cdot u + \varphi_{x_i} G_{x_i} v\cdot v + 2\varphi_0 Gv\cdot v\Big]\,dx\,dt. \tag{4.7}$$

Thus, we obtain under the same assumption:

(1) In the first case,

$$(T-T_0)\int_\Omega \Big[|u_t|^2 + |v_t|^2 + \phi(u) + \phi(v) + Gu\cdot u + Gv\cdot v\Big]\,dx \le C_1 \int_0^T \int_{\mathcal{D}} |u_t|^2\,dx\,dt + C_2 \int_0^T \Big(\|u\|^2_{L^2(\mathcal{D})} + \|v\|^2_{L^2(\mathcal{D})}\Big)\,dt. \tag{4.8}$$

(2) In the second case:

$$(T-T_0)\widetilde{E}(t) \le C_1 \int_0^T \int_{\mathcal{D}} |u_t|^2\,dx\,dt + C_3 \int_0^T \Big(\|u\|^2_{L^2(\mathcal{D})} + \|v\|^2_{L^2(\mathcal{D})}\Big)\,dt, \tag{4.9}$$

i.e., in this situation only the first case makes sense.

Thus, we have the following results of unique continuation, i.e., if $\mathcal{D} \subset \Omega$ is an arbitrary neighborhood of Γ_0 satisfying (i) and (ii), with support $(G) \subseteq \bar{\mathcal{D}}$, and the pair (u,v) is a strong solution of the initial boundary value problem for the coupled system (4.5) with homogeneous Dirichlet condition and

$$u \equiv v \equiv 0, \quad \text{in } \mathcal{D}\times]0,T[, \tag{4.10}$$

then

$$u \equiv v \equiv 0, \quad \text{in } \Omega\times]0,T[. \tag{4.11}$$

Acknowledgement. Kapitonov was supported by Russian Fund of Fundamental Research, project 94-01-0078; and Souza was supported by CNP_q grant Proc. 20.0692/97.6, Brazil.

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