Internat. J. Math. & Math. Sci.
Vol. 24, No. 6 (2000) 419–421
S0161171200003744

SEQUENTIAL COMPLETENESS OF INDUCTIVE LIMITS

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(Received 28 July 1999)

ABSTRACT. A regular inductive limit of sequentially complete spaces is sequentially complete. For the converse of this theorem we have a weaker result: if $\operatorname{ind} E_n$ is sequentially complete inductive limit, and each constituent space E_n is closed in $\operatorname{ind} E_n$, then $\operatorname{ind} E_n$ is α-regular.

Keywords and phrases. Regularity, α-regularity, sequential completeness of locally convex inductive limits.

2000 Mathematics Subject Classification. Primary 46A13; Secondary 46A30.

1. Introduction. In [2] Kučera proves that an LF-space is regular if and only if it is sequentially complete. In [1] Bosch and Kučera prove the equivalence of regularity and sequential completeness for bornivorously webbed spaces. It is natural to ask: can this result be extended to arbitrary sequentially complete spaces? We give a partial answer to this question in this paper.

2. Definitions. Throughout the paper $\{(E_n, \tau_n)\}_{n\in\mathbb{N}}$ is an increasing sequence of Hausdorff locally convex spaces with continuous identity maps $\mathrm{id} : E_n \to E_{n+1}$ and $E = \operatorname{ind} E_n$ its inductive limit.

A topological vector space E is sequentially complete if every Cauchy sequence $\{x_n\} \subset E$ is convergent to an element $x \in E$.

An inductive limit $\operatorname{ind} E_n$ is regular, resp. α-regular, if each of its bounded sets is bounded, resp. contained, in some constituent space E_n.

3. Main results

THEOREM 3.1. *Let each space (E_n, τ_n) be sequentially complete and $\operatorname{ind} E_n$ regular. Then $\operatorname{ind} E_n$ is sequentially complete.*

PROOF. Let $\{x_k\}$ be a Cauchy sequence in $\operatorname{ind} E_n$. Then the set $\bigcup\{x_k;\ k \in \mathbb{N}\}$ is bounded in $\operatorname{ind} E_n$ and since the inductive limit is regular, we may assume that it is also bounded in E_1. For every $n \in \mathbb{N}$, denote by P_n the τ_1-closure of the convex hull of $\bigcup\{x_k;\ k > n\}$. Let Q be τ_1-closed balanced hull of P_0 and F the span of Q with the topology generated by the filter basis $\{\lambda Q;\ \lambda > 0\}$. Denote this topology by μ.

Before we finish the proof, we prove two lemmas.

LEMMA 3.2. *(F, μ) is Banach.*

PROOF. Let $\{y_k\}$ be a Cauchy sequence in F. Since the set Q is bounded in E_1, each μ-neighborhood λQ is absorbed by any τ_1-neighborhood of zero in E_1. This implies that the topology μ is finer than the topology of F inherited from E_1. Hence $\{y_k\}$ is Cauchy in E_1 and as such, it converges to some $y \in E_1$ in the topology τ_1.

For any μ-neighborhood λQ of zero, there exists $k \in \mathbb{N}$ such that $y_p - y_q \in \lambda Q$ for any $p,q > k$. If we let $q \to \infty$, the τ_1-closedness of λQ implies that $y_p - y \in \lambda Q$ for any $p > k$, that is $y \in y_p + \lambda Q \subset F$ and $y_p \to y$ in the topology μ. □

Lemma 3.3. *The respective families of τ-bounded and μ-bounded sets in the space (F,μ) are the same.*

Proof. Since the topology μ is finer than τ_1, which in turn is finer than τ, any μ-bounded set is τ-bounded.

Let $A \subset F$ be a τ-bounded set. Denote by B the μ-closure of the balanced convex hull of A. Put $G = \bigcup\{nB;\ n \in \mathbb{N}\}$ and equip it with the topology, which we denote by γ, generated by the filter basis $\{\lambda B;\ \lambda > 0\}$. It follows from Lemma 3.2, that (G,γ) is a Banach space.

Let P_0 and Q be the same sets as above, we first show that the set $Q \bigcap G$ is closed in (G,γ). Let a sequence $\{z_k\} \subset Q \bigcap G$ be convergent to an element $z \in G$ with respect to the topology γ. Since γ is finer than τ_1, $z_k \to z$ also in the topology τ_1 and since Q is τ_1-closed, we have $z \in Q$ and $z \in Q \bigcap G$. Similarly, the sets $Q_n = n(Q \bigcap G)$, $n \in \mathbb{N}$, are closed in (G,γ) and $G = \bigcup\{Q_n;\ n \in \mathbb{N}\}$. By Baire's category theorem, there exists $n \in \mathbb{N}$ such that Q_n has nonempty interior and $Q_n - Q_n = Q_{2n}$ is a γ-neighborhood of zero. Since B is γ-bounded and Q_{2n} is a γ-neighborhood of zero, thus there exists $\beta > 0$ such that $B \subset \beta Q_{2n} \subset \beta 2nQ$, that is, B is μ-bounded. □

To continue the proof of the theorem, we observe that the weak topology σ on F is the weakest topology on F for which the family of all σ-bounded sets in F is the same as the family of all μ-bounded sets in F. So we have $\mu \supset \tau \supset \sigma$ and the τ-Cauchy sequence $\{x_k\}$ is also σ-Cauchy.

Let F'' be the strong second dual of F. Then F can be considered as a closed subspace of F''. Hence each $f \in F'$ can be continuously extended to F'' and the μ-closed convex set $P_0 \subset F''$ is also $\sigma(F'',F')$-closed in F''. Moreover, P_0 as a set bounded in F'', is equicontinuous on F'. Hence, by Alaoglu's theorem, it is relatively $\sigma(F'',F')$-compact. This, together with the $\sigma(F'',F')$-closedness implies that P_0 is $\sigma(F'',F')$-compact.

Similarly, as for P_0, all sets P_n, $n \in \mathbb{N}$, are $\sigma(F'',F')$-closed, and therefore $\sigma(F'',F')$-compact. Any finite intersection $\bigcap\{P_n;\ 0 \le n \le m\} = P_m$ is nonempty. Hence there exists $x \in \bigcap\{P_n;\ n \in \mathbb{N}\} \subset F$.

To show that $\bigcap\{P_n;\ n \in \mathbb{N}\}$ contains only one element, take $y \in F$, $y \ne x$. Then there exists a τ-neighborhood U of zero such that $y \notin x + U$.

Take a τ-closed, balanced, convex τ-neighborhood V of zero such that $V - V \subset U$. There exists $n \in \mathbb{N}$ such that $x_p - x_q \in V$ for any $p,q \ge n$. This implies that $x_p \in x_n + V$ for $p \ge n$, $P_n \subset x_n + V$, and $x_n \in x + V$.

Finally, $P_n \subset x_n + V \subset x + V + V \subset x + U$. But $y \notin x + U$, hence $y \notin P_n$ and $y \notin \bigcap\{P_n;\ n \in \mathbb{N}\}$.

This implies the existence of an upper-triangular matrix $\Lambda = (\lambda_{nm})$ with all $\lambda_{nm} \ge 0$, only a finite number of nonzero entries in each row, and the sum of all entries in each row is equal to 1, such that the sequence

$$\left\{ w_n = \sum_{m=n}^{\infty} \lambda_{nm} x_m : n \in \mathbb{N} \right\} \tag{3.1}$$

converges to x in the topology of F. Then $w_n \to x$ also in the weaker topology τ.

Given a balanced convex τ-neighborhood U of zero, there exist $p,\ q \in \mathbb{N}$ such that $w_n - x \in U$ for $n \geq p$ and $x_m - x_n \in U$ for $m \geq n \geq q$. Then for $n \geq \max(p,q)$, we have

$$x - x_n = (x - w_m) + (w_n - x_n) = (x - w_m) + \sum_{m=n}^{\infty} \lambda_{nm}(x_m - x_n) \in U + U, \tag{3.2}$$

that is, $x_n \to x$ in E. □

We have proved a little more: *if each E_n is sequentially complete, then any Cauchy sequence in $E = \operatorname{ind} E_n$ which is bounded in some E_n, converges to an element in E_n in the topology inherited from E, but not necessarily in the topology of E_n.*

In [3], Kučera and McKennon constructed a regular quasi-incomplete inductive limit of Banach spaces and they asked about the existence of a sequentially incomplete regular inductive limit of Banach spaces. Theorem 3.1 provides a negative answer.

We do not know whether sequentially complete inductive limit of sequentially complete spaces is regular. Nevertheless, we can at least claim the following.

Theorem 3.4. *Suppose E_n is closed in* $\operatorname{ind} E_n$ *for every $n \in \mathbb{N}$ and* $\operatorname{ind} E_n$ *is sequentially complete. Then* $\operatorname{ind} E_n$ *is α-regular.*

Proof. Let $B \subset \operatorname{ind} E_n$ be balanced, convex, closed and bounded. Then the space E_B, with the topology generated by the Minkowski functional of B, is Banach (see Lemma 3.2). Put $B_n = B \bigcap E_n$, $F_n = E_B \bigcap E_n$, $n \in \mathbb{N}$, and equip F_n with the topology generated by B_n.

Since E_n is closed in $\operatorname{ind} E_n$ and B_n is closed in E_n, F_n is a Banach subspace of E_B.

We have $E_B = \bigcup\{F_n;\ n \in \mathbb{N}\}$, hence by the Baire's Category theorem, there exists $n \in \mathbb{N}$ such that F_n contains an open set of E_B. This implies that B_n absorbs B and B is contained in E_n, i.e., $\operatorname{ind} E_n$ is α-regular. □

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