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ON QUASI h-PURE SUBMODULES OF QTAG-MODULES

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ABSTRACT. Different concepts and decomposition theorems have been done for QTAG-modules by number of authors. We introduce quasi h-pure submodules for QTAG-modules and we obtain several characterizations for quasi h-pure submodules and as a consequence we deduce a result done by Fuchs 1973.

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1. Introduction. Following [4] a module M_R is called QTAG-module if it satisfies the following condition:

(I) Any finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

The structure theory of such modules has been developed by various authors. Recently Singh and Khan [5] have characterized the modules in which h-neat submodules are h-pure. The main purpose of this paper is to introduce the concept of quasi h-pure submodules, a weaker version of h-pure submodules. In Section 3, some characterization of h-pure submodules are obtained (Theorems 3.2 and 3.4) for the subsequent use. In general it is known that $\mathrm{soc}(A+B) \neq \mathrm{soc}(A)+\mathrm{soc}(B)$. The equality for some submodules motivated to define the concept of quasi h-pure submodules. Several characterizations of quasi h-pure submodules are obtained (Theorems 4.6 and 4.7) and as a consequence we deduce [1, Theorem 66.3] as Corollary 4.9.

2. Preliminaries. Rings considered in this paper are with $1 \neq 0$ and modules are unital QTAG-module. A module in which the lattice of its submodule is totally ordered is called a serial module; in addition if it has finite composition length it is called a uniserial module. An $x \in M$ is called a uniform element if xR is a nonzero uniform (hence uniserial) submodule of M. For any module A_R with a composition series, $d(A)$ denotes its length. If $x \in M$ is uniform, then $e(x) = d(xR)$, $H_M(x) = \sup\{d(yR/xR) \mid y \in M$ and y is uniform with $x \in yR\}$ are called exponent of x and height of x, respectively. For any $n \geq 0$, $H_n(M) = \{x \in M \mid H(x) \geq n\}$. A submodule N of M is called h-pure in M if $N \cap H_k(M) = H_k(N)$ for every $k \geq 0$ and N is called h-neat if $N \cap H_1(M) = H_1(N)$. The module M is called h-divisible if $H_1(M) = M$. For any module K, $\mathrm{soc}(K)$ denotes the socle of K. For other basic concepts of QTAG-module one may refer to [2, 3, 4, 5].

3. h-pure submodules. In this section, we have obtained some characterizations of h-neat and h-pure submodules which are used in Section 4.

First, we prove the following proposition.

Proposition 3.1. *A submodule N of a QTAG-module M is h-neat if and only if* $\mathrm{soc}(M/N) = (\mathrm{soc}(M)+N)/N$.

Proof. Suppose N is h-neat in M. Let $\bar{y}$ be a uniform element in $\mathrm{soc}(M/N)$, where y may be chosen to be uniform in M. Then $\bar{y}R = (yR+N)/N \cong yR/(yR\cap N)$. Hence $d(yR/(yR\cap N)) = 1$. Put $yR\cap N = zR$, then due to h-neatness of N there exist a uniform element $w \in N$ such that $y \in wR$ and $d(wR/zR) = 1$. Appealing to [4, Lemma 2.3] we get $e(y-z) \le 1$, so $y - z \in \mathrm{soc}(M)$ and we get $\bar{y} \in (\mathrm{soc}(M)+N)/N$. Thus $\mathrm{soc}(M/N) = (\mathrm{soc}(M)+N)/N$. Conversely, let x be a uniform element in $N\cap H_1(M)$, then we can find a uniform element $y \in M$ such that $d(yR/xR) = 1$. Hence $e(\bar{y}) = 1$ and so $\bar{y} \in \mathrm{soc}(M/N)$. Therefore, $\bar{y} = \bar{z}$, where $z \in \mathrm{soc}(M)$. Now $xR = H_1(yR) = H_1((y-z)R) \subseteq H_1(N)$. Hence N is h-neat submodule of M.

It is well known that $H_n(M/N) = (H_n(M)+N)/N$ for all nonnegative integer n. □

Similar to Proposition 3.1, we have the following.

Theorem 3.2. *A submodule N of M is h-pure in M if and only if* $\mathrm{soc}(H_n(M/N)) = (\mathrm{soc}(H_n(M))+N)/N$, *for all nonnegative integers n.*

Proof. Suppose $\mathrm{soc}(H_n(M/N)) = (\mathrm{soc}(H_n(M))+N)/N$ holds for all $n \ge 0$. Then by Proposition 3.1 N is h-neat in M. Now suppose $N\cap H_n(M) = H_n(N)$ and x be a uniform element in $N\cap H_{n+1}(M)$, then there exists a uniform element $t \in H_n(M)$ such that $d(tR/xR) = 1$, so $e(\bar{t}) = 1$. Hence by assumption $\bar{t} = \bar{z}$, where $z \in \mathrm{soc}(H_n(m))$. Trivially $t - z \in N \cap H_n(M) = H_n(N)$. Therefore, $xR = H_1(tR) = H_1((t-z)R) \subseteq H_{n+1}(N)$. Hence by induction, N is h-pure submodule of M. Conversely, suppose N is h-pure in M, then by, Proposition 3.1, $\mathrm{soc}(M/N) = (\mathrm{soc}(M)+N)/N$. Now for applying induction suppose $\mathrm{soc}(H_k(M/N)) = (\mathrm{soc}(H_k(M))+N)/N$. Let $\bar{x}$ be a uniform element in $\mathrm{soc}(H_{k+1}(M/N)) = \mathrm{soc}((H_{k+1}(M)+N)/N)$, then x can be chosen to be a uniform element in $H_{k+1}(M)$. Now $\bar{x}R = (xR+N)/N \cong xR/(xR\cap N) = xR/yR$. Then $d(xR/yR) = 1$ which yields $y \in N\cap H_{k+2}(M) = H_{k+2}(N)$. Therefore we can find a uniform element $t \in H_{k+1}(N)$ such that $d(tR/yR) = 1$. Hence appealing to [4, Lemma 2.3] we get $e(x-t) \le 1$. Consequently, $x - t \in \mathrm{soc}(H_{k+1}(M))$ and $\bar{x} \in (\mathrm{soc}(H_{k+1}(M)+N)/N)$. Hence we get the equality. □

Notation 3.3. *For any nonnegative integer n, we denote by $S^n(M)$ the submodule* $\mathrm{soc}(H_n(M/N))$ *and by $S_n(M)$ the submodule* $(\mathrm{soc}(H_n(M))+N)/N$ *and by* $S_n(M,N) = S^n(M)/S_n(M)$.

In terms of the above notation and Theorem 3.2, we have the following.

Theorem 3.4. *A submodule N of M is h-pure if and only if $S_t(M,N) = 0$ for all $t \ge 0$.*

Now we prove the following which is of independent interest.

Theorem 3.5. *If N is a submodule of M and K is a proper h-pure submodule of M containing N, then the following holds*

(i) $S^t(M) = S^t(K) + S_t(M)$,

(ii) $S^t(K) \cap S_t(M) = S_t(K)$.

Proof. (i) Let $\bar{x} \in S^t(M)$ be a uniform element where x is uniform in $H_t(M)$. Then we can get a uniform element $y \in N$ such that $d(xR/yR) = 1$, then $y \in N \cap K \cap H_{t+1}(M)$. As K is h-pure, $y \in H_{t+1}(K)$. Therefore there is a uniform element $z \in H_t(K)$ such that $d(zR/yR) = 1$. Hence $e(x-z) \le 1$ and we get $x - z \in \operatorname{soc}(H_t(M))$. Consequently, $\bar{x} = \bar{z} + \bar{w}$, where $w \in \operatorname{soc}(H_t(M))$ and $\bar{x} \in S^t(K) + S_t(M)$. Hence $S^t(M) = S^t(K) + S_t(M)$.

(ii) Let $\bar{x} \in S^t(K) \cap S_t(M)$, then $\bar{x} = \bar{y} = \bar{z}$, $\bar{y} \in S^t(K)$ and $\bar{z} \in S_t(M)$. As $y - z \in N$, where $y \in H_t(K)$ and $z \in \operatorname{soc}(H_t(M))$ we have $y - z \in K \cap H_t(M) = H_t(K)$ and so $y - z = w \in H_t(K)$. Consequently, $z = y - w \in \operatorname{soc}(H_t(K))$. Hence $\bar{x} = \bar{z} = y - w + N \in (\operatorname{soc}(H_t(K)) + N)/N = S_t(K)$ and we get $S^t(K) \cap S_t(M) = S_t(K)$. □

4. Quasi h-pure submodules.

In this section, we introduce quasi h-pure submodule weakening the concept of h-pure submodules. As in Theorem 3.2, one can think of the equality of $\operatorname{soc}(N + H_n(M))$ and $\operatorname{soc}(N) + \operatorname{soc}(H_n(M))$. It is well known that the equality, in general does not hold. Here we examine, the consequences of the equality of the two expressions.

Notation 4.1. *For any nonnegative integer t, we denote by $N^t(M)$ the submodule $(N + H_{t+1}(M)) \cap \operatorname{soc}(H_t(M))$ and by $N_t(M)$ the submodule $N \cap \operatorname{soc}(H_t(M)) + \operatorname{soc}(H_{t+1}(M))$ and by $Q_t(M,N) = N^t(M)/N_t(M)$.*

Theorem 4.2. *If N and K are submodules of QTAG-module M such that $N \subseteq K$ and K is h-pure in M, then the module $Q_n(M,N)$ and $Q_n(K,N)$ are isomorphic.*

Proof. Define a map $\sigma : N^n(K)/N_n(K) \to N^n(M)/N_n(M)$ such that $\sigma(x + N_n(K)) = x + N_n(M)$. Obviously σ is an R-homomorphism. Now if for some $x \in N^n(K)$, $x \in N_n(M)$, then $x = y + z$, $y \in N \cap \operatorname{soc}(H_n(M))$ and $z \in \operatorname{soc}(H_{n+1(M)})$, then $y \in K \cap \operatorname{soc}(H_n(M)) \subseteq H_n(K)$ gives $y \in N \cap \operatorname{soc}(H_n(K))$. Also $z = x - y \in K \cap \operatorname{soc}(H_{n+1}(M))$ yields $z \in \operatorname{soc}(H_{n+1}(K))$. Hence $x \in N_n(K)$ and we get σ, a monomorphism. We now prove that σ is an epimorphism. Consider $s \in N^n(M)$ such that s is uniform and $s \notin N_n(M)$ then $s = a + b$, where $a \in N$, $b \in H_{n+1}(M)$. If $s \in N$ or $s \in H_{n+1}(M)$ we get $s \in N_n(M)$. Hence $aR \cap sR = 0 = bR \cap sR$. Consequently, $aR \subseteq bR \oplus sR$ with $a = -b + s$ gives $aR \cong bR$ under the correspondence $ar \leftrightarrow -br$. Then $H_1(aR) = H_1(bR)$ and the above correspondence is identity on $H_1(aR)$. Now $a = s - b \in K \cap H_n(M) = H_n(K)$, so that $H_1(aR) = H_1(bR) \subseteq H_{n+2}(M) \cap K = H_{n+2}(K)$ and we get $y \in H_{n+1}(K)$ such that $H_1(aR) = H_1(yR)$ and $\lambda : aR \to yR$ given by $\lambda(ar) = yr$ is identity on $H_1(aR)$. Consequently, $e(a - y) \le 1$. So that $a - y \in \operatorname{soc}(H_n(K))$. Then the mapping $\mu : bR \to yR$ such that $\mu(br) = -yr$ is also identity on $H_1(bR)$ and hence $b + y \in \operatorname{soc}(H_{n+1}(M))$. Therefore, $b + y \in N_n(M)$. Also $a - y \in (N + H_{n+1}(K)) \cap \operatorname{soc}(H_n(K))$. Hence

$$\sigma(a - y + N_n(K)) = a - y + N_n(M) = s - (b + y) + N_n(M) = s + N_n(M). \tag{4.1}$$

This proves that σ is an epimorphism. Hence the result follows. □

Theorem 4.3. *If N is h-neat submodule of M, then N is h-pure in M if and only if $Q_n(M,N)=0$ for every $n\geq 0$.*

Proof. Let N be h-pure in M then by, Theorem 4.2, $N^t(N)/N_t(N)\equiv N^t(M)/N_t(M)$ for all $t>0$, but $N^t(N)=N_t(N)$. Therefore, $N^t(M)=N_t(M)$ and we get $Q_t(M,N)=0$. Conversely, suppose $N\cap H_n(M)=H_n(N)$. Let x be a uniform element in $N\cap H_{n+1}(M)$ then there is a uniform element $y\in H_n(M)$ such that $d(yR/xR)=1$ and also as $x\in N\cap H_{n+1}(M)\subseteq N\cap H_n(M)=H_n(N)$ we can find a uniform element $x\in H_{n-1}(N)$ such that $d(zR/xR)=1$. Hence $e(y-z)\leq 1$ and so $y-z\in \operatorname{soc}(M)$ but $y-z\in N+H_n(M)$ and $y-z\in\operatorname{soc}(H_{n-1}(M))$. Therefore, $y-z\in (N+H_n(M))\cap\operatorname{soc}(H_{n-1}(M))$ but $N^{t-1}(M)=N_{t-1}(M)$, we get $y-z\in N\cap\operatorname{soc}(H_{n-1}(M))+\operatorname{soc}(H_n(M))$. So $y-z=a+b$, $a\in N\cap\operatorname{soc}(H_{n-1}(M))$, $b\in\operatorname{soc}(H_n(M))$, which gives $y-b=a+z\in N\cap H_n(M)=H_n(N)$. Hence $xR=H_1(yR)=H_1((y-b)R)\subseteq H_{n+1}(N)$. Therefore, N is h-pure in M. □

The question: what are the submodules for which $Q_n(M,N)=0$ for all $n\geq 0$? Gave the motivation to define the following.

Definition 4.4. A submodule N of a QTAG-module M is quasi h-pure in M if $Q_n(M,N)=0$ for all $n\geq 0$.

Proposition 4.5. *If N is h-pure submodule of M or if N is a subsocle of M, then N is quasi h-pure.*

Proof. If N is h-pure, then appealing to Theorem 4.3, we get N to be quasi h-pure. Now if $N\subseteq\operatorname{soc}(M)$, then trivially $N^t(M)=N_t(M)$ for all $t\geq 0$. Hence N is quasi h-pure submodule of M. □

Now we give the following nice characterization of quasi h-pure submodule.

Theorem 4.6. *If N is a submodule of a QTAG-module M, then the following are equivalent:*

(a) *N is quasi h-pure in M.*

(b) $\operatorname{soc}(N+H_n(M))=\operatorname{soc}(N)+\operatorname{soc}(H_n(M))$ *for all $n\geq 1$.*

(c) $H_1(N\cap H_n(M))=H_1(N)\cap H_{n+1}(M)$ *for all $n\geq 1$.*

Proof. (a)⇔(b). Suppose N is quasi h-pure in M then $Q_n(M,N)=0$ for all $n\geq 0$. Therefore, $N^t(M)=N_t(M)$ gives $\operatorname{soc}(N+H_1(M))=\operatorname{soc}(N)+\operatorname{soc}(H_1(M))$ for $t=0$. Now suppose (b) holds for all $t\leq m$, then $\operatorname{soc}(N+H_{m+1}(M))\subseteq\operatorname{soc}(N+H_m(M))=\operatorname{soc}(N)+\operatorname{soc}(H_m(M))$. Consequently,

$$\begin{aligned}\operatorname{soc}\left(N+H_{m+1}(M)\right)&=\left(N+H_{m+1}(M)\right)\cap\left[\operatorname{soc}(N)+\operatorname{soc}\left(H_m(M)\right)\right]\\&=\operatorname{soc}(N)+\left(N+H_{m+1}(M)\right)\cap\operatorname{soc}\left(H_m(M)\right)\\&=\operatorname{soc}(N)+N\cap\operatorname{soc}\left(H_m(M)\right)+\operatorname{soc}\left(H_{m+1}(M)\right)\\&=\operatorname{soc}(N)+\operatorname{soc}\left(H_{m+1}(M)\right).\end{aligned}\tag{4.2}$$

Hence (b) holds for all $n\geq 1$. Now suppose (b) holds then trivially

$$\left(N+H_{n+1}(M)\right)\cap\operatorname{soc}\left(H_n(M)\right)\subseteq N\cap\operatorname{soc}\left(H_n(M)\right)+\operatorname{soc}\left(H_{n+1}(M)\right).\tag{4.3}$$

Hence $Q_n(M,N) = 0$ for all $n \geq 1$. Therefore N is quasi h-pure in M.

(b)$\Leftrightarrow$(c). Suppose (b) holds. Trivially $H_1(N \cap H_n(M)) \subseteq H_1(N) \cap H_{n+1}(M)$. Let x be a uniform element in $H_1(N) \cap H_{n+1}(M)$, then we get uniform elements $y \in N$ and $z \in H_n(M)$ such that $d(yR/xR) = 1$ and $d(zR/xR) = 1$. Hence appealing to [4, Lemma 2.3] we get $e(y-z) \leq 1$, so $y-z \in \mathrm{soc}(N+H_n(M)) = \mathrm{soc}(N)+\mathrm{soc}(H_n(M))$. Hence $y-z = u+v$, $u \in \mathrm{soc}(N)$ and $v \in \mathrm{soc}(H_n(M))$. Thus $y-u = v+z \in N \cap H_n(M)$, consequently $xR = H_1((y-u)R) = H_1((v+z)R) \subseteq H_1(N \cap H_n(M))$. Hence (c) follows. Now suppose (c) holds. Let x be a uniform element in $\mathrm{soc}(N+H_n(M))$ then $x = w+t$, where $w \in N$ and $t \in H_n(M)$. Now $H_1(wR) = H_1((w-x)R) = H_1(-tR) \subseteq H_1(N) \cap H_{n+1}(M) = H_1(N \cap H_n(M))$. Hence, as done in the proof of Theorem 4.2, we get an element $s \in N \cap H_n(M)$ such that $H_1(wR) = H_1(-tR) = H_1(sR)$ and $e(w-s) \leq 1$ and $e(s+t) \leq 1$. Thus $x = w-s+s+t \in \mathrm{soc}(N)+\mathrm{soc}(H_n(M))$ and we get (b). □

Although the following result follows from Theorem 4.3, but using the above characterization we get a new proof.

Theorem 4.7. *If N is a submodule of M, then N is h-pure in M if and only if N is h-neat and quasi h-pure in M.*

Proof. If N is h-pure in M, then Theorem 4.3 implies that N is quasi h-pure in M. Now suppose N is h-neat and quasi h-pure in M and $N \cap H_n(M) = H_n(N)$, then $H_{n+1}(N) = H_1(N \cap H_n(M)) = H_1(N) \cap H_{n+1}(M)$ by above Theorem 4.6. But $H_1(N) \cap H_{n+1}(M) = (N \cap H_1(M)) \cap H_{n+1}(M) = N \cap H_{n+1}(M)$. Hence by induction N is h-pure in M. □

Now as an application of Theorem 4.6(b), we have the following.

Theorem 4.8. *If N is a submodeule of M, then the following hold:*

(i) *If $\mathrm{soc}(N)$ is h-dense in $\mathrm{soc}\,M$, then N is quasi h-pure in M.*

(ii) *If N is quasi h-pure in M, then every essential submodule of N is quasi h-pure in M.*

Proof. (i) Since $\mathrm{soc}(M) = \mathrm{soc}(N)+\mathrm{soc}(H_n(M))$ for all $n \geq 0$, so $\mathrm{soc}(N+H_n(M)) = \mathrm{soc}(N)+\mathrm{soc}(H_n(M))$ for all $n \geq 0$. Therefore N is quasi h-pure in M.

(ii) Let K be an essential submodule of N, then $\mathrm{soc}(K+H_n(M)) \subseteq \mathrm{soc}(N+H_n(M)) = \mathrm{soc}(N)+\mathrm{soc}(H_n(M))$. Hence $\mathrm{soc}(K+H_n(M)) = \mathrm{soc}(K)+\mathrm{soc}(H_n(M))$, consequently K is quasi h-pure in M. □

Corollary 4.9 (see [1, Theorem 66.3])**.** *If S is a h-dense subsocle of M, then any submodule N with $\mathrm{soc}(N) \subseteq S$ can be extended to an h-pure submodule K of M such that $\mathrm{soc}(K) = S$.*

Proof. Let K be an h-neat submodule such that $N \subseteq K$ and $S = \mathrm{soc}(K)$. Then by Theorem 4.8, K is quasi h-pure in M. Hence by Theorem 4.7, K is h-pure submodule of M. □

Proposition 4.10. *If N is a submodule of M, then the following hold:*

(i) $Q_{m+n}(M,N) = Q_m(H_n(M), N \cap H_n(M))$ *for all* $n,m \geq 0$.

(ii) $Q_j(M,N) = 0$ *for* $j = 0,1,\ldots,n$ *if and only if* $\mathrm{soc}(N+H_t(M)) = \mathrm{soc}(N)+\mathrm{soc}(H_t(M))$ *for* $t = 1,\ldots,n+1$.

(iii) *If N is quasi h-pure in M, then $N \cap H_n(M)$ is quasi h-pure in $H_n(M)$ for all n. Also if for some $n \geq 1$, $N \cap H_n(M)$ is quasi h-pure in $H_n(M)$ and* $\operatorname{soc}(N + H_t(M)) = \operatorname{soc}(N) + \operatorname{soc}(H_t(M))$ *for $t = 1,2,\ldots,n$, then N is quasi h-pure in M.*

Proof. (i) Is straightforward.

(ii) If $\operatorname{soc}(N + H_t(M)) = \operatorname{soc}(N) + \operatorname{soc}(H_t(M))$ for $t = 1,2,\ldots,n+1$, then trivially $Q_j(M,N) = 0$ for $j = 0,1,\ldots,n$. Conversely, as $Q_0(M,N) = 0$ we get $\operatorname{soc}(N + H_1(M)) = \operatorname{soc}(N) + \operatorname{soc}(H_1(M))$. Now suppose $\operatorname{soc}(N + H_t(M)) = \operatorname{soc}(N) + \operatorname{soc}(H_t(M))$ for $t < n+1$. Then $\operatorname{soc}(N + H_{t+1}(M)) \subseteq \operatorname{soc}(N) + \operatorname{soc}(H_t(M))$. As done in Theorem 4.6 we get $\operatorname{soc}(N + H_{t+1}(M)) = \operatorname{soc}(N) + \operatorname{soc}(H_{t+1}(M))$.

(iii) Due to (i), $N \cap H_n(M)$ is quasi h-pure in $H_n(M)$. Conversely, if $N \cap H_n(M)$ is quasi h-pure in $H_n(M)$, $Q_{m+n}(M,N) = 0$ for all $m \geq 0$. But from (ii) we have $Q_j(M,N) = 0$ for $j = 0,1,\ldots,n-1$. Hence $Q_t(M,N) = 0$ for all $t \geq 0$. So that N is quasi h-pure in M. □

Now we prove the following interesting result.

Proposition 4.11. *If N is a submodule of M and K is h-neat submodule of N. Then any submodule T of M maximal with respect to $T \cap N = K$, is h-neat and* $\operatorname{soc}(M) \subseteq T + \operatorname{soc}(N)$.

Proof. Trivially T/K is complement of N/K in M/K. Hence T/K is h-neat in M/K and $\operatorname{soc}(M/K) = \operatorname{soc}(T/K) = \operatorname{soc}(N/K)$. Using Proposition 3.1 we have $\operatorname{soc}(N/K) = (\operatorname{soc}(N) + K)/K$. Hence $\operatorname{soc}(M) \subseteq T + \operatorname{soc}(N)$. Let x be a uniform element in $T \cap H_1(M)$, then there exists a uniform element $y \in M$ such that $d(yR/xR = 1)$ if $y \in T$ we are done, otherwise h-neatness of T/K in M/K will result a uniform element $\bar{t} \in T/K$ such that $d(\bar{t}R/\bar{x}R) = 1$. Hence $e(\bar{y} - \bar{t}) \leq 1$. Therefore, $\bar{y} - \bar{t} \in \operatorname{soc}(M/K)$. Hence we can find $u \in \operatorname{soc}(N)$ and $v \in T$ such that $y - t - u - v \in K$. So $y = t + u + v + w$, $w \in K$. Hence $xR = H_1((t + u + v + w)R) = H_1((t + v + w)R) \subseteq H_1(T)$. Therefore T is h-neat in M. □

Theorem 4.12. *If K is h-pure submodule of $H_n(M)$, where $n \geq 0$. Then every submodule T of M maximal with respect to $T \cap H_n(M) = K$, is h-pure in M.*

Proof. Proposition 4.11 yields that T is h-neat in M and $\operatorname{soc}(M) \subseteq T + \operatorname{soc}(H_n(M))$. Hence $\operatorname{soc}(T + H_t(M)) = \operatorname{soc}(T) + \operatorname{soc}(H_t(M))$ for $t = 1,2,\ldots,n$. Trivially $T \cap H_n(M)$ is quasi h-pure in $H_n(M)$. Hence by Proposition 4.10(iii), T is quasi h-pure in M. Therefore by Theorem 4.7, T is h-pure in M. □

As in [3] a submodule N of M is called h-dense if M/N is h-divisible. From the notation of $N^t(M)$ and $N_t(M)$ it is easy to see that $N^t(M) = \operatorname{soc}(N \cap H_t(M) + H_{t+1}(M))$ and $N_t(M) = \operatorname{soc}(\operatorname{soc}(N) \cap H_t(M) + H_{t+1}(M))$. Now using Theorem 4.6 we establish the following results.

Proposition 4.13. *If N is a submodule of M and K is a quasi h-pure h-dense submodule of N, then $Q_t(M,K) = Q_t(M,N)$ for all $t \geq 0$.*

Proof. Due to h-divisibility of N/K, we have $N = K + H_t(N)$ for all $t \geq 0$. Hence $N^t(M) = K^t(M)$ for all t. Since K is quasi h-pure in N, so by Theorem 4.6, $\operatorname{soc}(N) =$

$\mathrm{soc}(K) = +\mathrm{soc}(H_t(N))$ for all $t \geq 0$. Now

$$\begin{aligned} N_t(M) &= \mathrm{soc}\left(\mathrm{soc}(N) \cap H_t(M) + H_{t+1}(M)\right) = \left(\mathrm{soc}(N)\right)^t(M) \\ &= \left(\mathrm{soc}(N) + H_{t+1}(M)\right) \cap \mathrm{soc}\left(H_t(M)\right) \\ &= \left(\mathrm{soc}(K) + \mathrm{soc}\left(H_{t+1}(N) + H_{t+1}(M)\right) \cap \mathrm{soc}\left(H_t(M)\right)\right) \\ &= \left(\mathrm{soc}(K) + H_{t+1}(M)\right) \cap \mathrm{soc}\left(H_t(M)\right) = \left(\mathrm{soc}(K)\right)^t(M) = K_t(M). \end{aligned} \tag{4.4}$$

Therefore, $Q_t(M,K) = Q_t(M,N)$. □

Proposition 4.14. *If N is quasi h-pure in M and* $\mathrm{soc}(N) \subseteq \cap_1^\infty H_n(M)$*, then* $N \subseteq \cap_1^\infty H_n(M)$.

Proof. Suppose every uniform element of N of exponent t lies inside $\cap H_n(M)$. Let x be a uniform element in N such that $e(x) = t+1$. Then we can find a uniform element $y \in xR$ such that $d(xR/yR) = 1$. Hence $y \in \cap H_n(M)$ and we get $y \in H_n(M)$ for every n. Consequently, there is a uniform element $z_i \in H_i(M)$ such that $d(z_iR/yR) = 1$ which in turn will give $e(x-z_i) \leq 1$. So $x - z_i \in \mathrm{soc}(N+H_i(M)) = \mathrm{soc}(N)+\mathrm{soc}(H_i(M))$. Let $x - z_i = u + v$, $u \in \mathrm{soc}(N)$ and $v \in \mathrm{soc}(H_i(M))$. Since $\mathrm{soc}(N) \subset \cap H_n(M)$, so $x \in \cap H_n(M)$ and we get $N \subseteq \cap_1^\infty H_n(M)$. □

Finally appealing to Theorem 4.2 and Proposition 4.13 we have the following.

Theorem 4.15. *If N is a submodule of M, then the following hold:*

(a) *If N is quasi h-pure in M and K is h-pure in M such that $N \subseteq K$, then N is quasi h-pure in K.*

(b) *If N is quasi h-pure in an h-pure submodule K of M, then N is quasi h-pure in M.*

(c) *If N is quasi h-pure in M, then every quasi h-pure and h-dense submodule K of N is quasi h-pure in M.*

(d) *If N has a quasi h-pure and h-dense submodule K such that K is also quasi h-pure in M, then N is quasi h-pure in M.*

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