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A SUBORDINATION THEOREM FOR SPIRALLIKE FUNCTIONS

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Abstract. We prove a subordination relation for a subclass of the class of λ-spirallike functions.

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1. Introduction. Let K denote the usual class of convex functions. Denote by $S_p(\lambda)$, $-\pi/2 < \lambda < \pi/2$, the class of functions $f(z) = z + a_2z^2 + \cdots$ which are analytic in E and satisfy therein the condition

$$\operatorname{Re}\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] > 0. \tag{1.1}$$

Spacek [3] proved that members of $S_p(\lambda)$, known as λ spirallike functions, are univalent in E. In 1989, Silverman [2] proved that if

$$\sum_{n=2}^{\infty}\left[1+(n-1)\sec\lambda\right]\left|a_n\right| \le 1 \quad \left(|\lambda| < \frac{\pi}{2}\right), \tag{1.2}$$

then the function $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ belongs to $S_p(\lambda)$. Let us denote by $G(\lambda)$, the class of function $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ whose coefficients satisfy the condition (1.2). Note that $G(0)$ is a subclass of the class of starlike functions (with respect to the origin) (see Silverman [1]).

In this paper, we prove a subordination theorem for the class $G(\lambda)$. To state and prove our main result we need the following definitions and lemma.

Definition 1.1. If $f(z) = \sum_{n=0}^{\infty} a_nz^n$ and $g(z) = \sum_{n=0}^{\infty} b_nz^n$ are analytic in $|z| < r$, then their Hadamard product/convolution, $f * g$ is the function defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_nb_nz^n. \tag{1.3}$$

The function $f * g$ is also analytic in $|z| < r$.

Definition 1.2. Let f be analytic in E, g analytic and univalent in E and $f(0) = g(0)$. Then by the symbol $f(z) \prec g(z)$ (f subordinate to g) in E, we shall mean that $f(E) \subset g(E)$.

Definition 1.3. A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z) = \sum_{k=1}^\infty a_k z^k$, $a_1 = 1$ is regular, univalent and convex in E, we have

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad \text{in } E. \tag{1.4}$$

Lemma 1.4. *The sequence $\{b_n\}_1^\infty$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left[1 + 2\sum_{n=1}^{\infty} b_n z^n\right] > 0, \quad (z \in E). \tag{1.5}$$

This lemma which gives a beautiful characterisation of a subordinating factor sequence is due to Wilf [4].

2. Main theorem

Theorem 2.1. *Let $f \in G(\lambda)$. Then*

$$\frac{1+\sec\lambda}{2(2+\sec\lambda)}(f * g)(z) \prec g(z), \quad (z \in E) \tag{2.1}$$

for every function g in the class K.

In particular

$$\operatorname{Re} f(z) > -\frac{2+\sec\lambda}{(1+\sec\lambda)}, \quad (z \in E). \tag{2.2}$$

The constant $(1+\sec\lambda)/2(2+\sec\lambda)$ cannot be replaced by any larger one.

Taking $\lambda = 0$, we obtain the following corollary.

Corollary 2.2. *If $f(z) = z + a_2 z^2 + \cdots$ is regular in E and satisfies therein the condition*

$$\sum_{n=2}^{\infty} n|a_n| \le 1, \tag{2.3}$$

then for every function g in K, we have

$$\frac{1}{3}(f * g)(z) \prec g(z), \quad (|z| < 1). \tag{2.4}$$

In particular, $\operatorname{Re} f(z) > -3/2$, $z \in E$. *The constant $1/3$ is best possible.*

Proof of Theorem 2.1. Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ be any member of the class $G(\lambda)$ and let $g(z) = z + \sum_{n=2}^\infty c_n z^n$ be any function in the class K. Then

$$\frac{1+\sec\lambda}{2(2+\sec\lambda)}(f * g)(z) = \frac{1+\sec\lambda}{2(2+\sec\lambda)}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right). \tag{2.5}$$

Thus, by Definition 1.3, the assertion of our theorem will hold if the sequence

$$\left(\frac{(1+\sec\lambda)a_n}{2(2+\sec\lambda)}\right)_{n=1}^{\infty} \tag{2.6}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of the lemma, this will be the case if and only if

$$\operatorname{Re}\left[1+2\sum_{n=1}^{\infty}\frac{1+\sec\lambda}{2(2+\sec\lambda)}a_n z^n\right]>0,\quad (z\in E). \tag{2.7}$$

Now

$$\begin{aligned}
\operatorname{Re}&\left[1+\frac{1+\sec\lambda}{2+\sec\lambda}\sum_{n=1}^{\infty}a_n z^n\right]\\
&=\operatorname{Re}\left[1+\frac{1+\sec\lambda}{2+\sec\lambda}z+\frac{1}{2+\sec\lambda}\sum_{n=2}^{\infty}(1+\sec\lambda)a_n z^n\right]\\
&>\left[1-\frac{1+\sec\lambda}{2+\sec\lambda}r-\frac{1}{2+\sec\lambda}\sum_{n=2}^{\infty}(1+(n-1)\sec\lambda)\,|a_n|\,r^n\right] \qquad (2.8)\\
&\qquad\text{(because } 1+\sec\lambda\le 1+(n-1)\sec\lambda \text{ for all } n\ge 2,\ |\lambda|<\pi/2)\\
&>\left[1-\frac{1+\sec\lambda}{2+\sec\lambda}r-\frac{1}{2+\sec\lambda}r\right]\quad (|z|=r)\\
&>0.
\end{aligned}$$

Thus (2.7) holds true in E. This proves the first assertion. That $\operatorname{Re}f(z)>-(2+\sec\lambda)/(1+\sec\lambda)$ for $f\in G(\lambda)$ follows by taking $g(z)=z/(1-z)$ in (2.1). To prove the sharpness of the constant $(1+\sec\lambda)/2(2+\sec\lambda)$, we consider the function f_0 defined by $f_0(z)=z-(1/(1+\sec\lambda))z^2(|\lambda|<\pi/2)$, which is a member of the class $G(\lambda)$. Thus from the relation (2.1) we obtain

$$\frac{1+\sec\lambda}{2(2+\sec\lambda)}f_0(z)\prec\frac{z}{1-z}. \tag{2.9}$$

It can be easily verified that

$$\min_{|z|\le 1}\operatorname{Re}\left[\frac{1+\sec\lambda}{2(2+\sec\lambda)}f_0(z)\right]=-\frac{1}{2}. \tag{2.10}$$

This shows that the constant $(1+\sec\lambda)/2(2+\sec\lambda)$ is best possible. □

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