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BIORTHOGONALITY CONDITION FOR CREEPING MOTION IN ANNULAR TRENCHES

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ABSTRACT. The biorthogonality condition for Stokes flow in annular trenches bounded by horizontal parallel planes and concentric vertical cylinders is derived. This condition, is needed to compute the coefficients of the eigenfunction expansion solution of the corresponding Stokes flow problem.

Keywords and phrases. Eigenvalues, eigenfunctions, eigenfunction expansion, biorthogonality conditions, Stokes flow.

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1. Introduction. Recently, the eigenfunction expansion method has been used extensively for solving problems of Stokes flow; (cf. [1, 2, 7, 8]). Most recently, biorthogonality conditions were used by Khuri to solve Stokes flow in a sectorial cavity [3] and by Khuri and Wang for solving Stokes flow around a bend [6]. The method leads to the development of a set of eigenfunctions, adjoint eigenfunctions, biorthogonality conditions and an algorithm for the computation of the eigenfunction expansions.

In the present paper, we derive the biorthogonality condition for Stokes flow in annular trenches bounded by horizontal parallel planes and concentric vertical cylinders. For biorthogonality conditions in other flow geometries, derived in a similar fashion, see [4, 5].

2. Biorthogonality condition. In this section, we state Theorem 2.1 which gives the biorthogonality condition satisfied by the eigenfunctions and adjoint eigenfunctions of the following fourth-order boundary value problem:

$$\left(P_0(r)y''(r)\right)'' + \left(P_1(r;\alpha)y'(r)\right)' + P_2(r;\alpha)y(r) = 0, \quad r \in [r_1, r_2]. \tag{2.1}$$

The boundary conditions are given by

$$y(r_1) = y(r_2) = y'(r_1) = y'(r_2) = 0. \tag{2.2}$$

This biorthogonality condition was proved by Khuri [3].

THEOREM 2.1 (biorthogonality condition). *Consider the boundary value problem given in (2.1) and (2.2) where $P_0(r)$, $P_1''(r;\alpha)$, $P_2(r;\alpha)$ are continuous and $P_0(r) \neq 0$ on $r_1 \le r \le r_2$. P_i in (2.1) is a polynomial of degree at most i in the parameter α, where $i = 0,1,2$, in particular, let $P_1(r;\alpha) = p_{11}(r)\alpha + p_{12}(r)$, and we require*

$$P_1^2(r;\alpha) - 4P_0(r)P_2(r;\alpha) = p_{31}(r)\alpha + p_{32}(r), \qquad p_{11}^2(r) + p_{31}^2(r) \not\equiv 0. \tag{2.3}$$

Then with P_n^* defined by

$$P_n^* = \int_{r_1}^{r_2} \left[\phi_2^{(n)}(r), \phi_1^{(n)}(r)\right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr, \tag{2.4}$$

we have the following biorthogonality condition:

$$\int_{r_1}^{r_2} \left[\phi_2^{(m)}(r), \phi_1^{(m)}(r)\right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr = P_n^* \delta_{mn}, \tag{2.5}$$

where δ_{mn} is the Kronecker's delta,

$$B(r) = \begin{pmatrix} -\dfrac{1}{2}\dfrac{p_{11}(r)}{P_0(r)} & 0 \\ \dfrac{1}{2}p_{11}''(r) + \dfrac{1}{4}\dfrac{p_{31}(r)}{P_0(r)} & -\dfrac{1}{2}\dfrac{p_{11}(r)}{P_0(r)} \end{pmatrix} \tag{2.6}$$

with

$$\begin{aligned} \phi_1^{(n)}(r) &= y_n(r), \\ \phi_2^{(n)}(r) &= P_0(r) y_n''(r) + \frac{1}{2} P_1(r; \alpha_n) y_n(r). \end{aligned} \tag{2.7}$$

Here y_i is an eigenfunction of (2.1) corresponding to the eigenvalue α_i. Assume the eigenvalues α_i are simple.

3. Stokes flow in annular trenches. Next, we derive the biorthogonality condition for Stokes flow in annular trenches bounded by horizontal parallel planes and concentric vertical cylinders which was studied by Yoo and Joseph [8]. The region is given by

$$v = \{r, z : 0 < r_1 \le r \le r_2,\ -z_1 \le z \le z_1\}. \tag{3.1}$$

The Stokes flow equation in v for axisymmetric flow in cylindrical coordinates is

$$E^4 \Psi(r,z) = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \Psi(r,z) = 0. \tag{3.2}$$

We require the velocity to vanish on the rigid boundaries of the two concentric cylinders where $r = r_1, r_2$ so we need

$$\Psi(r_1, z) = \Psi(r_2, z) = \frac{\partial \Psi}{\partial r}(r_1, z) = \frac{\partial \Psi}{\partial r}(r_2, z) = 0. \tag{3.3}$$

Other edge data are prescribed on $z = \pm z_1$ as in [8].

Separable solutions of (3.2) and (3.3) in the form

$$\Psi(r,z) \sim e^{\pm pz} T(r) \tag{3.4}$$

exist [8], when $T(r)$ satisfies the following equation:

$$T^{(4)} - \frac{2}{r}T^{(3)} + \left(2p^2 + \frac{3}{r^2}\right)T^{(2)} - \left(\frac{2p^2}{r} + \frac{3}{r^3}\right)T^{(1)} + p^4 T = 0 \tag{3.5}$$

and the boundary conditions

$$T(r_1) = T(r_2) = T'(r_1) = T'(r_2) = 0. \tag{3.6}$$

Clearly (3.5) can be rewritten in the following form:

$$\left(\frac{1}{r}T''\right)'' + \left(\left[\frac{2p^2}{r} + \frac{1}{r^3}\right]T'\right)' + \frac{p^4}{r}T = 0. \tag{3.7}$$

Comparing with Theorem 2.1, we require $\alpha_n \neq \alpha_m$ and we have

$$P_0(r) = \frac{1}{r}, \qquad P_1(r;\alpha) = \frac{2\alpha}{r} + \frac{1}{r^3}, \qquad P_2(r;\alpha) = \frac{\alpha^2}{r}, \tag{3.8}$$

where

$$\alpha = p^2. \tag{3.9}$$

Since

$$P_1^2(r;\alpha) - 4P_0(r)P_2(r;\alpha) = \frac{4\alpha}{r^4} + \frac{1}{r^6} \tag{3.10}$$

thus

$$p_{31}(r) = \frac{4}{r^4}, \qquad p_{32}(r) = \frac{1}{r^6}. \tag{3.11}$$

Clearly,

$$p_{11}(r) = \frac{2}{r}; \qquad p_{12}(r) = \frac{1}{r^3}. \tag{3.12}$$

By applying Theorem 2.1, the biorthogonality condition is given by

$$\int_{r_1}^{r_2} \left[\phi_1^{(m)}(r), \phi_2^{(m)}(r)\right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr = P_n^* \delta_{mn}, \quad (p_n^2 \neq p_m^2), \tag{3.13}$$

where

$$B(r) = \begin{pmatrix} -1 & 0 \\ \dfrac{3}{r^3} & -1 \end{pmatrix}. \tag{3.14}$$

The eigenfunctions satisfy

$$\begin{gathered} \phi_1^{(n)}(r) = T_n(r), \\ \phi_2^{(n)}(r) = \frac{1}{r}T_n''(r) + \left(\frac{1}{r}\alpha_n + \frac{1}{2r^3}\right)T_n(r) \end{gathered} \tag{3.15}$$

and the adjoint eigenfunctions satisfy

$$\begin{gathered} \psi_1^{(m)}(r) = \frac{1}{r}T_m''(r) + \left(\frac{1}{r}\alpha_m + \frac{2}{r^3}\right)T_m(r), \\ \psi_2^{(m)}(r) = T_m(r), \end{gathered} \tag{3.16}$$

where

$$\alpha_n = p_n^2. \tag{3.17}$$

A similar biorthogonality condition was derived by Yoo and Joseph [8], where $\phi^{(n)}$ and $\psi^{(n)}$ were defined through a two-dimensional eigenvalue problem. The eigenfunctions and adjoint eigenfunctions that we have derived are, however, given explicitly in terms of T_n.

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