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ON THE FEKETE-SZEGÖ PROBLEM

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ABSTRACT. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be an analytic function in the open unit disk. A sharp upper bound is obtained for $|a_3 - \mu a_2^2|$ by using the classes of strongly starlike functions of order β and type α when $\mu \geq 1$.

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1. Introduction. Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{A}$ is said to be strongly starlike of order β and type α in $\mathcal{U}$, and denoted by $\bar{\mathcal{S}}^*_\alpha(\beta)$ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathcal{U}) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$ and β $(0 < \beta \leq 1)$. If $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathcal{U}) \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$ and β $(0 < \beta \leq 1)$, then we say that $f(z)$ is strongly convex of order β and type α in $\mathcal{U}$, and we denote by $\bar{\mathcal{C}}_\alpha(\beta)$ the class of all such functions (see also Srivastava and Owa [16]). For the class $\mathcal{S}$ of analytic univalent functions, Fekete-Szegö [6] obtained the maximum value of $|a_3 - \mu a_2^2|$ when μ is real. For various functions of $\mathcal{S}$, the upper bound for $|a_3 - \mu a_2^2|$ is investigated by many different authors including [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 17].

In this paper, we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when f belonging to the classes of functions defined as follows.

DEFINITION 1.1. Let $0 \leq \alpha < 1$, $\beta > 0$ and let $f \in \mathcal{A}$. Then $f \in \mathcal{M}(\alpha,\beta)$ if and only if there exist $g \in \bar{\mathcal{S}}^*_\alpha(\beta)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathcal{U}), \tag{1.4}$$

and $f \in \mathcal{G}(\alpha,\beta)$ if and only if there exists $g \in \bar{\mathcal{C}}_\alpha(\beta)$ and satisfy condition (1.4) with $g(z) = z + b_2z^2 + b_3z^3 + \cdots$.

Note that $\mathcal{M}(0,\beta) = \mathcal{K}(\beta)$ is the class of close-to-convex functions defined in [3] and $\mathcal{M}(0,1) = \mathcal{K}(1)$ is the class of normalized close-to-convex functions defined by Kaplan [9].

2. Main results.

In order to derive our main results, we have to recall here the following lemma [15].

Lemma 2.1. *Let $h \in \mathcal{P}$, that is, h be analytic in $\mathcal{U}$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, and $\operatorname{Re} h(z) > 0$ for $z \in \mathcal{U}$, then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}. \tag{2.1}$$

Theorem 2.2. *Let $f(z) \in \mathcal{M}(\alpha,\beta)$ and be given by (1.1). Then for $0 \leq \alpha < 1$, $\beta \geq 1$, and $\mu \geq 1$ we have the sharp inequality*

$$|a_3 - \mu a_2^2| \leq \frac{2\beta^2(\mu-1) + \alpha\beta^2(8-2\alpha-3\mu)}{(1-\alpha)^2(2-\alpha)} + \frac{(2\beta+1-\alpha)(3\mu-2)}{3(1-\alpha)}. \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{M}(\alpha,\beta)$. It follows from (1.4) that

$$zf'(z) = g(z)q(z), \tag{2.3}$$

for $z \in \mathcal{U}$, with $q \in \mathcal{P}$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$. Equating coefficients, we obtain

$$2a_2 = q_1 + b_2, \qquad 3a_3 = q_2 + b_2 q_1 + b_3. \tag{2.4}$$

Also, it follows from (1.2) that

$$zg'(z) - \alpha g(z) = g(z)\big(p(z)\big)^{\beta}, \tag{2.5}$$

where $z \in \mathcal{U}$, $p \in \mathcal{P}$, and $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. Thus equating coefficients, we obtain

$$(1-\alpha)b_2 = \beta p_1, \qquad (2-\alpha)b_3 = \beta\left(p_2 + \frac{\beta(3-\alpha)+\alpha-1}{2(1-\alpha)} p_1^2\right). \tag{2.6}$$

From (2.4) and (2.6), we have

$$\begin{aligned} a_3 - \mu a_2^2 = {} & \frac{1}{3}\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{2-3\mu}{12} q_1^2 + \frac{\beta}{3(2-\alpha)}\left(p_2 - \frac{1}{2}p_1^2\right) \\ & + \frac{\beta^2[6(1-\mu)+\alpha(2\alpha+3\mu-8)]}{12(1-\alpha)^2(2-\alpha)} p_1^2 + \frac{\beta(2-3\mu)}{6(1-\alpha)} p_1 q_1. \end{aligned} \tag{2.7}$$

Assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\operatorname{Re}(a_3 - \mu a_2^2)$, so from (2.7) and by using Lemma 2.1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq R \leq 1$, $0 \leq \theta < 2\pi$, and $0 \leq \phi < 2\pi$, we obtain

$$
\begin{aligned}
3\operatorname{Re}(a_3-\mu a_2^2) &= \operatorname{Re}\left(q_2-\frac{1}{2}q_1^2\right)+\frac{2-3\mu}{4}\operatorname{Re}q_1^2+\frac{\beta}{(2-\alpha)}\operatorname{Re}\left(p_2-\frac{1}{2}p_1^2\right)\\
&\quad+\frac{\beta^2[(6+2\alpha^2+3\alpha\mu)-(6\mu+8\alpha)]}{4(1-\alpha)^2(2-\alpha)}\operatorname{Re}p_1^2+\frac{\beta(2-3\mu)}{2(1-\alpha)}\operatorname{Re}p_1q_1.\\
&\le 2(1-R^2)+(2-3\mu)R^2\cos 2\phi+\frac{2\beta}{2-\alpha}(1-r^2)\\
&\quad+\frac{\beta^2[(6+2\alpha^2+3\alpha\mu)-(6\mu+8\alpha)]}{(1-\alpha)^2(2-\alpha)}r^2\cos 2\theta+\frac{2\beta(2-3\mu)}{1-\alpha}rR\cos(\theta+\phi)\\
&\le (3\mu-4)R^2+\frac{2\beta(3\mu-2)}{1-\alpha}rR\\
&\quad+\frac{6\beta^2(\mu-1)+\alpha\beta^2(8-2\alpha-3\mu)-2\beta(1-\alpha)^2}{(1-\alpha)^2(2-\alpha)}r^2+\frac{2(\beta-\alpha)+4}{2-\alpha}\\
&=\Psi(r,R).
\end{aligned}
\tag{2.8}
$$

Letting α, β, and μ fixed and differentiating $\Psi(r,R)$ partially when $0\le\alpha<1$, $\beta\ge1$, and $\mu\ge1$, we observe that

$$
\begin{aligned}
\Psi_{rr}\Psi_{RR}-(\Psi_{rR})^2 &= 4\beta[4\beta+2+\alpha(2\alpha\beta+2\alpha-4-7\beta)]\\
&\quad-3\beta\mu[6\beta+2+\alpha(2\alpha\beta+2\alpha-4-8\beta)]<0.
\end{aligned}
\tag{2.9}
$$

Therefore, the maximum of $\Psi(r,R)$ occurs on the boundaries. Thus the desired inequality follows by observing that

$$
\Psi(r,R)\le\Psi(1,1)=\frac{6\beta^2(\mu-1)+\alpha\beta^2(8-2\alpha-3\mu)}{(1-\alpha)^2(2-\alpha)}+\frac{(2\beta+1-\alpha)(3\mu-2)}{1-\alpha}. \tag{2.10}
$$

The equality for (2.2) is attained when $p_1=q_1=2i$ and $q_1=q_2=-2$.

Letting $\alpha=0$ in Theorem 2.2, we have the result given by Jahangiri [8]. □

Corollary 2.3. *Let $f(z)\in\mathcal{K}(\beta)$ and be given by (1.1). Then for $\beta\ge1$, and $\mu\ge1$, we have the sharp inequality*

$$
|a_3-\mu a_2^2|\le\beta^2(\mu-1)+\frac{(2\beta+1)(3\mu-2)}{3}. \tag{2.11}
$$

Theorem 2.4. *Let $f(z)\in\mathcal{G}(\alpha,\beta)$ and be given by (1.1). Then for $0\le\alpha<1$, $\beta\ge1$, and $\mu\ge1$, we have the sharp inequality*

$$
|a_3-\mu a_2^2|\le\frac{6\beta^2(3\mu-4)+\alpha\beta^2(32-8\alpha-9\mu)}{36(1-\alpha)^2(2-\alpha)}+\frac{(\beta+1-\alpha)(3\mu-2)}{3(1-\alpha)}. \tag{2.12}
$$

Proof. Let $f(z)\in\mathcal{G}(\alpha,\beta)$. It follows from (1.3) that

$$
zg''(z)+(1-\alpha)g'(z)=g'(z)(p(z))^\beta, \tag{2.13}
$$

where $z\in\mathcal{U}$, $p\in\mathcal{P}$, and $p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots$. Thus equating coefficients, we obtain

$$
2(1-\alpha)b_2=\beta p_1,\qquad 3(2-\alpha)b_3=\beta\left(p_2+\frac{\beta(3-\alpha)+\alpha-1}{2(1-\alpha)}p_1^2\right). \tag{2.14}
$$

From (2.4) and (2.14) and proceeding as in the proof of Theorem 2.2, we get

$$\begin{aligned} 3\operatorname{Re}(a_3-\mu a_2^2) &\le (3\mu-4)R^2+\frac{\beta(3\mu-2)}{1-\alpha}rR+\frac{2(\beta-3\alpha)+12}{3(2-\alpha)} \\ &\quad+\frac{6\beta^2(3\mu-4)+\alpha\beta^2(32-8\alpha-9\mu)-8\beta(1-\alpha)^2}{12(1-\alpha)^2(2-\alpha)}r^2 \\ &=\Phi(r,R). \end{aligned} \tag{2.15}$$

Letting α, β and μ fixed and differentiating $\Phi(r,R)$ partially when $0\le\alpha<1$, $\beta\ge 1$, and $\mu\ge 1$, we have

$$\begin{aligned} \Phi_{rr}\Phi_{RR}-(\Phi_{rR})^2 &= 4\beta[18\beta+8+\alpha(8\alpha\beta+8\alpha-16-29\beta)] \\ &\quad-3\beta\mu[24\beta+8+\alpha(8\alpha\beta+8\alpha-16-32\beta)]<0. \end{aligned} \tag{2.16}$$

Therefore, the maximum of $\Phi(r,R)$ occurs on the boundaries. Thus the desired inequality (2.12) follows by observing that

$$\Phi(r,R)\le\Phi(1,1)=\frac{6\beta^2(3\mu-4)+\alpha\beta^2(32-8\alpha-9\mu)}{12(1-\alpha)^2(2-\alpha)}+\frac{(\beta+1-\alpha)(3\mu-2)}{1-\alpha}. \tag{2.17}$$

The equality in (2.12) is attained on choosing $p_1=q_1=2i$ and $q_1=q_2=-2$. This completes the proof of Theorem 2.4. □

Corollary 2.5. *Let $f(z)\in\mathcal{G}(0,\beta)$ and be given by (1.1). Then for $\beta\ge 1$, and $\mu\ge 1$, we have the sharp inequality*

$$|a_3-\mu a_2^2|\le\frac{1}{12}\left[(3\mu-2)(\beta+2)^2-2\beta^2\right]. \tag{2.18}$$

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