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# ON COINCIDENCE AND COMMON FIXED POINTS OF NEARLY DENSIFYING MAPPINGS

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ABSTRACT. Coincidence and common fixed point theorems for certain new classes of nearly densifying mappings are established. Our results extend, improve, and unify a lot of previously known theorems.

Keywords and phrases. Complete metric space, coincidence point, common fixed point, nearly densifying mapping, attractor, left reversible semigroup, $F$-diminishing orbital diameter.

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**1. Introduction.** Furi and Vignoli [4] established first the existence of fixed point for densifying mappings. Afterwards Chatterjee [1], Diviccaro, Khan and Sessa [2], Fisher and Khan [3], Iseki [6, 7], Jain and Jain [8], Janos, Ko, and Tan [9], Khan [10], Khan and Fisher [11], Khan and Liu [12], Khan and Rao [13], Khan [14], Liu [17, 18, 19, 20, 21, 16], Pande [24, 25], Rao [22], Ray and Fisher [26], Sastry and Naidu [27], Sharma [28], Sharma and Srivastava [29] and others obtained fixed and coincidence point theorems for densifying and nearly densifying mappings, respectively. Huang, Huang, and Jeng [5] proved a common fixed point theorem for a left reversible semigroup, which consists of a number of continuous self-mappings in compact metric spaces.

The purpose of this paper is to establish coincidence and common fixed point theorems for certain new classes of nearly densifying mappings in complete metric spaces. In Section 2, we introduce notation, terminology and prove a lemma, which plays an important role in the paper. In Section 3, we obtain some common fixed point theorems for families of mappings. In Section 4, we give general coincidence point theorems for two pairs of mappings. Our results extend, improve, and unify the corresponding results of Chatterjee [1], Diviccaro, Khan, and Sessa [2], Huang, Huang, and Jeng [5], Janos, Ko, and Tan [9], Khan [10], Khan and Liu [12], Khan and Rao [13]. Liu [17, 18, 19], Rao [22], Sharma and Srivastava [29] and others.

**2. Preliminaries.** Recall that a semigroup $G$ is said to be left reversible if for any $s,t \in G$ there exist $u,v \in G$ such that $su = tv$. It is easy to see that the notion of left reversibility is equivalent to the statement that any two right ideals of $G$ have nonempty intersection. A semigroup $G$ is called near-commutative if for any $s,t \in G$ there exists $u \in G$ such that $st = tu$. Clearly, every commutative semigroup is near-commutative, and every near-commutative semigroup is left reversible, but the converses are not true.

Throughout this paper, $(X,d)$ denotes a metric space, $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ denote the sets of positive integers, nonnegative real numbers, and real numbers, respectively, and $\omega = \mathbb{N}\bigcup\{0\}$. Define

$$\begin{aligned}
\mathfrak{I} &= \{F \mid F : X\times X \longrightarrow \mathbb{R}^+ \text{ and } F(x,y) = 0 \text{ if and only if } x = y\}, \\
\mathfrak{I}_1 &= \{F \mid F \in \mathfrak{I} \text{ and } F \text{ is upper semicontinuous in } X\times X\}, \\
\mathfrak{I}_2 &= \{F \mid F \in \mathfrak{I} \text{ and } F \text{ is lower semicontinuous in } X\times X\}.
\end{aligned} \tag{2.1}$$

Let $G$ be a family of self-mappings in $X$. A subset $Y$ of $X$ is called $G$-invariant if $gY \subseteq Y$ for all $g \in G$. Let

$$\begin{aligned}
NCI_G &= \{Y \mid Y \text{ is nonempty compact } G\text{-invariant subset of } X\}, \\
CIS_G &= \{g \mid g : X \longrightarrow X \text{ and } gY \subseteq Y,\ \forall Y \in NCI_G\},
\end{aligned} \tag{2.2}$$

and $G^*$ be the semigroup generated by $G$ under composition. Clearly, $CIS_G \supseteq G^* \supseteq \{g^n : n \in \omega\}$ for any $g \in G$. For $A,B \subseteq X$, $x,y \in X$, $f \in G$, and $F \in \mathfrak{I}$, define

$$\begin{aligned}
&\delta_F(A,B) = \sup\{F(a,b) : a \in A, b \in B\}, && \delta_F(A) = \delta_F(A,A), \\
&\delta_F(x,A) = \delta_F(\{x\},A), && \delta_F(x,y) = \delta_F(\{x\},\{y\}), \\
&O_f(x) = \{f^n x : n \in \omega\}, && O_f(x,y) = O_f(x)\bigcup O_f(y), \\
&C_f = \{h \mid h : X \longrightarrow X, fh = hf\}, && G^*x = \{x\}\bigcup\{gx : g \in G^*\}, \\
&CIS_f = CIS_{\{f\}}, && NCI_f = NCI_{\{f\}}.
\end{aligned} \tag{2.3}$$

$\bar{A}$ denotes the closure of $A$. $f$ is said to have diminishing orbital diameter if $\lim_{n\to\infty}\delta_d(O_f(f^n x)) < \delta_d(O_f(x))$ for all $x \in X$ with $\delta_d(O_f(x)) > 0$. $f$ is called contractive with respect to $d$ if $d(fx,fy) < d(x,y)$ for all distinct $x,y \in X$.

**Definition 2.1.** Let $G$ be a semigroup of self-mappings on a metric space $(X,d)$ and $F \in \mathfrak{I}$. $G$ is said to have $F$-diminishing orbital diameter, if for any $x \in X$ with $\delta_F(Gx) > 0$ there is $s \in G$ such that $\delta_F(Gsx) < \delta_F(Gx)$.

**Definition 2.2** (see [15])**.** Let $A$ be a bounded subset of a metric space $(X,d)$. Then $\alpha(A)$, the measure of noncompactness of $A$, is the infimum of all $\varepsilon > 0$ such that $A$ admits a finite covering consisting of subsets with diameters less than $\varepsilon$.

The following properties of $\alpha$ are well known.

**Lemma 2.3.** *Let $(X,d)$ be a metric space and $A,B$ be bounded subsets of $X$. Then*

$$\alpha\Big(A\bigcup B\Big) = \max\{\alpha(A),\alpha(B)\}; \tag{2.4}$$

$$\alpha(A) = 0 \Longleftrightarrow A \text{ is pre-compact, i.e., } A \text{ is totally bounded;} \tag{2.5}$$

$$\alpha(A) = \alpha(\bar{A}). \tag{2.6}$$

**Definition 2.4** (see [4])**.** A continuous self-mapping $f$ in a metric space $(X,d)$ is said to be densifying if $\alpha(f(A)) < \alpha(A)$ for every bounded subset $A$ of $X$ with $\alpha(A) > 0$.

**Definition 2.5** (see [27]). A self-mapping $f$ in a metric space $(X,d)$ is said to be nearly densifying if $\alpha(f(A)) < \alpha(A)$ for every bounded and $f$-invariant subset $A$ of $X$ with $\alpha(A) > 0$.

Obviously, each densifying mapping is nearly densifying, but the converse is false.

**Definition 2.6** (see [23]). Let $X$ be a topological space, $f$ a self-mapping in $X$, and $M$ a nonempty subset of $X$. $M$ is an attractor for compact sets under $f$ if

(i) $M$ is compact and $fM \subseteq M$,

(ii) given any compact set $C \subseteq X$ and any open neighborhood $U$ of $M$, there exists $k \in N$ such that $f^n C \subseteq U$ for all $n \ge k$.

Let $G$ be a left reversible semigroup. We define a relation $\ge$ on $G$ by $a \ge b$ if and only if $a \in bG \bigcup \{b\}$. It is easy to check that $(G,\ge)$ is a directed set.

**Lemma 2.7.** *Let $G$ be a left reversible semigroup of continuous self-mappings in a compact metric space $(X,d)$, $A = \bigcap_{f\in G} fx$, and $F \in \Im_1$. Then*

$$\lim_{f\in G} \delta_F(fx) = \delta_F(A); \qquad A \in NCI_G \quad \text{and} \quad fA = A, \quad \forall f \in G. \tag{2.7}$$

**Proof.** Note that $fX \subseteq gX$ for all $f,g \in G$ with $f \ge g$. Thus $\{\delta_F(fX)\}_{f\in G}$ is a bounded decreasing net in $\mathbb{R}$. Obviously, $\lim_{f\in G} \delta_f(fX)$ exists in $\mathbb{R}$ and

$$\delta_F(A) \le \lim_{f\in G} \delta_F(fx). \tag{2.8}$$

We now prove that $fX$ is a compact subset of $X$ for each $f \in G$. Let $x$ be in $X$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} fx_n = x$. The compactness of $X$ ensures that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that it converges to some point $t \in X$. Since $f$ is continuous, so $x = ft \in fX$. Therefore $fX$ is closed. That is, $fX$ is compact. This means that $A$ is compact.

We next prove that

$$\delta_F(A) \ge \lim_{f\in G} \delta_F(fX). \tag{2.9}$$

Let $f \in G$. Since $F$ is upper semicontinuous and $fX \times fX$ is compact, there exist $x_f, y_f \in fX$ with $F(x_f, y_f) = \delta_F(fX)$. From the compactness of $X$ we can choose two subnets $\{x_{f_k}\}$ and $\{y_{f_k}\}$ of $\{x_f\}$ and $\{y_f\}$, respectively, such that $x_{f_k} \to x$ and $y_{f_k} \to y$ for some $x,y \in X$. For every $g \in G$ and $f_k \ge g$, we get that $x_{f_k}$, $y_{f_k} \in gX$. By virtue of closedness of $gX$, we infer that $x,y \in gX$. This means that $x,y \in A$. Consequently,

$$\lim_{f\in G} \delta_F(fX) = \lim_{f\in G} F(x_f, y_f) = \lim_k F(x_{f_k}, y_{f_k}) = F(x,y) \le \delta_F(A). \tag{2.10}$$

Thus (i) follows from (2.8) and (2.9).

Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in G$. It follows from the left reversibility of $G$ that there exist $g_1, g_2, \ldots, g_n \in G$ with $f_1 g_1 = f_2 g_2 = \cdots = f_n g_n = h$ for some $h \in G$. Hence $\bigcap_{i=1}^n f_i X \supseteq hX \ne \emptyset$. The compactness of $X$ implies that $A \ne \emptyset$.

We finally prove that $fA = A$ for all $f \in G$. Let $f \in G$ and $x \in A$. For any $g \in G$ there exist $a, b \in G$ with $fa = gb$. Note that $x \in A \subseteq aX$. Thus there is $y \in X$ with $x = ay$. It follows that $fx = fay = gby \in gX$. This implies that $fA \subseteq \bigcap_{g\in G} gX = A$ for $f \in G$. For the reverse inclusion, let $f, g \in G$ and $y \in A$. It follows from $y \in fgX$ that there exists $x_g \in gX$ with $fx_g = y$. The compactness $X$ ensures that there exists a convergent subnet $\{x_{g_k}\}$ of $\{x_g\}$ such that $x_{g_k} \to x$ for some $x \in X$. Therefore $y = fx$. For any $h, g \in G$ with $g \geq h$, we obtain that $hX$ is closed and $x_g$ belongs to $hX$. Thus the limit point $x$ of $\{x_g\}$ lies in $hX$. That is, $x \in A$. Note that $y = fx \in fA$. Therefore, $A \subseteq fA$ for $f \in G$. This completes the proof. □

**Remark 2.8.** Lemma 2.7 generalizes Lemma 2.3 of Huang, Huang, and Jeng [5].

## 3. Common fixed point theorems for nearly densifying mappings

**Theorem 3.1.** *Let $G$ and $H$ be finite families of continuous and nearly densifying self-mappings in a complete metric space $(X, d)$. If there exist $g \in G^*$, $h \in H^*$, $F \in \Im_1$, $x_0, y_0 \in X$ such that*

$$F(gx, hy) < \delta_F\left(\bigcup_{s\in CIS_G} sG^*x, \bigcup_{t\in CIS_H} tH^*y\right), \quad \forall x, y \in X \text{ with } gx \neq hy; \tag{3.1}$$

*$G^*x_0, H^*y_0$ are bounded and $G^*, H^*$ are left reversible.Then the following statements hold:*

(i) *$G$ and $H$ have a unique common fixed point $w \in X$, and $w$ is also the only fixed point of $G$ and $H$, respectively;*

(ii) $\lim_{s\in G^*} F(sx_0, w) = \lim_{t\in H^*} F(ty_0, w) = \lim_{s\in G^*} \delta_F(s\overline{G^*x_0}) = \lim_{t\in H^*} \delta_F(t\overline{H^*y_0}) = 0$*;*

(iii) *for any $C \in NCI_{G^*}$ and any $D \in NCI_{H^*}$, $\bigcap_{s\in G^*} sC = \bigcap_{t\in H^*} tD = \{w\}$.*

**Proof.** Let $A = \bigcap_{s\in G^*} s\overline{G^*x_0}$ and $B = \bigcap_{t\in H^*} t\overline{H^*y_0}$. Since $G^*x_0 = \{x_0\} \bigcup_{s\in G} sG^*x_0$ and $G$ is finite, so

$$\alpha(G^*x_0) = \max\{\alpha(x_0), \alpha(sG^*x_0) : s \in G\} = \max\{\alpha(sG^*x_0) : s \in G\}. \tag{3.2}$$

Note that each $s \in G$ is nearly densifying. Thus, $\alpha(G^*x_0) = 0$. It follows from Lemma 2.3 that $G^*x_0$ is pre-compact. Completeness of $(X, d)$ ensures that $\overline{G^*x_0}$ is compact. Since every $s \in G^*$ is continuous, $s\overline{G^*x_0} \subseteq \overline{sG^*x_0} \subseteq \overline{G^*x_0}$. By Lemma 2.7 we immediately conclude that $A \in NCI_{G^*}$ and $fA = A$ for all $f \in G^*$. Similarly, $B \in NCI_{H^*}$ and $fB = B$ for all $f \in H^*$.

We assert that $\delta_F(A, B) = 0$. Otherwise $\delta_F(A, B) > 0$. Since $F$ is upper semicontinuous and $A \times B$ is compact, we can easily choose $a \in A$ and $b \in B$ with $F(a, b) = \delta_F(A, B)$. Therefore, there exist $x \in A$ and $y \in B$ such that $a = gx$ and $b = hy$. Using (3.1), we have

$$\begin{aligned} F(a, b) = F(gx, hy) &< \delta_F\left(\bigcup_{s\in CIS_G} sG^*x, \bigcup_{t\in CIS_H} tH^*y\right) \\ &\leq \delta_F(A, B) = F(a, b), \end{aligned} \tag{3.3}$$

which is a contradiction. Consequently, $A = B = a$ singleton, say, $\{w\}$ for some $w \in X$. Thus $w = fw$ for all $f \in G \bigcup H$. That is, $G$ and $H$ have a common fixed point $w \in X$. If $G$ has another fixed point $v \in X$ and $v \neq w$, by (3.1) we infer that

$$F(v,w) = F(gv,hw) < \delta_F\left(\bigcup_{s\in CIS_G} sG^*v, \bigcup_{t\in CIS_H} tH^*w\right) = F(v,w), \tag{3.4}$$

which is absurd. Hence $G$ has a unique fixed point $w$. Similarly, we conclude that $H$ has also a unique fixed point $w$.

It follows from Lemma 2.7 that

$$\lim_{s\in G^*} \delta_F(s\overline{G^*x_0}) = \delta_F(A) = \delta_F(\{w\}) = 0 = \delta_F(B) = \lim_{t\in H^*} \delta_F(t\overline{H^*y_0}). \tag{3.5}$$

Note that $sx_0 \in sG^*x_0$, $ty_0 \in tH^*y_0$ and $w \in s\overline{G^*x_0} \bigcap t\overline{H^*y_0}$ for all $s \in G^*$, $t \in H^*$. Thus Theorem 3.1(ii) follows immediately from (3.5).

Let $C \in NCI_{G^*}$ and $Y = \bigcap_{s\in G^*} sC$. Lemma 2.7 ensures that $Y \in NCI_{G^*}$ and $fY = Y$ for all $f \in G^*$. Suppose that $\delta_F(Y,w) > 0$. Then there exists $x \in Y$ such that $F(gx,w) = \delta_F(Y,w)$. In view of (3.1) and Theorem 3.1(i). We obtain that $F(gx,w) < \delta_F(\bigcup_{s\in CIS_G} sG^*x, w) \leq \delta_F(Y,w)$, which is impossible. Hence $\delta_F(y,w) = 0$. That is, $Y = \bigcap_{s\in G^*} sC = \{w\}$. Similarly, we obtain that $\bigcap_{t\in H^*} tD = \{w\}$ if $D \in NCI_{H^*}$. This completes the proof. □

**Theorem 3.2.** *Let $G$ and $H$ be finite families of continuous and nearly densifying self-mappings in a complete bounded metric space $(X,d)$ satisfying (3.1). Assume that $G^*$, $H^*$ are near commutative. Then Theorem 3.1(i), (iii), and the following statements hold:*

(i)

$$\begin{aligned}\lim_{s\in G^*} F(sx,w) = \lim_{t\in H^*} F(ty,w) &= \lim_{s\in G^*} \delta_F(s\overline{G^*x}) \\ &= \lim_{t\in H^*} \delta_F(t\overline{H^*y}) = 0, \quad \forall x,y \in X;\end{aligned} \tag{3.6}$$

(ii) *$G^*$ and $H^*$ have $F$-diminishing orbital diameter.*

**Proof.** Let $x,y$ be in $X$. Put $A = \bigcap_{s\in G^*} s\overline{G^*x}$ and $B = \bigcap_{t\in H^*} t\overline{H^*y}$. As in the proof of Theorem 3.1, we conclude that $A \in NCI_{G^*}$, $fA = A$ for all $f \in G^*$ and $B \in NCI_{H^*}$, $gB = B$ for all $g \in H^*$. It follows from Theorem 3.1(ii) that

$$A = \bigcap_{s\in G^*} sA = \{w\} = B = \bigcap_{t\in H^*} tB. \tag{3.7}$$

Thus (3.6) follows from Lemma 2.7 and the definitions of $G^*x$ and $H^*y$.

Given $s,t \in G^*$. Since $G^*$ is commutative, there is $g \in G^*$ with $ts = sg$. This means that

$$\delta_F(G^*sx) = \delta_F(\{sx\} \bigcup \{tsx : t \in G^*\}) \leq \delta_F(sG^*x) \leq \delta_F(s\overline{G^*x}). \tag{3.8}$$

Suppose that $\delta_F(G^*sx) > 0$. In view of (3.6) and (3.8) there exists $s \in G^*$ such that $\delta_F(G^*sx) < \delta_F(G^*x)$. That is, $G^*$ has $F$-diminishing orbital diameter. Analogously, $H^*$ has $F$-diminishing orbital diameter also. This completes the proof. □

We now state without proof analogues of Theorems 3.1 and 3.2.

**Theorem 3.3.** *Let $G$ be a finite family of continuous and nearly densifying self-mappings in a complete metric space $(X,d)$. If there exist $g,h \in G^*$, $F \in \Im_1$, $x_0 \in X$ such that*

$$F(gx,hy) < \delta_F\left(\bigcup_{s\in CIS_G}\left(sG^*x\bigcup sG^*y\right)\right), \quad \forall x,y \in X \text{ with } gx \neq hy; \tag{3.9}$$

*$G^*x_0$ is bounded and $G^*$ is left reversible.Then the following statements hold:*

(i) *$G$ has a unique common fixed point $w \in X$, and*

$$\lim_{s\in G^*} F(sx_0,w) = \lim_{s\in G^*}\delta_F(s\overline{G^*x_0}) = 0; \tag{3.10}$$

(ii) *for any $C \in NCI_{G^*}$, $\bigcap_{s\in G^*} sC = \{w\}$.*

**Theorem 3.4.** *Let $f$ and $g$ be continuous self-mappings in a complete metric space $(X,d)$. Assume that there exist $i,j,p,q \in \mathbb{N}$, $F \in \Im_1$, $x_0,y_0 \in X$ such that*

(i) $F(f^px,g^qy) < \delta_F(\bigcup_{s\in CIS_f} sO_f(x),\bigcup_{t\in CIS_g} tO_g(y))$, $\forall x,y \in X$ *with* $f^px \neq g^qy$;

(ii) *$f^i$ and $g^j$ are nearly densifying;*

(iii) *$O_f(x_0)$ and $O_g(y_0)$ are bounded.*

*Then the following statements hold:*

(1) *$f$ and $g$ have a unique common fixed point $w \in X$, and $w$ is also the only fixed point of $f$ and $g$, respectively;*

(2) $\lim_{n\to\infty} F(f^nx_0,w) = \lim_{n\to\infty} F(g^ny_0,w) = \lim_{n\to\infty}\delta_F(f^n\overline{O_f(x_0)}) = \lim_{n\to\infty}\delta_F(g^n\overline{O_g(y_0)}) = 0$;

(3) *for any $C \in NCI_f$ and any $D \in NCI_g$,*

$$\bigcap_{n\in\mathbb{N}} f^nC = \bigcap_{n\in\mathbb{N}} g^nD = \{w\}. \tag{3.11}$$

**Proof.** Set $A = \bigcap_{n\in\mathbb{N}} f^n\overline{O_f(x_0)}$ and $B = \bigcap_{n\in\mathbb{N}} g^n\overline{O_g(y_0)}$. In view of Theorem 3.4(ii), (iii) and

$$\alpha(O_f(x_0)) = \max\{\alpha\{f^kx_0 : 0 \le k \le i-1\},\alpha(f^iO_f(x_0))\}, \tag{3.12}$$

we conclude easily that $A \in NCI_f$ and $fA = A$. Similarly, $B \in NCI_g$ and $gB = B$. The rest of the proof is the same as that of Theorem 3.1. This completes the proof. □

**Remark 3.5.** Theorem 3.4 extends Theorems 3 and 4 of Liu [19], the theorem of Sharma and Srivastava [29]. Akin to Theorem 3.4, we have the following.

**Theorem 3.6.** *Let $f$ be continuous self-mapping in a complete metric space $(X,d)$. Assume that there exist $i,p,q \in \mathbb{N}$, $F \in \Im_1$, $x_0 \in X$ such that*

(i) $F(f^px,f^qy) < \delta_F(\bigcup_{s\in CIS_f} sO_f(x,y))$, $\forall x,y \in X$ *with* $f^px \neq f^qy$;

(ii) *$f^i$ is nearly densifying;*

(iii) *$O_f(x_0)$ is bounded.*

*Then the following statements hold:*

(1) *$f$ has a unique fixed point $w \in X$, and* $\lim_{n\to\infty} F(f^n x_0, w) = \lim_{n\to\infty} \delta_F(f^n \overline{O_f(x_0)}) = 0$*;*

(2) *for any $C \in NCI_f$, $\bigcap_{n\in\mathbb{N}} f^n C = \{w\}$.*

**Remark 3.7.** Theorem 4 of Khan [10] and Theorem 4 of Rao [22] are special cases of Theorem 3.6.

**Theorem 3.8.** *Let $f$ and $g$ be continuous self-mappings in a complete bounded metric space $(X,d)$. Assume that there exist $i,j,p,q \in \mathbb{N}$ satisfying Theorem 3.4(ii) and*

$$d(f^p x, g^q y) < \delta_d\left(\bigcup_{s\in CIS_f} sO_f(x), \bigcup_{t\in CIS_g} tO_g(y)\right), \quad \forall x,y \in X \text{ with } f^p x \neq g^q y. \tag{3.13}$$

*Then Theorem 3.4(1) and (3.11) and the following statements hold:*

(i) $\lim_{n\to\infty} d(f^n x, w) = \lim_{n\to\infty} d(g^n y, w) = \lim_{n\to\infty} \delta_d(f^n O_f(x)) = \lim_{n\to\infty} \delta_d(g^n O_g(y)) = 0$, $\forall x,y \in X$;

(ii) *there exist bounded complete metrics $d_1, d_2$ on $X$ which are equivalent to $d$ such that $f,g$ are contractive with respect to $d_1$ and $d_2$, respectively;*

(iii) *$CIS_f$ and $CIS_g$ have a unique common fixed point $w \in X$, and $w$ is also the only fixed point of $CIS_f$ and $CIS_g$, respectively;*

(iv) *$f$ and $g$ have diminishing orbital diameter.*

**Proof.** It follows from Theorem 3.4 that Theorem 3.4(1), (3.11), and Theorem 3.8(i) hold. By the definitions of $CIS_f$ and $CIS_g$, we conclude easily that Theorem 3.8(iii) holds. Since $f^n O_f(x) = O_f(f^n x)$ and $g^n O_g(y) = O_g(g^n y)$, so Theorem 3.6(iv) is satisfied. Now we prove that Theorem 3.8(ii) holds. Assume that $B$ be any nonempty compact subset of $X$. Using Lemma 2.3, we have

$$\begin{aligned}\alpha\left(\bigcup_{n\in\omega} f^n B\right) &= \max\left\{\alpha\left(\bigcup_{n=0}^{i-1} f^n B\right), \alpha\left(\bigcup_{n=i}^{\infty} f^n B\right)\right\} \\ &= \alpha\left(\bigcup_{n=i}^{\infty} f^n B\right) = \alpha\left(f^i \bigcup_{n\in\omega} f^n B\right).\end{aligned} \tag{3.14}$$

Thus $\bigcup_{n\in\omega} f^n B$ is totally bounded because $f^i$ is nearly densifying. Set $C = \overline{\bigcup_{n\in\omega} f^n B}$. Since $f$ is continuous and $X$ is complete, we infer that $C$ is compact and $fC \subseteq \overline{f\bigcup_{n\in\omega} f^n B} \subseteq C$. Hence (3.11) ensures that $\bigcap_{n\in\omega} f^n C = \{w\}$. This means that $\delta_d(f^n C) \downarrow 0$ as $n \to \infty$. For each open neighborhood $U$ of $w$, there exists an open ball $B(w,\varepsilon) = \{x : x \in X \text{ and } d(x,w) < \varepsilon\}$ with $B(w,\varepsilon) \subseteq U$. Note that $\delta_d(f^n C) \downarrow 0$ as $n \to \infty$. Thus there exists $k \in \mathbb{N}$ such that $\delta_d(f^n C) < \varepsilon$ for all $n \geq k$. Given $x \in f^n C$ and $n \geq k$, we obtain that $d(x,w) \leq \delta_d(f^n C) < \varepsilon$. Consequently, $f^n B \subseteq f^n C \subseteq B(w,\varepsilon) \subseteq U$ for all $n \geq k$. This shows that $\{w\}$ is an attractor for compact sets under $f$. Thus Theorem 3.8(ii) follows from theorem of [9] and Remark 1 of [9]. This completes the proof. □

Similarly, we have the following theorem.

**Theorem 3.9.** *Let $f$ be a continuous self-mapping in a complete bounded metric space $(X,d)$. Assume that there exist $i,p,q \in \mathbb{N}$ satisfying Theorem 3.6(ii) and*

$$d(f^p x, f^q y) < \delta_d\left(\bigcup_{s \in CIS_f} sO_f(x,y)\right), \quad \forall x,y \in X \text{ with } f^p x \neq f^q y. \tag{3.15}$$

*Then Theorem 3.6(2) and the following statements hold:*

(i) *$f$ has a unique fixed point $w \in X$, and has diminishing orbital diameter and $\lim_{n\to\infty} d(f^n x, w) = \lim_{n\to\infty} \delta_d(f^n O_f(x)) = 0$, $\forall x \in X$;*

(ii) *there exists a bounded complete metric $d_1$ on $X$ which is equivalent to $d$ such that $f$ is contractive with respect to $d_1$;*

(iii) *$CIS_f$ has a unique common fixed point $w \in X$.*

**Remark 3.10.** Theorem 3.8 generalizes Theorem 4 of [2] and Theorem 4 of [22]. Theorem 3.9 extends and improves Theorem 3 of [1], Corollary 2 of [9], Theorem 3.1 of [17], and Theorems 1 and 2 of [18]

## 4. Coincidence point theorems for two pairs of nearly densifying mappings

**Theorem 4.1.** *Let $f$, $g$, $s$, and $t$ be a continuous and nearly densifying mappings from a complete metric space $(X,d)$ into itself satisfying*

$$fgt = ftg = tfg \qquad \text{and} \qquad gst = sgt = stg. \tag{4.1}$$

*Let $G = \{f,g,s,t\}$. Assume that there exist $F_1, F_2 \in \mathfrak{I}$ and $x_0 \in X$ such that*

$$F_1 \text{ or } F_2 \in \mathfrak{I}_2; \tag{4.2}$$

$$\begin{aligned} F_1(fx,gy) < \max\Bigg\{ & F_2(sx,ty), F_2(sx,fx), F_1(ty,gy), \\ & \min\{F_2(sx,gy), F_1(fx,ty)\}, \frac{[F_2(sx,ty)]^2}{F_1(fx,gy)}, \\ & \frac{[F_2(sx,fx)]^2}{F_1(fx,gy)}, \frac{[F_1(ty,gy)]^2}{F_1(fx,gy)}, \\ & \frac{F_2(sx,ty)F_1(fx,ty)}{F_1(fx,gy)}, \frac{F_2(sx,fx)F_1(fx,ty)}{F_1(fx,gy)}, \\ & \frac{F_1(ty,gy)F_1(fx,ty)}{F_1(fx,gy)}, \frac{F_2(sx,gy)F_1(fx,ty)}{F_1(fx,gy)}, \\ & \frac{[F_2(sx,fx)]^2}{F_2(sx,ty)}, \frac{F_2(sx,fx)F_1(ty,gy)}{F_2(sx,ty)}, \\ & \frac{F_2(sx,fx)F_1(fx,gy)}{F_2(sx,ty)}, \frac{F_2(sx,fx)F_1(fx,ty)}{F_2(sx,ty)}, \\ & \frac{F_1(ty,gy)F_1(fx,ty)}{F_2(sx,ty)}, \frac{F_2(sx,gy)F_1(sx,ty)}{F_2(sx,ty)} \Bigg\} \end{aligned} \tag{4.3}$$

*for all $x,y \in X$ with $sx \neq ty$, $fx \neq gy$;*

$$F_2(gx,fy) < \max\Big\{F_1(tx,sy),F_1(tx,gx),F_2(sy,fy), \min\{F_1(gx,sy),F_2(tx,fy)\},\frac{[F_1(tx,sy)]^2}{F_2(gx,fy)}, \frac{[F_1(tx,gx)]^2}{F_2(gx,fy)},\frac{[F_2(sy,fy)]^2}{F_2(gx,fy)},\frac{F_1(tx,sy)F_1(gx,sy)}{F_2(gx,fy)}, \frac{F_1(tx,gx)F_1(gx,sy)}{F_2(gx,fy)},\frac{F_2(sy,fy)F_1(gx,sy)}{F_2(gx,fy)}, \frac{F_2(tx,fy)F_1(gx,sy)}{F_2(gx,fy)},\frac{[F_1(tx,gx)]^2}{F_1(tx,sy)},\frac{F_1(tx,gx)F_2(sy,fy)}{F_1(tx,sy)}, \frac{F_1(tx,gx)F_2(gx,fy)}{F_1(tx,sy)},\frac{F_1(tx,gx)F_1(gx,sy)}{F_2(tx,sy)}, \frac{F_2(sy,fy)F_1(gx,sy)}{F_1(tx,sy)},\frac{F_1(gx,sy)F_2(tx,fy)}{F_1(tx,sy)}\Big\} \tag{4.4}$$

*for all $x,y \in X$ with $gx \neq fy$, $tx \neq sy$*;

$$G^*x_0 \text{ is bounded and } G^* \text{ is left reversible.} \tag{4.5}$$

*Then $f$ and $s$ or $g$ and $t$ have a coincidence point in $X$.*

**Proof.** Put $A = G^*x_0$. It follows that $A = \{x_0\}\bigcup fA\bigcup gA\bigcup sA\bigcup tA$. This yields that

$$\alpha(A) = \max\{\alpha(fA),\alpha(gA),\alpha(sA),\alpha(tA)\}. \tag{4.6}$$

It is evident to see that $\alpha(A) = 0$. Thus $\bar{A}$ is compact by completeness of $X$. Set $B\bigcap_{h\in G^*} h\bar{A}$. Lemma 2.7 ensures that $fB = gB = sB = tB = B \neq \emptyset$ and $B$ is compact. Let $F_1$ be in $\Im_2$. Define $r : B \to \mathbb{R}^+$ by putting $r(x) = F_1(tx,gx)$. Since $r$ is a lower semi-continuous function on the compact set $B$, so there exists $b \in B$ with

$$r(b) = F_1(tb,gb) = \inf_{x\in B} F_1(tx,gx). \tag{4.7}$$

Suppose that neither $f$ and $s$ nor $g$ and $t$ have a coincidence point. Then

$$tfgc \neq gfgc, \qquad tstc \neq gstc, \qquad stgc \neq ftgc, \tag{4.8}$$

where $b = stc \in B$. In view of (4.1), (4.3), (4.4), (4.7) and (4.8), we have

$$r(fgc) = F_1(tfgc,gfgc) = F_1(ftgc,gfgc) < \max\Big\{F_2(stgc,tfgc),F_2(stgc,ftgc),F_1(tfgc,gfgc), \min\{F_2(stgc,gfgc),F_1(ftgc,tfgc)\}\frac{[F_2(stgc,tfgc)]^2}{F_1(ftgc,gfgc)}, \frac{[F_2(stgc,ftgc)]^2}{F_1(ftgc,gfgc)},\frac{[F_1(tfgc,gfgc)]^2}{F_1(ftgc,gfgc)},$$

$$\frac{F_2(stgc,tfgc)F_1(ftgc,tfgc)}{F_1(ftgc,gfgc)},\frac{F_2(stgc,ftgc)F_1(ftgc,tfgc)}{F_1(ftgc,gfgc)},$$

$$\frac{F_1(tfgc,gfgc)F_1(ftgc,tfgc)}{F_1(ftgc,gfgc)},\frac{F_2(stgc,gfgc)F_1(ftgc,tfgc)}{F_1(ftgc,gfgc)},$$

$$\frac{[F_2(stgc,ftgc)]^2}{F_2(stgc,tfgc)},\frac{F_2(stgc,ftgc)F_1(tfgc,gfgc)}{F_2(stgc,tfgc)},$$

$$\frac{F_2(stgc,ftgc)F_1(ftgc,gfgc)}{F_2(stgc,tfgc)},\frac{F_2(stgc,ftgc)F_1(ftgc,tfgc)}{F_2(stgc,tfgc)},$$

$$\frac{F_1(tfgc,gfgc)F_1(ftgc,tfgc)}{F_2(stgc,tfgc)},\frac{F_2(stgc,gfgc)F_1(ftgc,tfgc)}{F_2(stgc,tfgc)}\bigg\}$$

$$= \max\Big\{F_2(gstc,fgtc),F_2(gstc,fgtc),r(fgc),0,\frac{[F_2(gstc,fgtc)]^2}{r(fgc)},$$

$$\frac{[F_2(gstc,fgtc)]^2}{r(fgc)},r(fgc),0,0,0,0,F_2(gstc,fgtc),$$

$$r(fgc),r(fgc),0,0,0\Big\}$$

$$= \max\left\{F_2(gstc,fgtc),\frac{[F_2(gstc,fgtc)]^2}{r(fgc)}\right\}$$

$$= F_2(gstc,fgtc)$$

$$< \max\Big\{F_1(tstc,sgtc),F_1(tstc,gstc),F_2(sgtc,fgtc)$$

$$\min\{F_1(gstc,sgtc),F_2(tstc,fgtc)\},\frac{[F_1(tstc,sgtc)]^2}{F_2(gstc,fgtc)},$$

$$\frac{[F_1(tstc,gstc)]^2}{F_2(gstc,fgtc)},\frac{[F_2(sgtc,fgtc)]^2}{F_2(gstc,fgtc)},$$

$$\frac{F_1(tstc,sgtc)F_1(gstc,sgtc)}{F_2(gstc,fgtc)},\frac{F_1(tstc,gstc)F_1(gstc,sgtc)}{F_2(gstc,fgtc)},$$

$$\frac{F_2(sgtc,fgtc)F_1(gstc,sgtc)}{F_2(gstc,fgtc)},\frac{F_2(tstc,fgtc)F_1(gstc,sgtc)}{F_2(gstc,fgtc)},$$

$$\frac{[F_1(tstc,gstc)]^2}{F_1(tstc,sgtc)},\frac{F_1(tstc,gstc)F_2(sgtc,fgtc)}{F_1(tstc,sgtc)},$$

$$\frac{F_1(tstc,gstc)F_2(gstc,fgtc)}{F_1(tstc,sgtc)},\frac{F_1(tstc,gstc)F_1(gstc,sgtc)}{F_1(tstc,sgtc)},$$

$$\frac{F_2(sgtc,fgtc)F_1(gstc,sgtc)}{F_1(tstc,sgtc)},\frac{F_1(gstc,sgtc)F_2(tstc,fgtc)}{F_1(tstc,sgtc)}\bigg\}$$

$$= \max\Big\{r(b),r(b),F_2(gstc,fgtc),0,\frac{[r(b)]^2}{F_2(gstc,fgtc)},\frac{[r(b)]^2}{F_2(gstc,fgtc)},$$

$$F_2(gstc,fgtc),0,0,0,0,r(b),$$

$$F_2(gstc,fgtc),F_2(gstc,fgtc),0,0,0\Big\}$$

$$= r(b), \tag{4.9}$$

which implies that

$$r(b) \leq r(fgc) < r(b), \tag{4.10}$$

which is a contradiction. Hence $f$ and $s$ or $g$ and $t$ must have a coincidence point. The argument is similar for $F_2 \in \mathfrak{I}_2$. This completes the proof. □

**Theorem 4.2.** *Let $f, g, s$, and $t$ be continuous and nearly densifying mappings from a complete metric space $(X,d)$ into itself satisfying $f, g \in C_s \bigcap C_t$. Let $G = \{f,g,s,t\}$ and $H = \{s,t\}$. Assume that there exist $F_1, F_2 \in \mathfrak{I}$ and $x_0 \in X$ such that (4.2), (4.3), (4.4), and the following statement hold:*

$$G^* x_0 \text{ is bounded and } H^* \text{ is left reversible.} \tag{4.11}$$

*Then $f$ and $s$ or $g$ and $t$ have a coincidence point in $X$.*

**Proof.** Put $A = G^* x_0$ and $B = \bigcap_{h \in H^*} h\bar{A}$. As in the proof of Theorem 4.1, we infer that $B$ is nonempty compact subset of $\bar{A}$ and $sB = tB = B \supseteq fB \bigcup gB$. The remaining part of the proof is as in Theorem 4.1. This completes the proof. □

**Remark 4.3.** Theorem 3.1 of [12] and Theorem 3.1 of [13] are special cases of Theorem 4.2.

**Remark 4.4.** The following example reveals that $f, g, s$, and $t$ in Theorems 4.1 and 4.2 do not necessarily have a coincidence point and that if either $f$ and $s$ or $g$ and $t$ have a coincidence point, then the coincidence point may not be unique.

**Example 4.5.** Let $X = \{1,3,6\}$ with the usual metric $d$ and $F_1 = F_2 = d$. Define $f, g, s, t : X \to X$ by $f1 = g3 = g6 = 1$, $f3 = f6 = g1 = 3$ and $s = t = i_X$—the identity mapping on $X$. Take $G = \{f,g,s,t\}$ and $H = \{s,t\}$. Clearly, $g^2 = f = f^2$, $g = fg = gf = g^3$, $G = G^*$, $H = H^*$, and $G^*$ and $H^*$ are left reversible. It is easy to verify that

$$d(fx, gy) = 2 < 3 = d(sx, ty) \tag{4.12}$$

for all $x, y \in X$ with $sx \neq ty, fx \neq gy$, and

$$d(gx, fy) = 2 < 3 = d(tx, sy) \tag{4.13}$$

for all $x, y \in X$ with $tx \neq sy, gx \neq fy$. Thus the conditions of Theorems 4.1 and 4.2 are satisfied. However, $f$ and $s$ have two coincidence points 1 and 3, while $f, g, s$, and $t$ have none.

**Theorem 4.6.** *Let $f, g, s$, and $t$ be continuous and nearly densifying mappings from a complete metric space $(X,d)$ into itself satisfying $f, g, s \in C_t$ and $g \in C_s$. Assume that there exist $F_1, F_2 \in \mathfrak{I}$ and $x_0 \in X$ satisfying (4.2), (4.3), and (4.4). If $b$ is a common coincidence point of $f, g, s$, and $t$, then $tb$ is a unique common fixed point of $f, g, s$, and $t$.*

**Proof.** Since $f,g,s \in C_t$, $g \in C_s$, and $fb = gb = sb = tb$, we have $t^2b = tfb = ftb = tgb = gtb = tsb = stb$. Suppose that $t^2b \neq tb$. From (4.3) and (4.4) we conclude that

$$\begin{aligned}
F_1(t^2b,tb) &= F_1(ftb,gb)\\
&< \max\Big\{F_2(stb,tb),F_2(stb,ftb),F_1(tb,gb),\\
&\qquad \min\{F_2(stb,gb),F_1(ftb,tb)\},\frac{[F_2(stb,tb)]^2}{F_1(ftb,gb)},\\
&\qquad \frac{[F_2(stb,ftb)]^2}{F_1(ftb,gb)},\frac{[F_1(tb,gb)]^2}{F_1(ftb,gb)},\frac{F_2(stb,tb)F_1(ftb,tb)}{F_1(ftb,gb)},\\
&\qquad \frac{F_2(stb,ftb)F_1(ftb,tb)}{F_1(ftb,gb)},\frac{F_1(tb,gb)F_1(ftb,tb)}{F_1(ftb,gb)},\\
&\qquad \frac{F_2(stb,gb)F_1(ftb,tb)}{F_1(ftb,gb)},\frac{[F_2(stb,ftb)]^2}{F_2(stb,tb)},\frac{F_2(stb,ftb)F_1(tb,gb)}{F_2(stb,tb)},\\
&\qquad \frac{F_2(stb,ftb)F_1(ftb,gb)}{F_2(stb,tb)},\frac{F_2(stb,ftb)F_1(ftb,gb)}{F_2(stb,tb)},\\
&\qquad \frac{F_1(tb,gb)F_1(ftb,tb)}{F_2(stb,tb)},\frac{F_2(stb,gb)F_1(ftb,tb)}{F_2(stb,tb)}\Big\}\\
&= \max\Big\{F_2(t^2b,tb),\frac{[F_2(t^2b,tb)]^2}{F_1(t^2b,tb)},F_1(t^2b,tb)\Big\}\\
&= F_2(t^2b,tb) = F_2(gtb,fb)\\
&< \max\Big\{F_1(t^2b,sb),F_1(t^2b,gtb),F_2(sb,fb),\\
&\qquad \min\{F_1(gtb,sb),F_2(t^2b,fb)\},\frac{[F_1(t^2b,sb)]^2}{F_2(gtb,fb)},\\
&\qquad \frac{[F_1(t^2b,gtb)]^2}{F_2(gtb,fb)},\frac{[F_2(sb,fb)]^2}{F_2(gtb,fb)},\frac{F_1(t^2b,sb)F_1(gtb,sb)}{F_2(gtb,fb)},\\
&\qquad \frac{F_1(t^2b,gtb)F_1(gtb,sb)}{F_2(gtb,fb)},\frac{F_2(sb,fb)F_1(gtb,sb)}{F_2(gtb,fb)},\\
&\qquad \frac{F_2(t^2b,fb)F_1(gtb,sb)}{F_2(gtb,fb)},\frac{[F_1(t^2b,gtb)]^2}{F_1(t^2b,sb)},\\
&\qquad \frac{F_1(t^2b,gtb)F_2(sb,fb)}{F_1(t^2b,sb)},\frac{F_1(t^2b,gtb)F_2(gtb,fb)}{F_1(t^2b,sb)},\\
&\qquad \frac{F_1(t^2b,gtb)F_1(gtb,sb)}{F_1(t^2b,sb)},\frac{F_2(sb,fb)F_1(gtb,sb)}{F_1(t^2b,sb)},\\
&\qquad \frac{F_1(gtb,sb)F_2(t^2b,fb)}{F_1(t^2b,sb)}\Big\}\\
&= \max\Big\{F_1(t^2b,tb),\frac{[F_1(t^2b,tb)]^2}{F_2(t^2b,tb)},F_2(t^2b,tb)\Big\}\\
&= F_1(t^2b,tb),
\end{aligned} \tag{4.14}$$

which is a contradiction. Therefore $tb = t^2b = ftb = gtb = stb$. That is, $tb$ is a common fixed point of $f,g,s$, and $t$. The uniqueness of a common fixed point follows from (4.3) and (4.4). This completes the proof. □

**Remark 4.7.** Theorem 4.6 extends Theorem 3.2 of [12] and Theorem 3.2 of [13].

**Theorem 4.8.** *Let $f,g,s$, and $t$ be continuous and nearly densifying mappings from a complete metric space $(X,d)$ into itself and $G=\{f,g,s,t\}$. Suppose that there exist $F\in\Im_2$ and $x_0\in X$ such that (4.5) and the following hold:*

$$F(fx,gy)>\inf\{F(fz,sz),F(gz,tz):z\in G^*x\bigcup G^*y\},\qquad \forall x,y\in X \text{ with } fx\neq gy. \tag{4.15}$$

*Then $f$ and $s$ or $g$ and $t$ have a coincidence point in $X$.*

**Proof.** Define $A=G^*x_0$ and $B=\bigcap_{h\in G^*}h\bar{A}$. As in the proof of Theorem 4.1, we infer that $B$ is compact, $hB=B\neq\emptyset$ for all $h\in G^*$, and there are $a,b\in B$ such that

$$F(fa,sa)=\inf\{F(fx,sx):x\in B\},\qquad F(gb,tb)=\inf\{F(gx,tx):x\in B\}. \tag{4.16}$$

Without loss of generality, we assume that

$$F(fa,sa)\le F(gb,tb). \tag{4.17}$$

Since $f,g,s$, and $t\in G^*$, it follows that $fB=gB=sB=tB=B$. Thus there exist $v,w\in B$ with $a=gv$ and $sa=gw$. We claim that $fa=sa$. If not, then $fgv\neq gw$. By virtue of (4.15), (4.16), and (4.17), we have

$$\begin{aligned}F(fa,sa)&=F(fgv,gw)\\&>\inf\Big\{F(fz,sz),F(gz,tz):z\in G^*gv\bigcup G^*y\Big\}\\&\ge\inf\{F(fz,sz),F(gz,tz):z\in B\}\\&=F(fa,sa),\end{aligned} \tag{4.18}$$

which is a contradiction. Hence $fa=sa$. This completes the proof. □

**Theorem 4.9.** *Let $f$ and $g$ be continuous and nearly densifying mappings from a complete metric space $(X,d)$ into itself and $G=\{f,g\}$. Suppose that there exist $F\in\Im_2$ and $x_0\in X$ satisfying (4.5) and*

$$F(fx,gy)>\inf\Big\{F(fz,z),F(gz,z),F(hx,hy):z\in G^*x\bigcup G^*y,\ h\in C_f\bigcap C_g\bigcap G^*\Big\},\qquad \forall x,y\in X \text{ with } fx\neq gy. \tag{4.19}$$

*Then $f$ or $g$ has a fixed point in $X$.*

**Proof.** It may be completed following the proof of Theorem 4.8. □

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