DETERMINATION OF CONDUCTIVITY IN A HEAT EQUATION

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ABSTRACT. We consider the problem of determining the conductivity in a heat equation from overspecified non-smooth data. It is an ill-posed inverse problem. We apply a regularization approach to define and construct a stable approximate solution. We also conduct numerical simulation to demonstrate the accuracy of our approximation.

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1. Introduction. The problem of determining the conductivity $a(t)$ in

$$u_t(x,t) = a(t)u_{xx}(x,t), \quad 0 < x < 1, \, 0 < t < T \tag{1.1}$$

from overspecified smooth data has been studied by many people. For example, Jones in [4] proved existence and uniqueness of the solution of the inverse problem. Douglas and Jones provided in [3] numerical approach for determining the unknown coefficient. Around the same time, Cannon in [1] gave a different approach to the same problem. Later, in [2], Cannon considered (1.1) in which the conductivity was assumed to be an unknown constant and the heat flow rate was measured only for a single time.

For practical reasons, it is more interesting to consider the problem of determining the unknown conductivity coefficient from non-smooth data. In such case, the problem is ill posed, as we will demonstrate later. New way of determining the unknown coefficient is needed.

In this paper, we consider the problem of determining $a(t)$ in the following parabolic problem:

$$
\begin{aligned}
&u_t(x,t) = a(t)u_{xx}(x,t), \quad 0 < x, \, 0 < t < T; \\
&u(x,0) = 0, \quad 0 < x; \\
&u(0,t) = f(t), \quad 0 < t < T, \, f(0) = 0; \\
&-u_x(0,t) = g_0, \quad 0 < t < T,
\end{aligned}
\tag{1.2}
$$

where $u(x,t)$ and $a(t)$ are unknown and to be determined from known non-smooth data $f(t)$, g_0 (g_0 is a positive constant).

Let us first exam the ill-posedness of the problem.

For smooth function $f(t)$, if $u(x,t)$ is a classical solution of (1.2), then it can be shown that

$$u(x,t) = \frac{g_0}{\sqrt{\pi}} \int_0^t \frac{a(\tau)}{\sqrt{\theta(t)-\theta(\tau)}} e^{-x^2/(4(\theta(t)-\theta(\tau)))} \, d\tau, \tag{1.3}$$

$$\theta(t) = \int_0^t a(\tau)\,d\tau. \tag{1.4}$$

Letting x approach zero, we obtain from (1.3) that

$$f(t) = \frac{g_0}{\sqrt{\pi}} \int_0^t \frac{a(\tau)}{\sqrt{\theta(t) - \theta(\tau)}}\,d\tau, \tag{1.5}$$

which gives immediately

$$\theta(t) = \frac{\pi}{4g_0^2} f^2(t), \tag{1.6}$$

$$a(t) = \frac{\pi}{2g_0^2} f(t) f'(t). \tag{1.7}$$

The above computations are possible only under the assumption that the boundary term $f(t)$ is smooth.

In practice, unfortunately, the known data $f(t)$ (like boundary temperature) is obtained experimentally. It is generally not a smooth function in time. Then it is not possible to solve a in (1.4) in the classical sense.

Even if the data is obtained in such a way that the classical solution of (1.4) exists, this solution may not depend on the data continuously. To see this, we consider the following example.

EXAMPLE 1.1. Let $g_0 = \sqrt{\pi/2}$ and

$$f_T(t) = \begin{cases} t & 0 \le t \le \frac{1}{2}, \\ \left(2t - t^2 - \frac{1}{2}\right)^{1/2} & \frac{1}{2} < t \le 1. \end{cases} \tag{1.8}$$

It is easy then to obtain the exact solution of (1.4) as

$$a_T(t) = \begin{cases} t & 0 \le t \le \frac{1}{2}, \\ 1 - t & \frac{1}{2} < t \le 1. \end{cases} \tag{1.9}$$

Now take an approximate data function $f_\delta = f_T + (\pi/4)^{1/4}\delta \sin(t/\delta^3)$. While one has

$$\|f_\delta - f_T\|_{L^4} \le \delta^4, \tag{1.10}$$

the difference between the solutions

$$\|a_\delta(t) - a_T(t)\|_C \ge \frac{\pi}{4\delta}, \tag{1.11}$$

which shows that the problem of determining $a(t)$ in C from boundary data in L^4 is ill posed.

In this paper, we will apply a regularization method (cf. [5, 6, 7]) to define and construct a stable solution of (1.4), which is sometimes referred as a mapping:

$$A[a] = \int_0^t a(\tau)\,d\tau = \theta(t), \tag{1.12}$$

where θ is given in (1.6). We will also conduct numerical computations to verify the accuracy of our approximate approach.

2. Regularizing operator. Define the following functional:

$$M^\alpha[a,\theta] = \int_0^T \left[\int_0^t a(\tau)d\tau - \theta(t) \right]^2 dt + \alpha \int_0^T [a^2(\tau) + a'(\tau)^2]\,d\tau. \qquad (2.1)$$

THEOREM 2.1. *For every $\theta(t)$ in $L^2[0,T]$ and every positive number α, there exists a unique function $a_\alpha(t) \in C[0,T]$ that minimizes the functional (2.1).*

PROOF. Considering the first variation of the functional (2.1), we can see the minimizer of the functional is the solution of the following Euler integrodifferential equation:

$$\alpha(a'' - a) = \int_\tau^T \int_0^t a(\xi)d\xi\,dt - \int_\tau^T \theta(t)\,dt, \qquad (2.2)$$

subject to the boundary conditions $a(0) = 0$, $a(T) = 0$. It is trivial to show that there exists a unique solution of (2.2). We omit the details. $\square$

Based on Theorem 2.1, we now define an operator $R(\theta,\alpha)$ from the set of pairs: (θ,α), where $\theta \in L^2$, $\alpha > 0$, to the space $C[0,T]$ so that the element $a_\alpha = R(\theta,\alpha)$ minimizes the functional M^α. In what follows, we need to show that, for an appropriate α, a_α is a stable approximate solution of (1.4), namely, $R(\theta,\alpha)$ is a regularizing operator.

THEOREM 2.2. *Let a_T denote a solution of (1.4) with right-hand member θ_T and $a_\alpha = R(\theta_\delta,\alpha)$, where δ measures the error between θ_T and θ_δ. For any positive number ε, there exists a number $\delta(\varepsilon) > 0$, such that the inequality*

$$\|\theta_\delta - \theta_T\|_{L^2} \le \delta \le \delta(\varepsilon) \qquad (2.3)$$

implies the inequality

$$\|a_\alpha - a_T\|_C \le \varepsilon, \qquad (2.4)$$

where $\alpha = \alpha(\delta) = \delta^\lambda$, $0 < \lambda \le 2$.

PROOF. Since $a = a_\alpha$ is the minimizer of functional $M^\alpha[a,\theta_\delta]$, we have

$$M^\alpha[a_\alpha,\theta_\delta] \le M^\alpha[a_T,\theta_\delta]. \qquad (2.5)$$

Therefore,

$$\begin{aligned}
\alpha \int_0^T \left(a_\alpha^2(\tau) + (a_\alpha'(\tau))^2 \right)d\tau &\le \int_0^T \left(\int_0^t a_T(\tau)d\tau - \theta_\delta(t) \right)^2 dt \\
&\quad + \alpha \int_0^T \left(a_T^2(\tau) + (a_T'(\tau))^2 \right)d\tau \\
&\le \delta^2 + \alpha \int_0^T \left(a_T^2(\tau) + (a_T'(\tau))^2 \right)d\tau \le \delta^2 d,
\end{aligned} \qquad (2.6)$$

where $d = 1 + \int_0^T (a_T^2(\tau) + (a_T'(\tau))^2)d\tau$. Thus,

$$\int_0^T \left(a_\alpha^2(\tau) + (a_\alpha'(\tau))^2\right)d\tau \le d, \qquad \int_0^T \left(a_T^2(\tau) + (a_T'(\tau))^2\right)d\tau \le d. \qquad (2.7)$$

Consequently, the elements a_α, a_T belong to the following compact subset of space $C[0,T]$:

$$F = \left\{ a(\tau) : \int_0^T \left(a^2(\tau) + (a'(\tau))^2\right)d\tau \le d \right\}. \qquad (2.8)$$

Since the mapping $A : F \to AF$ (A is defined in (1.12)) is continuous and one-to-one, the inverse mapping $A^{-1} : AF \to F$ is also continuous. It means that, for arbitrary $\varepsilon > 0$, there exists a number $\gamma(\varepsilon) > 0$ such that the inequality

$$\|\theta_\alpha - \theta_T\|_{L^2} \le \gamma(\varepsilon), \quad \theta_\alpha = A(a_\alpha),\ \theta_T = A(a_T) \in AF \qquad (2.9)$$

implies the inequality

$$\|a_\alpha - a_T\|_{C[0,T]} \le \varepsilon. \qquad (2.10)$$

On the other hand, for $\theta_\delta, \theta_\alpha$, we have

$$\|\theta_\alpha - \theta_\delta\|_{L^2}^2 = \int_0^T \left(\int_0^t a_\alpha(\tau)d\tau - \theta_\delta(t)\right)^2 dt \\ \le M^\alpha[a_\alpha, \theta_\delta] \le M^\alpha[a_T, \theta_\delta] \le \delta^\lambda d. \qquad (2.11)$$

Obviously,

$$\|\theta_\alpha - \theta_T\|_{L^2} \le \|\theta_\alpha - \theta_\delta\|_{L^2} + \|\theta_\delta - \theta_T\|_{L^2}, \qquad (2.12)$$

which implies that

$$\|\theta_\alpha - \theta_T\|_{L^2} \le \delta^{\lambda/2}\sqrt{d} + \delta \le \delta^{\lambda/2}(1+\sqrt{d}). \qquad (2.13)$$

To end the proof of the theorem, let

$$\delta(\varepsilon) = \left(\frac{\gamma(\varepsilon)}{1+\sqrt{d}}\right)^{2/\lambda}. \qquad (2.14)$$
$\square$

Theorem 2.2 shows that a_α can be taken as an approximate solution of (1.4) with approximate right-hand member $\theta = \theta_\delta$.

Next, we need to show that θ depends on f, g_0 continuously.

THEOREM 2.3. *Suppose that* $\|f_\delta - f_T\|_{L^4[0,T]} \le \delta$, $|g_\delta - g_0| \le \delta$, *then*

$$\|\theta_\delta - \theta_T\|_{L^2} \le D\delta, \qquad (2.15)$$

where

$$D = \frac{4\pi}{g_0^3}\|f_T\|_{L^4}\left(8\left(g_0^4 + \|f_T\|_{L^4}^4\right)\right)^{1/4}. \qquad (2.16)$$

PROOF. From (1.6) and Cauchy inequality,

$$\|\theta_\delta - \theta_T\|_{L^2}^2 = \left(\frac{\pi}{4g_0^2 g_\delta^2}\right)^2 \int_0^T \left(g_0^2 f_\delta^2(t) - g_d t^2 f_T^2(t)\right)^2 dt$$

$$= \left(\frac{\pi}{4g_0^2 g_\delta^2}\right)^2 \int_0^T \left(g_0 f_\delta(t) - g_\delta f_T(t)\right)^2 \left(g_0 f_\delta(t) + g_\delta f_T(t)\right)^2 dt$$

$$\le \left(\frac{\pi}{g_0^4}\right)^2 \left(\int_0^T \left(g_0 f_\delta(t) - g_\delta f_T(t)\right)^4 dt\right)^{1/2} \cdot \left(\int_0^T \left(g_0 f_\delta(t) + g_\delta f_T(t)\right)^4 dt\right)^{1/2},$$

(2.17)

where $g_0 \le 2g_\delta$. The result in this theorem follows from the following estimates:

$$\int_0^T \left(g_0 f_\delta(t) + g_\delta f_T^{(t)}\right)^4 dt \le 8\left(g_0^4 \|f_\delta\|_{L^4}^4 + g_\delta^4 \|f_T\|_{L^4}^4\right)$$

$$\le 8^2 g_0^4 \left(\|f_\delta - f_T\|_{L^4}^4 + 3\|f_T\|_{L^4}^4\right) \quad (g_\delta < 2g_0)$$

$$\le 8^2 g_0^4 \left(\delta^4 + 3\|f_T\|_{L^4}^4\right)$$

$$\le 8^2 4 g_0^4 \|f_T\|_{L^4}^4 \quad \left(\delta < \|f_T\|_{L^4}^4\right),$$

(2.18)

$$\int_0^T \left(g_0 f_\delta(t) - g_\delta f_T^{(t)}\right)^4 dt = \int_0^T \left(g_0 (f_\delta(t) - f_T(t)) + (g_0 - g_d t) f_T^{(t)}\right)^4 dt$$

$$\le 8\left(g_0^4 \|f_\delta - f_T\|_{L^4}^4 + |g_0 - g_\delta|^4 \|f_T\|_{L^4}^4\right)$$

$$\le 8\left(g_0^4 + \|f_T\|_{L^4}^4\right)\delta^4. \qquad \square$$

(2.19)

Combing Theorems 2.2 and 2.3, we have the following stability result.

THEOREM 2.4. *Given $\varepsilon > 0$, there exists $\delta > 0$, $\alpha = \alpha(\delta)$, such that, for the approximate solution a_α corresponding to f_δ, g_δ and the exact solution a_T corresponding to f_T, g_0, inequalities*

$$\|f_\delta - f_T\|_{L^4[0,T]} \le \delta, \qquad |g_\delta - g_0| \le \delta, \tag{2.20}$$

imply

$$\|a_\alpha - a_T\|_C \le \varepsilon. \tag{2.21}$$

Therefore, a_α can be taken as a stable approximate solution of (1.2).

3. A numerical example. To demonstrate the applicability of our approximation approach, we consider the example in Section 2. Replacing the Euler integral equation (2.2) by its finite difference approximation on a uniform grid with step $h = T/(n+1)$, $T = 1$, we obtain the following system of linear equations:

$$\alpha\left[\frac{a_{j+1} - 2a_j + a_{j-1}}{h^2} - a_j\right] = h^2 \sum_{i=j}^{n} \sum_{k=1}^{i} a_k - h \sum i = j^n \theta_i, \quad j = 1,\ldots,n, \tag{3.1}$$

with $a_0 = a_{n+1} = 0$, $a_j = a(\tau_j)$, $\theta_i = \theta(t_i)$.

Take the regularization parameter α in the form of

$$\alpha = \alpha(\delta) = (D\delta)^\lambda, \tag{3.2}$$

where $0 < \lambda \le 2$, D is given by (2.16). With $(\theta_\delta)_i = \theta_\delta(t_i) = (\pi/g_\delta^2)f_\delta(t_i)^2$, $g_\delta = g_0 + \delta$, we want to recover a_T from (3.1). The numerical comparison between the exact solution a_T and the approximate solution are given in the following table ($n + 79$, $\delta = 10^{-8}$, $\lambda = 1.84$):

t	0.025	0.05	0.075	0.1	0.125	0.15	0.175
$a_T(t)$	0.025	0.05	0.075	0.1	0.125	0.15	0.175
$a_\alpha(t)$	0.025005	0.050030	0.074908	0.0100105	0.125084	0.149973	0.175053
t	0.2	0.225	0.25	0.275	0.3	0.325	0.35
$a_T(t)$	0.2	0.225	0.25	0.2750	0.3	0.325	0.35
$a_\alpha(t)$	0.199973	0.225043	0.250111	0.275142	0.300033	0.325053	0.350123
t	0.375	0.4	0.425	0.45	0.475	0.5	
$a_T(t)$	0.375	0.4	0.425	0.45	0.475	0.5	
$a_\alpha(t)$	0.374837	0.400034	0.425039	0.449937	0.45057	0.493663	

One can see that, unlike the discussion in Section 2, our approximate solution a_α and the exact solution a_T match very well. This shows that the regularization approach discussed in this work is an effective way of determining the unknown conduction coefficient $a(t)$ in the heat equation (1.2).

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