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ON THE SOLVABILITY OF A VARIATIONAL INEQUALITY PROBLEM AND APPLICATION TO A PROBLEM OF TWO MEMBRANES

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ABSTRACT. The purpose of this work is to give a continuous convex function, for which we can characterize the subdifferential, in order to reformulate a variational inequality problem: find $u = (u_1, u_2) \in K$ such that for all $v = (v_1, v_2) \in K$, $\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + (f, v - u) \geq 0$ as a system of independent equations, where f belongs to $L^2(\Omega) \times L^2(\Omega)$ and $K = \{v \in H_0^1(\Omega) \times H_0^1(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\}$.

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1. Introduction. We are interested in the following variational inequality problem: find $u = (u_1, u_2) \in K$ such that for all $v = (v_1, v_2) \in K$,

$$\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + (f, v - u) \geq 0, \tag{1.1}$$

where f belongs to $L^2(\Omega) \times L^2(\Omega)$ and K is a closed convex set of $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$K = \{v = (v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\}. \tag{1.2}$$

Thanks to the orthogonal projection of the space $L^2(\Omega) \times L^2(\Omega)$ onto the cone $\mathcal{K}$ defined by

$$\mathcal{K} = \{v = (v_1, v_2) \in L^2(\Omega) \times L^2(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\}, \tag{1.3}$$

we construct a functional φ for which we can characterize the subdifferential at a point u, in order to reformulate problem (1.1) to a variational inequality without constraints; that is, find $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega) \times H_0^1(\Omega)$,

$$\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + \varphi(v) - \varphi(u) + (h, v - u) \geq 0, \tag{1.4}$$

where φ is a continuous convex function from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $\mathbb{R}$ and h is an element of $L^2(\Omega) \times L^2(\Omega)$ depending only on f.

We prove that the solution $u = (u_1, u_2)$ can be obtained as a solution of a system of independent two Dirichlet problems

$$u_1, u_2 \in H_0^1(\Omega), \quad \Delta u_1 = g_1, \quad \Delta u_2 = g_2 \text{ in } \Omega, \tag{1.5}$$

where g_1 and g_2 are two functions of $L^2(\Omega)$ determined in terms of f_1 and f_2. We will give an algorithm for computing these functions.

This approach can be applied to study a variational inequality arising from a problem of two membranes [2].

2. Formulation of the problem. Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We equip $H_0^1(\Omega)\times H_0^1(\Omega)$ with the norm

$$a(u,v)=\int_\Omega \nabla u_1\nabla v_1+\int_\Omega \nabla u_2\nabla v_2, \tag{2.1}$$

where

$$u=(u_1,u_2),\ v=(v_1,v_2)\in H_0^1(\Omega)\times H_0^1(\Omega). \tag{2.2}$$

For $r\in L^2(\Omega)$, we let

$$r^+=\max\{r,0\},\qquad r^-=\min\{r,0\}. \tag{2.3}$$

For $f=(f_1,f_2)\in L^2(\Omega)\times L^2(\Omega)$, we let

$$f^+=(f_1^+,f_2^-),\qquad f^-=(f_1^-,f_2^+). \tag{2.4}$$

For $v=(v_1,v_2)\in H_0^1(\Omega)\times H_0^1(\Omega)$, we let

$$v_+=\left(v_1+\frac{(v_2-v_1)^+}{2},v_2-\frac{(v_2-v_1)^+}{2}\right),\qquad v_-=\left(-\frac{(v_2-v_1)^+}{2},\frac{(v_2-v_1)^+}{2}\right) \tag{2.5}$$

the projection of v onto the cone $\mathcal{K}$ given by (1.3) with respect to the scalar product of $L^2(\Omega)\times L^2(\Omega)$ (respectively, the projection with respect to the scalar product of $L^2(\Omega)\times L^2(\Omega)$ on the polar cone of $\mathcal{K}$ defined by $\mathcal{K}^0=\{v=(-r,r)\in L^2(\Omega)\times L^2(\Omega):r\geq 0$ a.e. on $\Omega\}$). We easily verify that

$$a(v_+,v_-)=0 \tag{2.6}$$

for all $v\in H_0^1(\Omega)\times H_0^1(\Omega)$. A function φ defined from $H_0^1(\Omega)\times H_0^1(\Omega)$ to $\mathbb{R}$ is called lower semi-continuous (l.s.c.) if its epigraph defined by

$$\operatorname{epi}(\varphi)=\{v=(v_1,v_2)\in H_0^1(\Omega)\times H_0^1(\Omega),\ \lambda\in\mathbb{R}:\varphi(v)\leq\lambda\} \tag{2.7}$$

is closed in $H_0^1(\Omega)\times H_0^1(\Omega)\times\mathbb{R}$. Let $u\in H_0^1(\Omega)\times H_0^1(\Omega)$, we denote by $\partial\varphi(u)$ the subdifferential of φ at u, defined by

$$\partial\varphi(u)=\{\mu\in H^{-1}(\Omega)\times H^{-1}(\Omega):\varphi(u)-\varphi(v)\leq\langle\mu,u-v\rangle\ \forall v\in H_0^1(\Omega)\times H_0^1(\Omega)\}. \tag{2.8}$$

If φ is a convex l.s.c. function, then for all $v\in H_0^1(\Omega)\times H_0^1(\Omega)$, $\partial\varphi(v)\neq\emptyset$.

Let $f=(f_1,f_2)\in L^2(\Omega)\times L^2(\Omega)$. We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm of $L^2(\Omega)\times L^2(\Omega)$, respectively. We consider the following variational inequality problem: find $u=(u_1,u_2)\in K$ such that

$$a(u,v-u)+(f,v-u)\geq 0\quad\forall v=(v_1,v_2)\in K. \tag{2.9}$$

It admits a unique solution. The functional φ defined from $L^2(\Omega)\times L^2(\Omega)$ to $\mathbb{R}$ by $v\mapsto(f^+,v_+)$ is continuous on $H_0^1(\Omega)\times H_0^1(\Omega)$ and convex.

Proposition 2.1. *$u = (u_1, u_2)$ is a solution of the problem (2.9) if and only if u is the solution of the following problem: find $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that*

$$a(u, v - u) + \varphi(v) - \varphi(u) + (f^-, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{2.10}$$

Proof. It is well known in the general theory of variational inequalities that problem (2.10) admits a unique solution. So, it is sufficient to show that the solution u of (2.10) is an element of K. Let $v = u_+$, then the inequality of (2.10) becomes

$$a(u, -u_-) + \varphi(u) - \varphi(u) + (f^-, -u_-) \geq 0. \tag{2.11}$$

By the relation (2.6) we deduce that $u_- = 0$, hence $u \in K$. □

Proposition 2.2. *Problem (2.10) is equivalent to the following problem: find $\mu = (\mu_1, \mu_2) \in L^2(\Omega) \times L^2(\Omega)$, $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$,*

$$a(u, v) + (\mu, v) + (f^-, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega),\ \mu \in \partial\varphi(u). \tag{2.12}$$

Proof. If $u \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $\mu \in L^2(\Omega) \times L^2(\Omega)$ are the solution of (2.12), then by definition of $\mu \in \partial\varphi(u)$, we have

$$a(u, v - u) + \varphi(v) - \varphi(u) + (f^-, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{2.13}$$

Conversely, let u be the solution of problem (2.10). For $v = u \pm w$, with $w \in H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality of (2.10) gives

$$\begin{aligned} a(u, w) + (f^-, w) &\geq -(f^+, w^+) \geq -\|f^+\|\|w\|, \\ a(u, w) + (f^-, w) &\leq (f^+, (-w)^+) \leq \|f^+\|\|w\|. \end{aligned} \tag{2.14}$$

We deduce that

$$|a(u, w) + (f^-, w)| \leq \|f^+\|\|w\|. \tag{2.15}$$

So the linear form

$$w \longmapsto a(u, w) + (f^-, w) \tag{2.16}$$

is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the norm of $L^2(\Omega) \times L^2(\Omega)$. Where μ is an element of $L^2(\Omega) \times L^2(\Omega)$. □

We set

$$C = \{\nu \in L^2(\Omega) \times L^2(\Omega),\ (\nu, v) \leq \varphi(v)\ \forall v \in L^2(\Omega) \times L^2(\Omega)\}. \tag{2.17}$$

Lemma 2.3. *Let $u \in L^2(\Omega) \times L^2(\Omega)$, then the following properties are equivalent:*
(a) $\mu \in \partial\varphi(u)$.
(b) $\mu \in C$ *and* $(\mu, u) = \varphi(u)$.
(c) $\mu \in C$ *and* $(\nu - \mu, u) \leq 0$ *for all* $\nu \in C$.

Proof. (a)⇒(b). Let $\mu \in \partial\varphi(u)$, we have

$$\varphi(v) - \varphi(u) \geq (\mu, v - u) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \tag{2.18}$$

We put $v = 0$, next $v = 2u$ in (2.18). Since φ is positively homogeneous of degree 1, we obtain $\varphi(u) = (\mu, u)$ and consequently

$$\varphi(v) \geq (\mu, v) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \tag{2.19}$$

(c)⇒(a). For all $v \in V$, we have

$$(\mu, v-u) \leq \varphi(v) - (\mu, u) \leq \varphi(v) - (\nu, u) \quad \forall \nu \in C. \tag{2.20}$$

Hence for $\nu \in \partial\varphi(u)$, we have $(\nu, u) = \varphi(u)$, consequently $\mu \in \varphi(u)$. □

We deduce from Lemma 2.3 the following relations:

$$\mu_1 + \mu_2 = f_1^+ + f_2^-, \quad f_2^- \leq \mu_2 \leq \mu_1 \leq f_1^+ \text{ a.e. in } \Omega. \tag{2.21}$$

Indeed, the function φ being positively homogeneous of degree 1, $\mu \in \partial\varphi(u)$ implies

$$(\mu, u) = \varphi(u), \tag{2.22}$$

$$(\mu, v) \leq \varphi(v) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \tag{2.23}$$

Finally, it is sufficient to take in (2.23) elements $v = (v_1, v_2)$ with suitable choices on the components v_1 and v_2.

Let $V = H_0^1(\Omega) \times H_0^1(\Omega)$, and taking into account Lemma 2.3, we can write problem (2.12) as follows: find $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, $\mu \in C$,

$$\begin{gathered} a(u,v) + (\mu, v) + (f^-, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega), \\ (\nu - \mu, u) \leq 0 \quad \forall \nu \in C. \end{gathered} \tag{2.24}$$

Let A be the Riesz-Fréchet representation of $H^{-1}(\Omega) \times H^{-1}(\Omega)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. We set $M = A(C)$, this is a closed convex subset in $H_0^1(\Omega) \times H_0^1(\Omega)$ characterized by

$$M = \{w \in H_0^1(\Omega) \times H_0^1(\Omega) : a(w,v) \leq \varphi(v) \ \forall v \in H_0^1(\Omega) \times H_0^1(\Omega)\}. \tag{2.25}$$

Problem (2.24) can be written in the following form: find $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, $z \in M$,

$$\begin{gathered} a(u+z+t, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega), \\ a(w - z, u) \leq 0 \quad \forall w \in M. \end{gathered} \tag{2.26}$$

with $z = A(\mu)$ and $t = A(f^-)$. Hence

$$u = -z - t, \quad z = P_M(-t), \tag{2.27}$$

where $P_M(-t)$ is the projection of $-t$ onto the closed convex set M with respect to the scalar product $a(\cdot,\cdot)$ of $H_0^1(\Omega) \times H_0^1(\Omega)$.

From the equality of Proposition 2.2, we deduce that the solution u of problem (2.9) verifies the following equations:

$$\Delta u_1 = \mu_1 + f_1^-, \quad \Delta u_2 = \mu_2 + f_2^+ \quad \text{in } \Omega. \tag{2.28}$$

We notice that the prior knowledge of $\mu = (\mu_1, \mu_2)$ in terms of data of problem (2.9) yields the solutions u_1 and u_2 as solutions of two independent Dirichlet problems given by the system (2.28). We recall that for each element f of $L^p(\Omega)$, the solution of the problem

$$u \in H_0^1(\Omega), \quad -\Delta u = f \quad \text{in } \Omega, \tag{2.29}$$

verifies the following properties (see [2]):

$$u \in H^{2,p}(\Omega), \quad \|u\|_{H^{2,p}} \le C\|f\|_{L^p}, \tag{2.30}$$

where C is a constant depending only on p and Ω. We deduce from (2.28) that u_1, u_2 are in $H^2(\Omega)$ and

$$\begin{aligned} ||u_1||_{H^2(\Omega)} &\le c_1||\mu_1 + f_1^-||_{L^2(\Omega)}, \\ ||u_2||_{H^2(\Omega)} &\le c_2||\mu_2 + f_2^-||_{L^2(\Omega)}, \\ ||u_1 + u_2||_{H^2(\Omega)} &\le c||f_1 + f_2||_{L^2(\Omega)}, \end{aligned} \tag{2.31}$$

where c, c_1, and c_2 are constants depending only on Ω. We define the domain of non-coincidence [2] by

$$\Omega^+ = \{x \in \Omega : u_1(x) > u_2(x)\}. \tag{2.32}$$

From relations (2.21), (2.22), and (2.23) we deduce that

$$\mu_1 = f_1^+, \quad \mu_2 = f_2^- \quad \text{a.e. in } \Omega^+. \tag{2.33}$$

When u_1 and u_2 are continuous on Ω, the following relations are verified:

$$\Delta u_1 = f_1, \quad \Delta u_2 = f_2 \quad \text{in } \Omega^+. \tag{2.34}$$

2.1. Algorithm for computing z. We consider the following projection problem:

$$z \in H_0^1(\Omega) \times H_0^1(\Omega), \quad z = P_M(t'), \quad \text{where } t' = -t. \tag{2.35}$$

Let z_0 belong to M, we compute the element w_0 of M which verifies the following inequality:

$$a(w - w_0, z_0 - t') \ge 0 \quad \forall w \in M. \tag{2.36}$$

Next we compute

$$z_1 = P_{[z_0, w_0]}(t'). \tag{2.37}$$

So, the algorithm is: z_n being given in M, we construct w_n verifying

$$a(w - w_n, z_n - t') \ge 0 \quad \forall w \in M. \tag{2.38}$$

Next $z_{n+1} = P_{[z_n, w_n]}(t')$. The sequence $\{z_n\}$ converges in $H_0^1(\Omega) \times H_0^1(\Omega)$ strongly to the solution of problem (2.35) [1]. Since $M = A(C)$, then the inequality (2.38) implies that there exists $\{v_n\}$ in C which verifies

$$(v - v_n, t' - z_n) \le 0 \quad \forall v \in C \tag{2.39}$$

and Lemma 2.3 shows that v_n is an element of $\partial\varphi(t' - z_n)$.

2.2. Application. This method of solvability can be applied to the study of a variational inequality arising from a problem of two membranes [2],

$$\begin{aligned} &\Delta u_1 + \lambda u_1 = f_1, \quad \Delta u_2 = f_2 \text{ in } \Omega^+, \quad u_1 = u_2, \\ &\frac{\partial u_1}{\partial x_i} = \frac{\partial u_2}{\partial x_i}, \quad 1 \le i \le n, \\ &\Delta u_1 + \left(\frac{\lambda}{2}\right) u_1 = \frac{1}{2}(f_1 + f_2) \quad \text{in } \Omega^-, \end{aligned} \tag{2.40}$$

where Ω^+ and Ω^-, are two parts of Ω (unknown) separated by a hypersurface Γ of $\mathbb{R}^n$ such that $\Omega = \Omega^+ \cup \Gamma \cup \Omega^-$; f_1, f_2 are two regular functions and $\lambda \in \mathbb{R}$. Formally, Ω^+ is the non-coincidence domain given by (2.32).

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