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COMPUTATIONS OF NAMBU-POISSON COHOMOLOGIES

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ABSTRACT. We want to associate to an n-vector on a manifold of dimension n a cohomology which generalizes the Poisson cohomology of a 2-dimensional Poisson manifold. Two possibilities are given here. One of them, the Nambu-Poisson cohomology, seems to be the most pertinent. We study these two cohomologies locally, in the case of germs of n-vectors on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

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1. Introduction. A way to study a geometrical object is to associate to it a cohomology. In this paper, we focus on the n-vectors on an n-dimensional manifold M.

If $n = 2$, the 2-vectors on M are the Poisson structures thus, we can consider the Poisson cohomology. In dimension 2, this cohomology has three spaces. The first one, H^0, is the space of functions whose Hamiltonian vector field is zero (Casimir functions). The second one, H^1, is the quotient of the space of infinitesimal automorphisms (or Poisson vector fields) by the subspace of Hamiltonian vector fields. The last one, H^2, describes the deformations of the Poisson structure. In a previous paper (see [9]) we have computed the cohomology of germs at 0 of Poisson structures on $\mathbb{K}^2$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

In order to generalize this cohomology to the n-dimensional case ($n \geq 3$), we can follow the same reasoning. These spaces are not necessarily of finite dimension and it is not always easy to describe them precisely.

Recently, a team of Spanish researchers has defined a cohomology, called Nambu-Poisson cohomology, for the Nambu-Poisson structures (see [6]). In this paper, we adapt their construction to our particular case. We will see that this cohomology generalizes in a certain sense the Poisson cohomology in dimension 2. Then we compute locally this cohomology for germs at 0 of n-vectors $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$ on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with the assumption that f is a quasihomogeneous polynomial of finite codimension ("most of" the germs of n-vectors have this form). This computation is based on a preliminary result that we have shown, in the formal case and in the analytical case (so, the $\mathscr{C}^\infty$ case is not entirely solved). The techniques we use in this paper are quite the same as in [9].

2. Nambu-Poisson cohomology. Let M be a differentiable manifold of dimension n ($n \geq 3$), admitting a volume form ω. We denote by $\mathscr{C}^\infty(M)$ the space of $\mathscr{C}^\infty$ functions on M, by $\Omega^k(M)$ ($k = 0,\dots,n$) the $\mathscr{C}^\infty(M)$-module of k-forms on M, and by $\chi^k(M)$ ($k = 0,\dots,n$) the $\mathscr{C}^\infty(M)$-module of k-vectors on M.

We consider an n-vector Λ on M. Note that Λ is a Nambu-Poisson structure on M.

Recall that a Nambu-Poisson structure on M of order r is a skew-symmetric r-linear map $\{,\dots,\}$

$$\mathscr{C}^\infty(M)\times\cdots\times\mathscr{C}^\infty(M)\longrightarrow\mathscr{C}^\infty(M),\qquad (f_1,\dots,f_r)\longmapsto\{f_1,\dots,f_r\}, \tag{2.1}$$

which satisfies

$$\begin{gathered}\{f_1,\dots,f_{r-1},gh\}=\{f_1,\dots,f_{r-1},g\}h+g\{f_1,\dots,f_{r-1},h\},\\ \{f_1,\dots,f_{r-1},\{g_1,\dots,g_r\}\}=\sum_{i=1}^{r}\{g_1,\dots,g_{i-1},\{f_1,\dots,f_{r-1},g_i\},g_{i+1},\dots,g_r\},\end{gathered} \tag{2.2}$$

for any $f_1,\dots,f_{r-1},g,h,g_1,\dots,g_r$ in $\mathscr{C}^\infty(M)$. It is clear that we can associate to such a bracket an r-vector on M. If $r=2$, we rediscover Poisson structures. Thus, Nambu-Poisson structures can be seen as a kind of generalization of Poisson structures. The notion of Nambu-Poisson structures was introduced in [14] by Takhtajan in order to give a formalism to an idea of Y. Nambu (see [12]).

Here, we suppose that the set $\{x\in M;\ \Lambda_x\neq 0\}$ is dense in M. We are going to associate a cohomology to (M,Λ).

2.1. The choice of the cohomology. If M is a differentiable manifold of dimension 2, then the Poisson structures on M are the 2-vectors on M. If Π is a Poisson structure on M, then we can associate to (M,Π) the complex

$$0\longrightarrow\mathscr{C}^\infty(M)\xrightarrow{\ \partial\ }\chi^1(M)\xrightarrow{\ \partial\ }\chi^2(M)\longrightarrow 0 \tag{2.3}$$

with $\partial(g)=[g,\Pi]=X_g$ (Hamiltonian of g) if $g\in\mathscr{C}^\infty(M)$ and $\partial(X)=[X,\Pi]$ ($[\,,]$ indicates Schouten's bracket) if $X\in\chi^1(M)$. The cohomology of this complex is called the Poisson cohomology of (M,Π). This cohomology has been studied for instance in [9, 10, 15].

Now if M is of dimension n with $n\geq 3$, we want to generalize this cohomology. Our first approach was to consider the complex

$$0\longrightarrow(\mathscr{C}^\infty(M))^{n-1}\xrightarrow{\ \partial\ }\chi^1(M)\xrightarrow{\ \partial\ }\chi^n(M)\longrightarrow 0 \tag{2.4}$$

with $\partial(X)=[X,\Lambda]$ and $\partial(g_1,\dots,g_{n-1})=i_{dg_1\wedge\cdots\wedge dg_{n-1}}\Lambda=X_{g_1,\dots,g_{n-1}}$ (Hamiltonian vector field) where we adopt the convention $i_{dg_1\wedge\cdots\wedge dg_{n-1}}\Lambda=\Lambda(dg_1,\dots,dg_{n-1},\bullet)$. We denote by $H^0_\Lambda(M)$, $H^1_\Lambda(M)$, and $H^2_\Lambda(M)$ the three spaces of cohomology of this complex. With this cohomology, we rediscover the interpretation of the first spaces of the Poisson cohomology, that is, $H^2_\Lambda(M)$ describes the infinitesimal deformations of Λ and $H^1_\Lambda(M)$ is the quotient of the algebra of vector fields which preserve Λ by the ideal of Hamiltonian vector fields.

In [6], the authors associate to any Nambu-Poisson structure on M a cohomology. The second idea is then to adapt their construction to our particular case.

Let $\#_\Lambda$ be the morphism of $\mathscr{C}^\infty(M)$-modules $\Omega^{n-1}(M)\to\chi^1(M):\alpha\mapsto i_\alpha\Lambda$. Note that $\ker\#_\Lambda=\{0\}$ (because the set of regular points of Λ is dense). We can define (see [7]) an $\mathbb{R}$-bilinear operator $[[\,,]]:\Omega^{n-1}(M)\times\Omega^{n-1}(M)\to\Omega^{n-1}(M)$ by

$$[[\alpha,\beta]]=\mathscr{L}_{\#_\Lambda\alpha}\beta+(-1)^n(i_{d\alpha}\Lambda)\beta. \tag{2.5}$$

The vector space $\Omega^{n-1}(M)$ equipped with $[[\,,\,]]$ is a Lie algebra (for any Nambu-Poisson structure, it is a Leibniz algebra). Moreover, this bracket verifies that $\#_\Lambda[[\alpha,\beta]] = [\#_\Lambda\alpha,\#_\Lambda\beta]$ for any α,β in $\Omega^{n-1}(M)$. The triple $(\Lambda^{n-1}(T^*(M)),[[\,,\,]],\#_\Lambda)$ is then a Lie algebroid and the Nambu-Poisson cohomology of (M,Λ) is the Lie algebroid cohomology of $\Lambda^{n-1}(T^*(M))$ (for any Nambu-Poisson structure, it is more elaborate see [6]). More precisely, for every $k \in \{0,\dots,n\}$, we consider the vector space $C^k(\Omega^{n-1}(M);\mathscr{C}^\infty(M))$ of the skew-symmetric and $\mathscr{C}^\infty(M)$-k-multilinear maps $\Omega^{n-1}(M)\times\cdots\times\Omega^{n-1}(M) \to \mathscr{C}^\infty(M)$. The cohomology operator $\partial : C^k(\Omega^{n-1}(M);\mathscr{C}^\infty(M)) \to C^{k+1}(\Omega^{n-1}(M);\mathscr{C}^\infty(M))$ is defined by

$$\begin{aligned}\partial c(\alpha_0,\dots,\alpha_k) = &\sum_{i=0}^{k}(-1)^i(\#_\Lambda\alpha_i)\cdot c(\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_k)\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}c([[\alpha_i,\alpha_j]],\alpha_0,\dots,\hat{\alpha}_i,\dots,\hat{\alpha}_j,\dots,\alpha_k)\end{aligned} \tag{2.6}$$

for all $c \in C^k(\Omega^{n-1}(M);\mathscr{C}^\infty(M))$ and $\alpha_0,\dots,\alpha_k$ in $\Omega^{n-1}(M)$.

The Nambu-Poisson cohomology of (M,Λ), denoted by $H^\bullet_{NP}(M,\Lambda)$, is the cohomology of this complex.

2.2. An equivalent cohomology. So defined, the Nambu-Poisson cohomology is quite difficult to manipulate. We are going to give an equivalent cohomology which is more accessible.

Recall that we assume that M admits a volume form ω.

Let $f \in \mathscr{C}^\infty(M)$, we define the operator

$$d_f : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \qquad \alpha \longmapsto f\,d\alpha - k\,df\wedge\alpha. \tag{2.7}$$

It is easy to prove that $d_f \circ d_f = 0$. We denote by $H^\bullet_f(M)$ the cohomology of this complex. Let $\flat$ be the isomorphism $\chi^1(M) \to \Omega^{n-1}(M)$, $X \mapsto i_X\omega$.

Lemma 2.1. (1) *If $X \in \chi^1(M)$, then $\#_\Lambda(\flat(X)) = (-1)^{n-1}fX$, where $f = i_\Lambda\omega$.*

(2) *If X and Y are in $\chi^1(M)$, then*

$$(-1)^{n-1}[[\flat(X),\flat(Y)]] = f\flat([X,Y]) + (X\cdot f)\flat(Y) - (Y\cdot f)\flat(X). \tag{2.8}$$

Proof. (1) Obvious.

(2) We have $\#_\Lambda([[\flat(X),\flat(Y)]]) = [\#_\Lambda(\flat(X)),\#_\Lambda(\flat(Y))]$ (property of the Lie algebroid), which implies that

$$\begin{aligned}\#_\Lambda([[\flat(X),\flat(Y)]]) &= f(X\cdot f)Y - f(Y\cdot f)X + f^2[X,Y]\\ &= (-1)^{n-1}\#_\Lambda\big((X\cdot f)\flat(Y) - (Y\cdot f)\flat(X) + f\flat([X,Y])\big).\end{aligned} \tag{2.9}$$

The result follows via the injectivity of $\#_\Lambda$. □

Proposition 2.2. *If We put $f = i_\Lambda\omega$, then $H^\bullet_{NP}(M,\Lambda)$ is isomorphic to $H^\bullet_f(M)$.*

Proof. For every k, we consider the application $\varphi : C^k(\Omega^{n-1}(M);\mathscr{C}^\infty(M)) \to \Omega^k(M)$ defined by

$$\varphi(c)(X_1,\dots,X_k) = c\big((-1)^{n-1}\flat(X_1),\dots,(-1)^{n-1}\flat(X_k)\big), \tag{2.10}$$

where $c \in C^k(\Omega^{n-1}(M);\mathscr{C}^\infty(M))$ and $X_1,\dots,X_k \in \chi^1(M)$. It is easy to see that φ is an isomorphism of vector spaces. We show that it is an isomorphism of complexes.

Let $c \in C^k(\Omega^{n-1}(M);\mathscr{C}^\infty(M))$. We put $\alpha = \varphi(c)$. If $X_0,\dots,X_k$ are in $\chi^1(M)$, then $\varphi(\partial c)(X_0,\dots,X_k) = (-1)^{(n-1)(k+1)}\partial c(\flat(X_0),\dots,\flat(X_k)) = A+B$, where

$$A = (-1)^{(n-1)(k+1)}\sum_{i=0}^{k}(-1)^i \#_\Lambda(\flat(X_i))\cdot c\Big(\flat(X_0),\dots,\widehat{\flat(X_i)},\dots,\flat(X_k)\Big),$$

$$B = (-1)^{(n-1)(k+1)}\sum_{0\le i<j\le k}(-1)^{i+j}c\Big([[\flat(X_i),\flat(X_j)]],\flat(X_0),\dots,\widehat{\flat(X_i)},\dots,\widehat{\flat(X_j)},\dots,\flat(X_k)\Big). \tag{2.11}$$

We have $A = f\sum_{i=0}^{k}(-1)^i X_i\cdot\alpha(X_0,\dots,\hat{X}_i,\dots,X_k)$ and

$$\begin{aligned} B = f&\sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}(X_i\cdot f)\alpha(X_j,X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\\ &-\sum_{0\le i<j\le k}(-1)^{i+j}(X_j\cdot f)\alpha(X_i,X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\\ =f&\sum_{0\le i<j\le k}(-1)^{i+j}\alpha([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\\ &-k\sum_{i=0}^{k}(-1)^i(X_i\cdot f)\alpha(X_0,\dots,\hat{X}_i,\dots,X_k). \end{aligned} \tag{2.12}$$

Consequently, $\varphi(\partial c) = d_f\alpha = d_f(\varphi(c))$. □

Remark 2.3. We claim that this cohomology is a "good" generalization of the Poisson cohomology of a 2-dimensional Poisson manifold. Indeed, if (M,Π) is an orientable Poisson manifold of dimension 2, we consider the volume form ω on M and we put

$$\phi^2:\chi^2(M)\longrightarrow\Omega^2(M),\qquad \phi^1:\chi^1(M)\longrightarrow\Omega^1(M), \tag{2.13}$$

defined by

$$\phi^2(\Gamma) = (i_\Gamma\omega)\omega,\qquad \phi^1(X) = -i_X\omega, \tag{2.14}$$

for every 2-vector Γ and vector field X.

We also put $\phi^0 = id:\mathscr{C}^\infty(M)\to\mathscr{C}^\infty(M)$.

If we denote by ∂ the operator of the Poisson cohomology, and $f = i_\Pi\omega$, it is quite easy to see that

$$\phi:(\chi^\bullet(M),\partial)\longrightarrow(\Omega^\bullet(M),d_f) \tag{2.15}$$

is an isomorphism of complexes.

Remark 2.4. (1) The definitions we have given make sense if we work in the holomorphic case or in the formal case.

(2) Important: if h is a function on M which does not vanish on M, then the cohomologies $H_f^\bullet(M)$ and $H_{fh}^\bullet(M)$ are isomorphic.

Indeed, the applications $(\Omega^k(M),d_{fh})\to(\Omega^k(M),d_f)$, $\alpha\mapsto\alpha/h^k$ give an isomorphism of complexes.

In particular, if f does not vanish on M then $H_f^\bullet(M)$ is isomorphic to the de Rham's cohomology.

2.3. Other cohomologies. We can construct other complexes which look like $(\Omega^\bullet(M), d_f)$. More precisely we denote, for $p \in \mathbb{Z}$,

$$d_f^{(p)} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M), \qquad \alpha \longmapsto f\, d\alpha - (k-p)\, df \wedge \alpha. \tag{2.16}$$

We denote by $H_{f,p}^\bullet(M)$ the cohomology of these complexes. We will see in Section 3 some relations between these different cohomologies.

Using the contraction $i_\bullet \omega$, it is quite easy to prove the following proposition.

Proposition 2.5. *The spaces $H_\Lambda^1(M)$ and $H_\Lambda^2(M)$ are isomorphic to $H_{f,n-2}^{n-1}(M)$ and $H_{f,n-2}^n(M)$.*

Remark 2.6. The two properties of Remark 2.4 are valid for $H_{f,p}^\bullet(M)$ with $p \in \mathbb{Z}$.

3. Computation. Henceforth, we will work locally. Let Λ be a germ of n-vectors on $\mathbb{K}^n$ ($\mathbb{K}$ indicates $\mathbb{R}$ or $\mathbb{C}$) with $n \geq 3$. We denote by $\mathcal{F}(\mathbb{K}^n)$ $(\Omega^k(\mathbb{K}^n), \chi(\mathbb{K}^n))$ the space of germs at 0 of (holomorphic, analytic, $\mathscr{C}^\infty$, formal) functions (k-forms, vector fields). We can write Λ (with coordinates $(x_1,\dots,x_n)$) $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$, where $f \in \mathcal{F}(\mathbb{K}^n)$. We assume that the volume form ω is $dx_1 \wedge \cdots \wedge dx_n$.

We suppose that $f(0) = 0$ (see Remark 2.4) and that f is of finite codimension, which means that $Q_f = \mathcal{F}(\mathbb{K}^n)/I_f$ (I_f is the ideal spanned by $\partial f/\partial x_1, \dots, \partial f/\partial x_n$) is a finite-dimensional vector space.

Remark 3.1. It is important to note that, according to Tougeron's theorem (cf. [3]), if f is of finite codimension, then the set $f^{-1}(\{0\})$ is, from the topological point of view, the same as the set of the zeros of a polynomial.

Therefore, if g is a germ at 0 of functions which satisfies $fg = 0$, then $g = 0$.

Moreover, we suppose that f is a quasihomogeneous polynomial of degree N (for a justification of this additional assumption, see Section 4). We are going to recall the definition of the quasihomogeneity.

3.1. Quasihomogeneity. Let $(w_1,\dots,w_n) \in (\mathbb{N}^*)^n$. We denote by W the vector field $w_1x_1(\partial/\partial x_1) + \cdots + w_nx_n(\partial/\partial x_n)$ on $\mathbb{K}^n$. We say that a nonzero tensor T is quasihomogeneous with weights $w_1,\dots,w_n$ and of (quasi)degree $N \in \mathbb{Z}$ if $\mathcal{L}_W T = NT$ ($\mathcal{L}$ indicates the Lie derivative operator). Note that T is then polynomial.

If f is a quasihomogeneous polynomial of degree N, then $N = k_1w_1 + \cdots + k_nw_n$ with $k_1,\dots,k_n \in \mathbb{N}$; so, an integer is not necessarily the quasidegree of a polynomial. If $f \in \mathbb{K}[[x_1,\dots,x_n]]$, we can write $f = \sum_{i=0}^\infty f_i$ with f_i quasihomogeneous of degree i (we adopt the convention that $f_i = 0$ if i is not a quasidegree); f is said to be of order d ($\operatorname{ord}(f) = d$) if all of its monomials have a degree d or higher. For more details, see [3].

Since $\mathcal{L}_W$ and the exterior differentiation d commute, if α is a quasihomogeneous k-form, then $d\alpha$ is a quasihomogeneous $(k+1)$-form of degree $\deg\alpha$. In particular, it is important to notice that dx_i is a quasihomogeneous 1-form of degree w_i (note that $\partial/\partial x_i$ is a quasihomogeneous vector field of degree $-w_i$). Thus, the volume form

$\omega = dx_1 \wedge \cdots \wedge dx_n$ is quasihomogeneous of degree $w_1 + \cdots + w_n$. Note that a quasihomogeneous nonzero k-form ($k \geq 1$) has a degree strictly positive.

Note that if f is a quasihomogeneous polynomial of degree N, then the n-vector $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$ is quasihomogeneous of degree $N - \sum_i w_i$.

In what follows, the degrees will be quasidegrees with respect to $W = w_1 x_1(\partial/\partial x_1) + \cdots + w_n x_n(\partial/\partial x_n)$.

We will need the following result.

Lemma 3.2. *Let $k_1, \ldots, k_n \in \mathbb{N}$ and put $p = \sum k_i w_i$. Assume that $g \in \mathcal{F}(\mathbb{K}^n)$ and $\alpha \in \Omega^i(\mathbb{K}^n)$ verify $\operatorname{ord}(j_0^\infty(g)) > p$ and $\operatorname{ord}(j_0^\infty(\alpha)) > p$ (j_0^∞ indicates the ∞-jet at 0). Then*

(1) *there exists $h \in \mathcal{F}(\mathbb{K}^n)$ such that $W \cdot h - ph = g$,*

(2) *there exists $\beta \in \Omega^i(\mathbb{K}^n)$ such that $\mathcal{L}_W \beta - p\beta = \alpha$.*

Proof. The first claim is only a generalization of Lemma 3.5 in [9] (it also appears in Lemma 2 in [2]) and it can be proved in the same way. The second claim is a consequence of the first. □

Now, we compute the spaces $H_f^k(\mathbb{K}^n)$ (i.e., $H_{NP}^K(\mathbb{K}^n, \Lambda)$) for $k = 0, \ldots, n$. We denote by $Z_f^k(\mathbb{K}^n)$ and $B_f^k(\mathbb{K}^n)$ the spaces of k-cocycles and k-cobords. We also compute some spaces $H_{f,p}^k(\mathbb{K}^n)$ with particular interest in the spaces $H_{f,n-2}^n(\mathbb{K}^n)$ (i.e., $H_\Lambda^2(\mathbb{K}^n)$) and $H_{f,n-2}^{n-1}(\mathbb{K}^n)$ (i.e., $H_\Lambda^1(\mathbb{K}^n)$). We denote by $Z_{f,p}^k(\mathbb{K}^n)$ ($B_{f,p}^k(\mathbb{K}^n)$) the spaces of k-cocycles (k-cobords) for the operator $d_f^{(p)}$.

3.2. Two useful preliminary results. In the computation of these spaces of cohomology, we need the two following propositions. The first is only a corollary of the de Rham's division lemma (see [4]).

Proposition 3.3. *Let $f \in \mathcal{F}(\mathbb{K}^n)$ of finite codimension. If $\alpha \in \Omega^k(\mathbb{K}^n)$ $(1 \leq k \leq n-1)$ verifies $df \wedge \alpha = 0$, then there exists $\beta \in \Omega^{k-1}(\mathbb{K}^n)$ such that $\alpha = df \wedge \beta$.*

Proposition 3.4. *Let $f \in \mathcal{F}(\mathbb{K}^n)$ of finite codimension. Let α be a k-form $(2 \leq k \leq n-1)$ which verifies $d\alpha = 0$ and $df \wedge \alpha = 0$, then there exists $\gamma \in \Omega^{k-2}(\mathbb{K}^n)$ such that $\alpha = df \wedge d\gamma$.*

Proof. We prove this result in the formal case and in the analytical case.

Formal case: let α be a quasihomogeneous k-form of degree p which verifies the hypotheses. Since $df \wedge \alpha = 0$, we have $\alpha = df \wedge \beta_1$, where β_1 is a quasihomogeneous $(k-1)$-form of degree $p - N$. Now, since $d\alpha = 0$, we have $df \wedge d\beta_1 = 0$ and so $d\beta_1 = df \wedge \beta_2$, where β_2 is a quasihomogeneous $(k-1)$-form of degree $p - 2N$. This way, we can construct a sequence (β_i) of quasihomogeneous $(k-1)$-forms with $\deg \beta_i = p - iN$ which verifies that $d\beta_i = df \wedge \beta_{i+1}$. Let $q \in \mathbb{N}$ such that $p - qN \leq 0$. Thus, we have $\beta_q = 0$ and so $d\beta_{q-1} = 0$, that is, $\beta_{q-1} = d\gamma_{q-1}$, where γ_{q-1} is a $(k-2)$-form. Consequently, $d\beta_{q-2} = df \wedge d\gamma_{q-1}$ which implies that $\beta_{q-2} = -df \wedge \gamma_{q-1} + d\gamma_{q-2}$, where γ_{q-2} is a $(k-2)$-form. In the same way, $d\beta_{q-3} = df \wedge d\gamma_{q-2}$ so $\beta_{q-3} = -df \wedge \gamma_{q-2} + d\gamma_{q-3}$, where γ_{q-3} is a $(k-2)$-form. This way, we can show that $\beta_1 = -df \wedge \gamma_2 + d\gamma_1$, where γ_1 and γ_2 are $(k-2)$-forms. Therefore, $\alpha = df \wedge d\gamma_1$.

Analytical case: in [8], Malgrange gave a result on the relative cohomology of a germ of an analytical function. In particular, he showed that in our case, if β is a germ at

0 of analytical r-forms ($r < n-1$) which verifies $d\beta = df \wedge \mu$ (μ is an r-form) then there exist two germs of analytical $(r-1)$-forms γ and ν such that $\beta = d\gamma + df \wedge \nu$.

Now, we prove our proposition. Let α be a germ of analytical k-forms $(2 \le k \le n-1)$ which verifies the hypotheses of the proposition. Then there exists a $(k-1)$-form β such that $\alpha = df \wedge \beta$ (Proposition 3.3). But since $0 = d\alpha = -df \wedge d\beta$, we have $d\beta = df \wedge \mu$ and so (see [8]) $\beta = d\gamma + df \wedge \nu$, where γ and ν are analytical $(k-2)$-forms. We deduce that $\alpha = df \wedge d\gamma$, where γ is analytic. □

Remark 3.5. Important: in fact, some results which appear in [13] lead us to think that this proposition is not true in the real $\mathscr{C}^\infty$ case.

The computation of the spaces $H^n_{f,p}(\mathbb{K}^n)$, $H^{n-1}_{f,p}(\mathbb{K}^n)$ $(p \neq n-2)$, and $H^0_{f,p}(\mathbb{K}^n)$ does not use this proposition, so it still holds in the $\mathscr{C}^\infty$ case.

The results we find on the other spaces should be the same in the $\mathscr{C}^\infty$ case as in the analytical case but another proof need to be found.

3.3. Computation of $H^0_{f,p}(\mathbb{K}^n)$.

We consider the application $d_f^{(p)} : \Omega^0(\mathbb{K}^n) \to \Omega^1(\mathbb{K}^n)$, $g \mapsto f\,dg + p\,df \wedge g$.

Theorem 3.6. (1) *If $p > 0$ then $H^0_{f,p}(\mathbb{K}^n) = \{0\}$.*

(2) *If $p \le 0$ then $H^0_{f,p}(\mathbb{K}^n) = \mathbb{K} \cdot f^{-p}$.*

Proof. (1) If $g \in \mathscr{F}(\mathbb{K}^n)$ verifies $d_f^{(p)} g = 0$, then $d(f^p g) = 0$, and so $f^p g$ is constant. But as $f(0) = 0$, $f^p g$ must be 0, that is, $g = 0$ (because f is of finite codimension; see Remark 3.1).

(2) We use an induction to show that for any $k \ge 0$, if g satisfies $f\,dg = kg\,df$ then $g = \lambda f^k$, where $\lambda \in \mathbb{K}$.

For $k = 0$ it is obvious.

Now we suppose that the property is true for $k \ge 0$. We show that it is still valid for $k+1$. Let $g \in \mathscr{F}(\mathbb{K}^n)$ be such that

$$f\,dg = (k+1)g\,df. \tag{3.1}$$

Then $df \wedge dg = 0$ and so there exists $h \in \mathscr{F}(\mathbb{K}^n)$ such that $dg = h\,df$ (Proposition 3.3). Replacing dg by $h\,df$ in (3.1), we get $fh\,df = (k+1)g\,df$, that is, $g = (1/(k+1))fh$. Now, this former relation gives, on one hand, $f\,dg = (1/(k+1))(f^2\,dh + fh\,df)$ and on the other hand, using (3.1), $f\,dg = fh\,df$. Consequently, $f\,dh = kh\,df$ and so $h = \lambda f^k$ with $\lambda \in \mathbb{K}$. □

3.4. Computation of $H^k_f(\mathbb{K}^n)$ $1 \le k \le n-2$

Lemma 3.7. *Let $\alpha \in Z^k_{f,p}(\mathbb{K}^n)$ with $1 \le k \le n-2$. Then α is cohomologous to a closed k-form.*

Proof. We have $f\,d\alpha - (k-p)\,df \wedge \alpha = 0$. If $k = p$ then α is closed. Now we suppose that $k \neq p$. We put $\beta = d\alpha \in \Omega^{k+1}(\mathbb{K}^n)$. We have

$$0 = df \wedge (f\,d\alpha - (k-p)\,df \wedge \alpha) = f\,df \wedge \alpha, \tag{3.2}$$

so $df \wedge \alpha = 0$.

Now, since $d\beta = 0$ and $df \wedge \beta = 0$, Proposition 3.4 gives $\beta = df \wedge d\gamma$ with $\gamma \in \Omega^{k-1}(\mathbb{K}^n)$. Then, if we consider $\alpha' = \alpha - (1/(k-p))(f\,d\gamma - (k-p-1)\,df \wedge \gamma)$, we have $d\alpha' = 0$ and $f\,d\gamma - (k-p-1)\,df \wedge \gamma \in B^k_{f,p}(\mathbb{K}^n)$. □

Theorem 3.8. *If $k \in \{2,\dots,n-2\}$ then $H^k_f(\mathbb{K}^n) = \{0\}$.*

Proof. Let $\alpha \in Z^k_f(\mathbb{K}^n)$. Then $\alpha \in \Omega^k(\mathbb{K}^n)$ and verifies $f\,d\alpha - k\,df \wedge \alpha = 0$.

According to Lemma 3.7 we can assume that α is closed. Now we show that $\alpha \in B^k_f(\mathbb{K}^n)$. Since $d\alpha = 0$ and $df \wedge \alpha = 0$, there exists $\beta \in \Omega^{k-2}(\mathbb{K}^n)$ such that $\alpha = df \wedge d\beta$ (Proposition 3.4). Thus, $\alpha = d_f((-1/(k-1))d\beta)$. □

Remark 3.9. It is possible to adapt this proof to show that $H^k_{f,p}(\mathbb{K}^n) = \{0\}$ if $k \in \{2,\dots,n-2\}$ and $p \neq k,\ k-1$.

Lemma 3.10. *Let $\alpha \in Z^1_f(\mathbb{K}^n)$. If $\operatorname{ord}(j^\infty_0(\alpha)) > N$ then $\alpha \in B^1_f(\mathbb{K}^n)$.*

Proof. According to Lemma 3.7, we can assume that $d\alpha = 0$.

Since $df \wedge \alpha = 0$ we have $\alpha = g\,df$ (see Proposition 3.3), where g is in $\mathcal{F}(\mathbb{K}^n)$ and verifies $\operatorname{ord}(j^\infty_0(g)) > 0$. We show that f divides g.

Let $\bar{g} \in \mathcal{F}(\mathbb{K}^n)$ be such that $W \cdot \bar{g} = g$ (see Lemma 3.2); note that $\operatorname{ord}(j^\infty_0(\bar{g})) > 0$.

We have $\mathcal{L}_W(df \wedge d\bar{g}) = N\,df \wedge d\bar{g} + df \wedge dg$, and since $df \wedge dg = -d\alpha = 0$, $df \wedge d\bar{g}$ verifies

$$\mathcal{L}_W(df \wedge d\bar{g}) = N\,df \wedge d\bar{g}, \tag{3.3}$$

which means that $df \wedge d\bar{g}$ is either 0 or quasihomogeneous of degree N.

But since $\operatorname{ord}(j^\infty_0(df \wedge d\bar{g})) > N$, $df \wedge d\bar{g}$ must be 0.

Consequently, there exists $\nu \in \mathcal{F}(\mathbb{K}^n)$ such that $\partial\bar{g}/\partial x_i = \nu(\partial f/\partial x_i)$ for any i. Thus, $W \cdot \bar{g} = \nu W \cdot f$ and so $g = \nu f$.

We deduce that $\alpha = f\beta$ with $\beta \in \Omega^1(\mathbb{K}^n)$.

Now, we have

$$0 = d\alpha = df \wedge \beta + f\,d\beta, \qquad 0 = df \wedge \alpha = f\,df \wedge \beta, \tag{3.4}$$

which implies that $d\beta = 0$.

Therefore, $\alpha = f\,dh = d_f(h)$ with $h \in \mathcal{F}(\mathbb{K}^n)$. □

Theorem 3.11. *The space $H^1_f(\mathbb{K}^n)$ is of dimension 1 and spanned by df.*

Proof. Let $\alpha \in Z^1_f(\mathbb{K}^n)$. According to Lemma 3.10, we only have to study the case where α is quasihomogeneous with $\deg(\alpha) \leq N$. We have $f\,d\alpha - df \wedge \alpha = 0$, so $df \wedge d\alpha = 0$. We deduce that $d\alpha = df \wedge \beta$, where β is a quasihomogeneous 1-form of degree $\deg(\alpha) - N \leq 0$. But since dx_i is quasihomogeneous of degree $w_i > 0$ for any i, every quasihomogeneous nonzero 1-form has a strictly positive degree. We deduce that $\beta = 0$ and so $d\alpha = 0$. Therefore, $df \wedge \alpha = 0$ which implies that $\alpha = g\,df$, where g is a quasihomogeneous function of degree $\deg(\alpha) - N \leq 0$. Consequently, if $\deg(\alpha) < N$ then $g = 0$; otherwise g is constant. To conclude, note that df is not a cobord because f does not divide df. □

3.5. Computation of $H^n_{f,p}(\mathbb{K}^n)$. We compute the spaces $H^n_{f,p}(\mathbb{K}^n)$ for $p \neq n-1$. We consider the application

$$d_f^{(n-q)} : \Omega^{n-1}(\mathbb{K}^n) \longrightarrow \Omega^n(\mathbb{K}^n), \qquad \alpha \longmapsto f\,d\alpha - (q-1)\,df \wedge \alpha, \tag{3.5}$$

with $q \neq 1$ (note that if $q = n$ then we obtain the space $H^n_{NP}(M,\Lambda)$ and if $q = 2$ then we have $H^2_\Lambda(\mathbb{K}^n)$).

We denote $\mathcal{I}^n = \{df \wedge \alpha;\ \alpha \in \Omega^{n-1}(\mathbb{K}^n)\}$. It is clear that $\mathcal{I}^n \simeq I_f$ (recall that I_f is the ideal of $\mathcal{F}(\mathbb{K}^n)$ spanned by $\partial f/\partial x_1,\dots,\partial f/\partial x_n$) and that $\Omega^n(\mathbb{K}^n)/\mathcal{I}^n \simeq Q_f = \mathcal{F}(\mathbb{K}^n)/I_f$.

We put $\sigma = i_W\omega$ (recall that $W = w_1x_1(\partial/\partial x_1) + \cdots + w_nx_n(\partial/\partial x_n)$ and that ω is the standard volume form on $\mathbb{K}^n$). Note that σ is a quasihomogeneous $(n-1)$-form of degree $\sum_i w_i$ and that $dg \wedge \sigma = (W\cdot g)\omega$ if $g \in \mathcal{F}(\mathbb{K}^n)$.

If $\alpha \in \Omega^{n-1}(\mathbb{K}^n)$, we use the notation $\operatorname{div}(\alpha)$ for $d\alpha = \operatorname{div}(\alpha)\omega$; for example, $\operatorname{div}(\sigma) = \sum_i w_i$. Note that if α is quasihomogeneous, then $\operatorname{div}(\alpha)$ is quasihomogeneous of degree $\deg\alpha - \sum_i w_i$.

Lemma 3.12. (1) *If the ∞-jet at 0 of γ does not contain a component of degree qN (in particular if $q \le 0$) then $\gamma \in B^n_{f,n-q}(\mathbb{K}^n) \Leftrightarrow \gamma \in \mathcal{I}^n$.*

(2) *If γ is a quasihomogeneous n-form of degree qN, then $\gamma \in B^n_{f,n-q}(\mathbb{K}^n) \Rightarrow \gamma \in \mathcal{I}^n$.*

Proof. If $\gamma = f\,d\alpha - (q-1)\,df \wedge \alpha \in B^n(d_f^{(n-q)})$, where $\alpha \in \Omega^{n-1}$ then $\gamma = df \wedge \beta$ with $\beta = -(q-1)\alpha + (\operatorname{div}(\alpha)/N)\sigma$. This shows the second claim and the first part of the first one.

Now we prove the reverse of the first claim.

Formal case: let $\gamma = \sum_{i>0}\gamma^{(i)}$ and $\beta = \sum \beta^{(i-N)}$ (with $\gamma^{(i)}$ of degree i, $\gamma^{(qN)} = 0$ and $\beta^{(i-N)}$ of degree $i-N$) such that $\gamma = df \wedge \beta$. If we put $\alpha = (-1/(q-1))\beta + \sum_i(\operatorname{div}(\beta^{(i-N)})/(q-1)(i-qN))\sigma$, we have $d_f^{(n-q)}(\alpha) = \gamma$.

Analytical case: if β is analytic at 0, the function $\operatorname{div}(\beta)$ is analytic too, and since $\lim_{i\to+\infty}(1/(i-qN)) = 0$, the $(n-1)$-form defined above is also analytic at 0.

$\mathcal{C}^\infty$ case: we suppose that $\gamma = df \wedge \beta$. If we denote $\tilde{\gamma} = j_0^\infty(\gamma)$, then there exists a formal $(n-1)$-form $\tilde{\alpha}$ such that $\tilde{\gamma} = f\,d\tilde{\alpha} - (q-1)\,df \wedge \tilde{\alpha}$. Let α be a $\mathcal{C}^\infty - (n-1)$-form such that $\tilde{\alpha} = j_0^\infty(\alpha)$. This form verifies $f\,d\alpha - (q-1)\,df \wedge \alpha = \gamma + \epsilon$, where ϵ is flat at 0. Since $B^n_{f,n-q}(\mathbb{K}^n) \subset \mathcal{I}^n$, $\epsilon \in \mathcal{I}^n$ so that $\epsilon = df \wedge \mu$, where μ is flat at 0. Let $g \in \mathcal{F}(\mathbb{K}^n)$ be such that $W\cdot g - ((q-1)N - \sum w_i)g = \operatorname{div}(\mu)/(q-1)$ (Lemma 3.2). Then the form $\theta = (-1/(q-1))\mu + g\sigma$ verifies $d_f^{(n-q)}(\theta) = \epsilon$. □

Remark 3.13. (1) Lemma 3.12 gives $B^n_{f,n-q}(\mathbb{K}^n) \subset \mathcal{I}^n$. Thus, there is a surjection from $H^n_{f,n-q}(\mathbb{K}^n)$ onto Q_f. Therefore, if f is not of finite codimension, then $H^n_{f,n-q}(\mathbb{K}^n)$ is an infinite-dimensional vector space.

(2) According to this lemma, if γ is in $\mathcal{I}^n$ then there exits a quasihomogeneous n-form θ, of degree qN, such that $\gamma + \theta \in B^n_{f,n-q}(\mathbb{K}^n)$. Note that θ is in $\mathcal{I}^n$.

The first claim of this lemma allows us to state the following theorem.

Theorem 3.14. *If $q \le 0$ then $H^n_{f,n-q}(\mathbb{K}^n) \simeq Q_f$.*

Now we suppose that $q > 1$.

Lemma 3.15. *Let $\alpha \in \Omega^k(\mathbb{K}^n)$ and $p \in \mathbb{Z}$. Then $f\,d_f^{(p)}(\alpha) = d_f^{(p-1)}(f\alpha)$.*

Proof. The proof is obvious. □

Lemma 3.16. (1) *Let $q > 2$. If $\alpha \in \Omega^n(\mathbb{K}^n)$ is quasihomogeneous of degree $(q-1)N$ and verifies $f\alpha \in B^n_{f,n-q}(\mathbb{K}^n)$, then $\alpha \in B^n_{f,n-q+1}(\mathbb{K}^n)$.*

(2) *If α is quasihomogeneous of degree N with $f\alpha \in B^n_{f,n-2}(\mathbb{K}^n)$, then $\alpha = 0$.*

Proof. (1) We suppose that $\alpha = g\omega$ with $g \in \mathcal{F}(\mathbb{K}^n)$ quasihomogeneous of degree $(q-1)N - \sum w_i$. We have $fg\omega = f\,d\beta - (q-1)\,df \wedge \beta$, where β is a quasihomogeneous $(n-1)$-form of degree $(q-1)N$.

If we put $\theta = -(q-1)\beta + ((\operatorname{div}(\beta) - g)/N)\sigma$, then $df \wedge \theta = 0$, and so $\theta = df \wedge \gamma$, where γ is a quasihomogeneous $(n-2)$-form of degree $(q-2)N$. Consequently, $\beta = (-1/(q-1))\,df \wedge \gamma + ((\operatorname{div}(\beta) - g)/(q-1)N)\sigma$. Now, a computation shows that $f\,d\beta - (q-1)\,df \wedge \beta = (1/(q-1))f\,df \wedge d\gamma$, that is, $f\alpha = (1/(q-1))f\,df \wedge d\gamma$. Therefore, $\alpha = (1/(q-1))\,df \wedge d\gamma = (1/(q-1))\,d_f^{(n-q+1)}((-1/(q-2))\,d\gamma)$.

(2) As in (1) (with $q = 2$), we have $f\alpha = fg\omega = d_f^{(n-2)}(\beta)$ with $\deg g = N$ and $\deg \beta = N$. We put $\theta = -\beta + ((\operatorname{div}(\beta) - g)/N)\sigma$.

If $\theta \neq 0$ then $\theta = df \wedge \gamma$, where γ is a quasihomogeneous $(n-2)$-form of degree 0 which is not possible. So $\theta = 0$, that is, $\beta = ((\operatorname{div}(\beta) - g)/N)\sigma$.

We deduce that $f\,d\beta - df \wedge \beta = 0$, that is, $\alpha = 0$. □

Let $\mathcal{B}$ be a monomial basis of Q_f (for the existence of such a basis, see [3]). We denote by r_j $(j = 2, \dots, q-1)$ the number of monomials of $\mathcal{B}$ whose degree is $jN - \sum w_i$ (this number does not depend on the choice of $\mathcal{B}$). We also denote by s the dimension of the space of quasihomogeneous polynomials of degree $N - \sum w_i$ and c the codimension of f.

Theorem 3.17. *Let $\alpha \in \Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials $h_1, \dots, h_q$ (possibly zero) such that*

(a) *h_1 is quasihomogeneous of degree $N - \sum w_i$,*

(b) *h_j $(2 \le j \le q-1)$ is a linear combination of monomials of $\mathcal{B}$ of degree $jN - \sum w_i$,*

(c) *h_q is a linear combination of monomials of $\mathcal{B}$, and*

$$\alpha = (h_q + fh_{q-1} + \cdots + f^{q-1}h_1)\omega \mod B^n_{f,n-q}(\mathbb{K}^n). \tag{3.6}$$

In particular, the dimension of $H^n_{f,n-q}(\mathbb{K}^n)$ is $c + r_{q-1} + \cdots + r_2 + s$.

Proof

Existence. We suppose that $\alpha = g\omega$ with $g \in \mathcal{F}(\mathbb{K}^n)$. There exists h_q, a linear combination of the monomials of $\mathcal{B}$, such that $g = h_q \mod I_f$. So, according to Lemma 3.12 (see Remark 3.13), $g\omega = h_q\omega + df \wedge \beta \mod B^n_{f,n-q}(\mathbb{K}^n)$, where β is a quasihomogeneous $(n-1)$-form of degree $(q-1)N$.

Consequently, $g\omega = h_q\omega + (1/(q-1))f\,d\beta - (1/(q-1))[f\,d\beta - (q-1)\,df \wedge \beta] \mod B^n_{f,n-q}(\mathbb{K}^n)$, so we can write

$$g\omega = h_q\omega + fg_{q-1}\omega \mod B^n_{f,n-q}(\mathbb{K}^n), \tag{3.7}$$

where g_{q-1} is quasihomogeneous of degree $(q-1)N - \sum w_i$.

In the same way,

$$g_{q-1}\omega = h_{q-1}\omega + f g_{q-2}\omega \mod B^n_{f,n-q+1}(\mathbb{K}^n), \tag{3.8}$$

where h_{q-1} is a linear combination of the monomials of $\mathcal{B}$ of degree $(q-1)N - \sum w_i$ and g_{q-2} is quasihomogeneous of degree $(q-2)N - \sum w_i, \dots,$

$$g_2\omega = h_2\omega + f h_1\omega \mod B^n_{f,n-2}(\mathbb{K}^n), \tag{3.9}$$

where h_2 is a linear combination of the monomials of $\mathcal{B}$ of degree $2N - \sum w_i$ and h_1 is quasihomogeneous of degree $N - \sum w_i$.

Using Lemma 3.15, we get

$$\alpha = g\omega = (h_q + h_{q-1} + f^2 h_{q-2} + \cdots + f^{q-1}h_1)\omega \mod B^n\left(d_f^{(n-q)}\right). \tag{3.10}$$

Unicity. Let $g = h_q + f h_{q-1} + \cdots + f^{q-1}h_1$ with $h_1,\dots,h_q$ as in the statement of the theorem. We suppose that $g\omega \in B^n_{f,n-q}(\mathbb{K}^n)$. Then $g\omega \in \mathcal{I}^n$, that is, $g \in I_f$. But since $f h_{q-1} + \cdots + f^{q-1}h_1 \in I_f$ (because $f \in I_f$) we have $h_q \in I_f$ and so $h_q = 0$.

Now, according to Lemma 3.16, $(h_{q-1} + f h_{q-2} + \cdots + f^{q-2}h_1)\omega$ is in $B^n_{f,n-q+1}(\mathbb{K}^n)$ and so, in the same way, $h_{q-1} = 0$.

This way, we get $h_q = h_{q-1} = \cdots = h_2 = 0$ and $f h_1\omega \in B^n_{f,n-2}(\mathbb{K}^n)$. Lemma 3.16 gives $h_1 = 0$. □

This theorem allows us to give the dimension of the spaces $H^n_{NP}(\mathbb{K}^n,\Lambda)$ and $H^2_\Lambda(\mathbb{K}^n)$.

Corollary 3.18. *Let $\alpha \in \Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials $h_1,\dots,h_n$ (Possibly zero) such that*

(a) *h_1 is quasihomogeneous of degree $N - \sum w_i$,*

(b) *h_j $(2 \le j \le n-1)$ is a linear combination of monomials of $\mathcal{B}$ of degree $jN - \sum w_i$,*

(c) *h_n is a linear combination of monomials of $\mathcal{B}$, and*

$$\alpha = (h_n + f h_{n-1} + \cdots + f^{n-1}h_1)\omega \mod B^n_f(\mathbb{K}^n). \tag{3.11}$$

In particular, the dimension of $H^n_{NP}(\mathbb{K}^n,\Lambda)$ is $c + r_{n-1} + \cdots + r_2 + s$.

Corollary 3.19. *Let $\alpha \in \Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials h_1, h_2 (possibly zero) such that*

(a) *h_1 is quasihomogeneous of degree $N - \sum w_i$,*

(b) *h_2 is a linear combination of monomials of $\mathcal{B}$, and*

$$\alpha = (h_2 + f h_1)\omega \mod B^n_{f,n-2}(\mathbb{K}^n). \tag{3.12}$$

In particular, the dimension of $H^2_\Lambda(\mathbb{K}^n)$ is $c + s$.

Remark 3.20. If $q = 1$, then the space $H^n_{f,n-1}(\mathbb{K}^n)$ is $\Omega^n(\mathbb{K}^n)/f\Omega^n(\mathbb{K}^n)$ which is of infinite dimension.

3.6. Computation of $H^{n-1}_{f,p}(\mathbb{K}^n)$. We compute the spaces $H^{n-1}_{f,p}(\mathbb{K}^n)$ with $p \neq n-1$. We consider the piece of complex

$$\Omega^{n-2}(\mathbb{K}^n) \longrightarrow \Omega^{n-1}(\mathbb{K}^n) \longrightarrow \Omega^n(\mathbb{K}^n), \tag{3.13}$$

with

$$d_f^{(n-q)}(\alpha) = f\,d\alpha - (q-2)\,df \wedge \alpha \quad \text{if } \alpha \in \Omega^{n-2}(\mathbb{K}^n),$$
$$d_f^{(n-q)}(\alpha) = f\,d\alpha - (q-1)\,df \wedge \alpha \quad \text{if } \alpha \in \Omega^{n-1}(\mathbb{K}^n), \tag{3.14}$$

with $q \neq 1$.

Remember that if $q = n$, we obtain $H_{NP}^{n-1}(K^n, \Lambda)$ and if $q = 2$ we have $H_\Lambda^1(\mathbb{K}^n)$.

Lemma 3.21. *If $\alpha \in Z_{f,n-q}^{n-1}(\mathbb{K}^n)$, then $\alpha = (\operatorname{div}(\alpha)/(q-1)N)\sigma + df \wedge \beta$ with $\beta \in \Omega^{n-2}(\mathbb{K}^n)$ and so, $d\alpha$ verifies $\mathcal{L}_W(d\alpha) - (q-1)N\,d\alpha = (q-1)N\,df \wedge d\beta$.*

Proof. It is sufficient to notice that $df \wedge (\alpha - (\operatorname{div}(\alpha)/(q-1)N)\sigma) = 0$ (see Proposition 3.3). For the second claim, we have $(q-1)N\,d\alpha = (W \cdot \operatorname{div}(\alpha) + (\sum w_i)\operatorname{div}(\alpha))\omega - (q-1)N\,df \wedge d\beta$ and the conclusion follows. □

Lemma 3.22. *If $\alpha \in Z_{f,n-q}^{n-1}(\mathbb{K}^n)$ with $\operatorname{ord}(j_0^\infty(\alpha)) > (q-1)N$, then α is cohomologous to a closed $(n-1)$-form. In particular, if $q \leq 0$ then every $(n-1)$-cocycle for $d_f^{(n-q)}$ is cohomologous to a closed $(n-1)$-form.*

Proof. We have $\alpha = (\operatorname{div}(\alpha)/(q-1)N)\sigma + df \wedge \beta$ (Lemma 3.21) with

$$\mathcal{L}_W(d\alpha) - (q-1)N\,d\alpha = (q-1)N\,df \wedge d\beta. \tag{3.15}$$

Now, let $\gamma \in \Omega^{n-2}(\mathbb{K}^n)$ such that $\mathcal{L}_W\gamma - (q-2)N\gamma = (q-1)N\beta$ (γ exists because $\operatorname{ord}(j_0^\infty(\beta)) > (q-2)N$, see Lemma 3.2).

We have $\mathcal{L}_W\,d\gamma - (q-2)N\,d\gamma = (q-1)N\,d\beta$. Thus $df \wedge d\gamma$ verifies

$$\mathcal{L}_W(df \wedge d\gamma) - (q-1)N\,df \wedge d\gamma = (q-1)N\,df \wedge d\beta. \tag{3.16}$$

From (3.15) and (3.16) we get $d\alpha = df \wedge d\gamma$.

Indeed, $\mathcal{L}_W(d\alpha - df \wedge d\gamma) = (q-1)N(d\alpha - df \wedge d\gamma)$ but $d\alpha - df \wedge d\gamma$ is not quasihomogeneous of degree $(q-1)N$.

Now, if we put $\theta = \alpha - (1/(q-1))(f\,d\gamma - (q-2)\,df \wedge \gamma)$, we have $d\theta = 0$ and $\theta = \alpha \mod B_{f,n-q}^{n-1}(\mathbb{K}^n)$. □

Lemma 3.22 allows us to state the following theorem.

Theorem 3.23. *If we suppose that $q \leq 0$ then $H_{f,n-q}^{n-1}(\mathbb{K}^n) = \{0\}$.*

Proof. Let $\alpha \in Z_{f,n-q}^{n-1}(\mathbb{K}^n)$. We can suppose (according to Lemma 3.22) that $d\alpha = 0$. Thus we have $df \wedge \alpha = 0$. Proposition 3.4 gives then, $\alpha = df \wedge d\gamma$ with $\gamma \in \Omega^{n-3}(\mathbb{K}^n)$. Therefore, $\alpha = d_f^{(n-q)}(-(1/(q-2))\,d\gamma)$. □

Now, we assume that $q > 1$.

Lemma 3.24. *If $\alpha \in Z_{f,n-q}^{n-1}(\mathbb{K}^n)$ is a quasihomogeneous $(n-1)$-form whose degree is strictly lower than $(q-1)N$, then α is cohomologous to a closed $(n-1)$-form.*

Proof. According to Lemma 3.21, we have $\alpha = (\operatorname{div}(\alpha)/(q-1)N)\sigma + df \wedge \beta$, and so

$$d\alpha = \frac{(q-1)N}{\deg(\alpha) - (q-1)N}\,df \wedge d\beta. \tag{3.17}$$

We deduce that, if we put $\theta = \alpha - d_f^{(n-q)}((N/(\deg(\alpha) - (q-1)N))\,d\beta)$, we have $d\theta = 0$. □

Remark 3.25. A consequence of Lemmas 3.22 and 3.24 is that, if $q > 1$, every cocycle $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ is cohomologous to a cocycle $\eta+\theta$, where η is in $Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ and is closed, and θ is quasihomogeneous of degree $(q-1)N$.

Lemma 3.26. *Let $\alpha = g\sigma$, where g is a quasihomogeneous polynomial of degree $(q-1)N - \sum w_i$. Then*

(1) *if $q > 2$, then $\alpha \in B^{n-1}_{f,n-q}(\mathbb{K}^n) \Leftrightarrow g\omega \in B^{n}_{f,n-q+1}(\mathbb{K}^n)$,*

(2) *if $q = 2$, $\alpha \in B^{n-1}_{f,n-2}(\mathbb{K}^n) \Leftrightarrow \alpha = 0$.*

Proof. (1) (a) We suppose that $\alpha \in B^{n-1}_{f,n-q}(\mathbb{K}^n)$, that is, $\alpha = f\,d\beta - (q-2)\,df \wedge \beta$ with $\beta \in \Omega^{n-2}(\mathbb{K}^n)$. Then $d\alpha = (q-1)\,df \wedge d\beta$.

On the other hand, $d\alpha = (q-1)Ng\omega$ so $g\omega = (1/N)\,df \wedge d\beta = d_f^{(n-q+1)}(-d\beta/(q-2)N)$.

(b) Now we suppose that $g\omega \in B^{n}_{f,n-q+1}(\mathbb{K}^n)$, that is, $g\omega = f\,d\beta - (q-2)\,df \wedge \beta$, where β is a quasihomogeneous $(n-1)$-form of degree $(q-2)N$. We put $\gamma = i_W\beta \in \Omega^{n-2}(\mathbb{K}^n)$. We have

$$\begin{aligned} d_f^{(n-q)}(\gamma) &= f\,d\gamma - (q-2)\,df \wedge \gamma = f\,d(i_W\beta) - (q-2)\,df \wedge (i_W\beta) \\ &= f(\mathcal{L}_W\beta - i_W\,d\beta) - (q-2)[-i_W(df \wedge \beta) + (i_W\,df) \wedge \beta] \\ &= f(q-2)N\beta - i_W[f\,d\beta - (q-2)\,df \wedge \beta] - (q-2)(W \cdot f)\beta \\ &= -i_W[f\,d\beta - (q-2)\,df \wedge \beta]. \end{aligned} \tag{3.18}$$

Consequently, $d_f^{(n-q)}(\gamma) = i_W(g\omega) = -g\sigma$.

(2) If $\alpha = f\,d\beta$, where β is a quasihomogeneous $(n-2)$-form of degree $\deg\alpha - N = 0$, then $\beta = 0$ and so $\alpha = 0$. □

We recall that $\mathcal{B}$ indicates a monomial basis of Q_f. We adopt the same notations as for Theorem 3.17.

Theorem 3.27. *We suppose that $q > 2$. Let $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$. There exist unique polynomials $h_1,\dots,h_{q-1}$ (possibly zero) such that*

(a) *h_1 is quasihomogeneous of degree $N - \sum w_i$,*

(b) *h_k $(k \geq 2)$ is a linear combination of monomials of $\mathcal{B}$ of degree $kN - \sum w_i$, and*

$$\omega = (h_{q-1} + fh_{q-2} + \cdots + f^{q-2}h_1)\sigma \mod B^{n-1}_{f,n-q}(\mathbb{K}^n). \tag{3.19}$$

In particular, the dimension of the space $H^{n-1}_{f,n-q}(\mathbb{K}^n)$ is $r_{q-1} + \cdots + r_2 + s$.

Proof. If $\alpha \in Z^{n-1}_{f,n-q}(\mathbb{K}^n)$, then α is cohomologous to $\eta+\theta$, where η is in $Z^{n-1}_{f,n-q}(\mathbb{K}^n)$ and is closed, and θ is quasihomogeneous of degree $(q-1)N$ (see Remark 3.25).

The same proof as of Theorem 3.23 shows that η is a cobord.

Now, we have to study θ. According to Lemma 3.21, we can write $\theta = (\operatorname{div}(\theta)/(q-1)N)\sigma + df \wedge \beta$ $(\beta \in \Omega^{n-2}(\mathbb{K}^n))$ with $\mathcal{L}_W(d\theta) - (q-1)N\,d\theta = (q-1)N\,df \wedge d\beta$. Since θ is quasihomogeneous of degree $(q-1)N$, the former relation gives $df \wedge d\beta = 0$. Consequently, if we put $\gamma = df \wedge \beta$, Proposition 3.4 gives $\gamma = df \wedge d\xi$.

Therefore, $\gamma = d_f^{(n-q)}(-(1/(q-2))\,d\xi)$ and so $\theta = (\operatorname{div}(\theta)/(q-1)N)\sigma \mod B^{n-1}_{f,n-q}(\mathbb{K}^n)$. The conclusion follows using Lemma 3.26 and Theorem 3.17. □

Corollary 3.28. *We suppose that $q = n$. Let $\alpha \in Z_f^{n-1}(\mathbb{K}^n)$. There exist unique polynomials $h_1,\dots,h_{n-1}$ (possibly zero) such that*

(a) *h_1 is quasihomogeneous of degree $N-\sum w_i$,*

(b) *h_k $(k \geq 2)$ is a linear combination of monomials of $\mathcal{B}$ of degree $kN-\sum w_i$, and*

$$\omega = (h_{n-1}+fh_{n-2}+\cdots+f^{n-2}h_1)\sigma \mod B_f^{n-1}(\mathbb{K}^n). \tag{3.20}$$

In particular, the dimension of the space $H_{NP}^{n-1}(\mathbb{K}^n,\Lambda)$ is $r_{n-1}+\cdots+r_2+s$.

Remark 3.29. If $q = 2$, the description of the space $H_{f,n-2}^{n-1}(\mathbb{K}^n)$ (and so $H_\Lambda^1(\mathbb{K}^n)$) is more difficult. It is possible to show that this space is not of finite dimension. Indeed, we consider the case $n = 3$ for simplicity (but it is valid for any $n \geq 3$). We put $\alpha = g((\partial f/\partial x)\,dx\wedge dz+(\partial f/\partial y)\,dy\wedge dz)$, where g is a function which depends only on z. We have $d\alpha = 0$ and $df\wedge\alpha = 0$, so $\alpha \in Z_{f,n-1}^{n-1}(\mathbb{K}^n)$ but $\alpha \notin B_{f,n-2}^{n}(\mathbb{K}^n)$ because f does not divide α.

We can yet give more precisions on the space $H_{f,n-2}^{n-1}(\mathbb{K}^n)$.

Theorem 3.30. *Let E be the space of $(n-1)$-forms $h\sigma$, where h is a quasihomogeneous polynomial of degree $N-\sum w_i$, and F the quotient of the vector space $\{df\wedge d\gamma;\ \gamma\in\Omega^{n-3}(\mathbb{K}^n)\}$ by the subspace $\{df\wedge d(f\beta);\ \beta\in\Omega^{n-3}(\mathbb{K}^n)\}$.*

Then $H_{f,n-2}^{n-1}(\mathbb{K}^n) = E\oplus F$.

Proof. Let α in $Z_{f,n-2}^{n-1}(\mathbb{K}^n)$.

According to Remark 3.25, there exist a closed $(n-1)$-form η with $\eta \in Z_{f,n-2}^{n-1}(\mathbb{K}^n)$ and a quasihomogeneous $(n-1)$-form θ of degree N, such that α is cohomologous to $\eta+\theta$.

We have (Lemma 3.21) $\theta = (\operatorname{div}(\theta)/N)\sigma+df\wedge\beta$ with β quasihomogeneous of degree 0 which is possible only if $\beta = 0$. So, $\theta = g\sigma$, where g is a quasihomogeneous polynomial of degree $N-\sum w_i$. Lemma 3.26 says that $\theta \in B_{f,n-2}^{n-1}(\mathbb{K}^n)$ if and only if $\theta = 0$.

Now we study η. Proposition 3.4 gives $\eta = df\wedge d\gamma$, where γ is an $(n-3)$-form. If we suppose that $\eta \in B_{f,n-2}^{n-1}(\mathbb{K}^n)$, then $df\wedge d\gamma = f\,d\xi$ with $\xi\in\Omega^{n-2}(K^n)$, and so $df\wedge d\xi = 0$. Now we apply Proposition 3.4 to $d\xi$ and we obtain $d\xi = df\wedge d\beta$ with $\beta\in\Omega^{n-3}(\mathbb{K}^n)$. Consequently, $df\wedge d\gamma = f\,df\wedge d\beta$ which implies that $d\gamma = f\,d\beta+df\wedge\mu$ with $\mu\in\Omega^{n-3}(\mathbb{K}^n)$, and so $d\gamma = d(f\beta)+df\wedge\nu$ with $\nu\in\Omega^{n-3}(\mathbb{K}^n)$.

Therefore, $\eta\in B_{f,n-2}^{n-1}(\mathbb{K}^n) \Leftrightarrow \eta = df\wedge d(f\beta)$. □

3.7. Summary. It is time to sum up the results we have found.

The cohomology $H_f^\bullet(\mathbb{K}^n)$ (and so the Nambu-Poisson cohomology $H_{NP}^\bullet(\mathbb{K}^n,\Lambda)$) has been entirely computed (see Theorems 3.6, 3.8, 3.11, and Corollaries 3.18 and 3.28).

The spaces of this cohomology are of finite dimension and only the "extremal" ones (i.e., H^0,H^1,H^{n-1}, and H^n) are possibly different to $\{0\}$. The spaces $H_{NP}^0(\mathbb{K}^n,\Lambda)$ and $H_{NP}^1(\mathbb{K}^n,\Lambda)$ are always of dimension 1. The dimensions of the spaces $H_{NP}^{n-1}(\mathbb{K}^n,\Lambda)$ and $H_{NP}^n(\mathbb{K}^n,\Lambda)$ depend, on one hand, on the type of the singularity of Λ (via the role played by Q_f), and on the other hand, on the "polynomial nature" of Λ.

Concerning the cohomology $H_{f,n-2}^\bullet(\mathbb{K}^n)$, we have computed H^n, that is, $H_\Lambda^n(\mathbb{K}^n)$ (see Corollary 3.19) and we have given a sketch of description of H^{n-1} (see Theorem 3.30).

We have also computed the spaces $H^0_{f,n-2}(\mathbb{K}^n)$ (see Theorem 3.6) and $H^k_{f,n-2}(\mathbb{K}^n)$ (see Theorem 3.8) for $k \neq n-2, n-1$, but these spaces are not particularly interesting for our problem. The space $H^2_\Lambda(\mathbb{K}^n)$, which describes the infinitesimal deformations of Λ is of finite dimension and its dimension has the same property as the dimension of $H^n_{NP}(\mathbb{K}^n, \Lambda)$. On the other hand, the space $H^1_\Lambda(\mathbb{K}^n)$ which is the space of the vector fields preserving Λ modulo the Hamiltonian vector fields, is not of finite dimension.

It is interesting to compare the results we have found on these two cohomologies with the ones given in [9] on the computation of the Poisson cohomology in dimension 2.

Finally, if $p \neq 0, n-2, n-1$, we have computed the spaces $H^0_{f,p}(\mathbb{K}^n), H^{n-1}_{f,p}(\mathbb{K}^n)$, $H^n_{f,p}(\mathbb{K}^n)$, and $H^k_{f,p}(\mathbb{K}^n)$ with $k \neq p, p+1$.

If $p = n-1$, we have computed the spaces $H^0_{f,n-1}(\mathbb{K}^n)$ and $H^k_{f,n-1}(\mathbb{K}^n)$ for $2 \leq k \leq n-2$, $k \neq p, p+1$ (the space $H^n_{f,n-1}(\mathbb{K}^n)$ is of infinite dimension).

4. Examples.

In this section, we explicit the cohomology of some particular germs of n-vectors.

4.1. Normal forms of n-vectors.

Let $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$ be a germ at 0 of n-vectors on $\mathbb{K}^n$ $(n \geq 3)$ with f of finite codimension (see the beginning of Section 3) and $f(0) = 0$ (if $f(0) \neq 0$, then the local triviality theorem, see [1, 5] or [11], allows us to write, up to a change of coordinates, that $\Lambda = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n$).

Proposition 4.1. *If 0 is not a critical point for f, then there exist local coordinates $y_1, \ldots, y_n$ such that*

$$\Lambda = y_1 \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}. \tag{4.1}$$

Proof. A similar proposition is shown for instance in [9] in dimension 2. The proof can be generalized to the n-dimensional $(n \geq 3)$ case.

Now we suppose that 0 is a critical point of f. Moreover, we suppose that the germ f is simple, which means that a sufficiently small neighbourhood (with respect to Whitney's topology; see [3]) of f intersects only a finite number of R-orbits (two germs g and h are said to be R-equivalent if there exits φ, a local diffeomorphism at 0, such that $g = h \circ \varphi$). Simple germs are those who present a certain kind of stability under deformation. □

The following theorem can be found in [2].

Theorem 4.2. *Let f be a simple germ at 0 of finite codimension. Suppose that f has at 0 a critical point with critical value 0. Then there exist local coordinates $y_1, \ldots, y_n$ such that the germ $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$ can be written, up to a multiplicative constant, $g(\partial/\partial y_1) \wedge \cdots \wedge \partial/\partial y_n$, where g is in the following list:*

$$\begin{gathered} A_k : y_1^{k+1} \pm y_2^2 \pm \cdots \pm y_n^2, \quad k \geq 1, \quad D_k : y_1^2 y_2 \pm y_2^{k-1} \pm y_3^2 \pm \cdots \pm y_n^2, \quad k \geq 4, \\ E_6 : y_1^3 + y_2^4 \pm y_3^2 \pm \cdots \pm y_n^2, \qquad E_7 : y_1^3 + y_1 y_2^3 \pm y_3^2 \pm \cdots \pm y_n^2, \\ E_8 : y_1^3 + y_2^5 \pm y_3^2 \pm \cdots \pm y_n^2. \end{gathered} \tag{4.2}$$

Proposition 4.1 and Theorem 4.2 describe most of the germs at 0 of n-vectors on $\mathbb{K}^n$ vanishing at 0.

We can notice that the models given in the former list are all quasihomogeneous polynomials; which justifies the assumption we made in Section 3.

4.2. Some examples. (1) The regular case: $f(x_1,\dots,x_n)=x_1$.

It is easy to see that $Q_f=\{0\}$ and that f is quasihomogeneous of degree $N=1$, with respect to $w_1=\cdots=w_n=1$. We have $N-\sum w_i<0$, so $H_f^0(\mathbb{K}^n)\simeq\mathbb{K}$, $H_f^1(\mathbb{K}^n)=\mathbb{K}\cdot dx_1$ and $H_f^k(\mathbb{K}^n)=\{0\}$ for any $k\geq 2$.

(2) Nondegenerate singularity: $f(x_1,\dots,X_n)=x_1^2+\cdots+x_n^2$ with $n\geq 3$.

We have $N=2$ and $w_1=\cdots=w_n=1$. The space Q_f is isomorphic to $\mathbb{K}$ and is spanned by the constant germ 1, which is of degree 0.

We deduce that $H_f^0(\mathbb{K}^n)\simeq\mathbb{K}$, $H_f^1(\mathbb{K}^n)=\mathbb{K}\cdot(x_1\,dx_1+\cdots+x_n\,dx_n)$ and $H_f^k=\{0\}$ for $2\leq k\leq n-2$.

In order to describe the spaces $H_f^{n-1}(\mathbb{K}^n)$ and $H_f^n(\mathbb{K}^n)$, we look for an integer $k\in\{1,\dots,n-1\}$ such that $kN-\sum w_i=\deg 1$, that is, $2k-n=0$.

Therefore,

(a) if n is even, then $\{\omega,f^{n/2}\omega\}$ is a basis of $H_f^n(\mathbb{K}^n)$ and $H_f^{n-1}(\mathbb{K}^n)$ is spanned by $\{f^{n/2-1}\sigma\}$,

(b) if n is odd, then $H_f^{n-1}(\mathbb{K}^n)=\{0\}$ and the space $H_f^n(\mathbb{K}^n)$ is spanned by $\{\omega\}$.

We recall that $\omega=dx_1\wedge\cdots\wedge dx_n$ and

$$\sigma=i_W\omega=\sum_{i=1}^{n}(-1)^{i-1}x_i\,dx_i\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n. \tag{4.3}$$

(3) The case A_2 with $n=3: f(x_1,x_2,x_3)=x_1^3+x_2^2+x_3^2$.

Here, $w_1=2$, $w_2=w_3=3$, and $N=6$. Thus, $N-\sum w_i=-2$, $2N-\sum w_i=4$, and $3N-\sum w_i=10$.

Moreover, $\mathcal{B}=\{1,x_1\}$ is a monomial basis of Q_f. But as $\deg 1=0$ and $\deg x_1=3$, we have

$$\begin{aligned} H_f^0(\mathbb{K}^3)\simeq\mathbb{K},\qquad & H_f^1(\mathbb{K}^3)=\mathbb{K}\cdot(3x_1\,dx_1+2x_2\,dx_2+2x_3\,dx_3),\\ & H_f^2(\mathbb{K}^3)=H_f^3(\mathbb{K}^3)=\{0\}. \end{aligned} \tag{4.4}$$

(4) The case D_5 with $n=4: f(x_1,x_2,x_3,x_4)=x_1^2x_2+x_2^4+x_3^2+x_4^2$.

We have $w_1=3$, $w_2=2$, $w_3=w_4=4$, and $N=8$, then $N-\sum w_i=-5$, $2N-\sum w_i=3$, $3N-\sum w_i=11$, and $4N-\sum w_i=19$.

Now, $\mathcal{B}=\{1,x_1,x_2,x_2^2,x_2^3\}$ is a monomial basis of Q_f. Here, $\deg 1=0$, $\deg x_1=3$, $\deg x_2=2$, $\deg x_2^2=4$, and $\deg x_2^3=6$. Thus, the only element of $\mathcal{B}$ whose degree is of type $kN-\sum w_i$ is x_1.

Consequently,

$$\begin{aligned} H_f^0(\mathbb{K}^4)\simeq\mathbb{K},\qquad & H_f^1(\mathbb{K}^4)=\mathbb{K}\cdot(2x_1x_2\,dx_1+(x_1^2+4x_2^3)\,dx_2+2x_3\,dx_3+2x_4\,dx_4),\\ & H_f^2(\mathbb{K}^4)=\{0\},\qquad H_f^3(\mathbb{K}^4)=\mathbb{K}\cdot(x_1\sigma), \end{aligned} \tag{4.5}$$

and $\{\omega,x_1\omega,x_2\omega,x_2^2\omega,x_2^3\omega,x_1f\omega\}$ is a basis of $H_f^4(\mathbb{K}^4)$.

Here, we have $W = 3x_1(\partial/\partial x_1) + 2x_2(\partial/\partial x_2) + 4x_3(\partial/\partial x_3) + 4x_4(\partial/\partial x_4)$ and

$$\begin{aligned}\sigma &= 3x_1\, dx_2 \wedge dx_3 \wedge dx_4 - 2x_2\, dx_1 \wedge dx_3 \wedge dx_4 \\ &\quad + 4x_3\, dx_1 \wedge dx_2 \wedge dx_4 - 4x_4\, dx_1 \wedge dx_2 \wedge dx_3.\end{aligned} \tag{4.6}$$

References

[1] D. Alekseevsky and P. Guha, *On decomposability of Nambu-Poisson tensor*, Acta Math. Univ. Comenian. (N.S.) **65** (1996), no. 1, 1–9. MR 97j:58044. Zbl 864.70012.

[2] V. I. Arnol'd, *Poisson structures on the plane and on other powers of volume forms*, J. Soviet Math. **47** (1989), no. 3, 2509–2516. Zbl 694.58015.

[3] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of Differentiable Maps. Vol. I*, Monograph in Mathematics, vol. 82, Birkhäuser, Massachusetts, 1985, the classification of critical points, caustics, and wave fronts. MR 86f:58018. Zbl 554.58001.

[4] G. de Rham, *Sur la division de formes et de courants par une forme linéaire*, Comment. Math. Helv. **28** (1954), 346–352 (French). MR 16,402d. Zbl 056.31601.

[5] P. Gautheron, *Some remarks concerning Nambu mechanics*, Lett. Math. Phys. **37** (1996), no. 1, 103–116. MR 97f:58055. Zbl 849.70014.

[6] R. Ibáñez, M. de León, B. López, J. C. Marrero, and E. Padrón, *Duality and modular class of a Nambu-Poisson structure*, preprint, http://xxx.lanl.gov/abs/math.SG/0004065.

[7] R. Ibáñez, M. de León, J. C. Marrero, and E. Padrón, *Leibniz algebroid associated with a Nambu-Poisson structure*, J. Phys. A **32** (1999), no. 46, 8129–8144. MR 2001b:53102. Zbl 991.23560.

[8] B. Malgrange, *Frobenius avec singularités. I. Codimension un*, Inst. Hautes Études Sci. Publ. Math. (1976), no. 46, 163–173 (French). MR 58#22685a. Zbl 355.32013.

[9] P. Monnier, *Poisson cohomology in dimension 2*, preprint, http://xxx.lanl.gov/abs/math.DG/0005261.

[10] N. Nakanishi, *Poisson cohomology of plane quadratic Poisson structures*, Publ. Res. Inst. Math. Sci. **33** (1997), no. 1, 73–89. MR 98d:58063. Zbl 997.48807.

[11] ———, *On Nambu-Poisson manifolds*, Rev. Math. Phys. **10** (1998), no. 4, 499–510. MR 99g:53036. Zbl 929.70015.

[12] Y. Nambu, *Generalized Hamiltonian dynamics*, Phys. Rev. D (3) **7** (1973), 2405–2412. MR 56#13847.

[13] C. A. Roche, *Cohomologie relative dans le domaine réel*, Ph.D. thesis, University of Grenoble, 1973.

[14] L. Takhtajan, *On foundation of the generalized Nambu mechanics*, Comm. Math. Phys. **160** (1994), no. 2, 295–315. MR 95b:58067. Zbl 808.70015.

[15] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics, vol. 118, Birkhäuser, Basel, 1994. MR 95h:58057. Zbl 810.53019.

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