LATTICE MODULES HAVING SMALL COFINITE IRREDUCIBLES

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ABSTRACT. We introduce the concept of small cofinite irreducibles in Noetherian lattice modules and obtain several characterizations of this property.

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Let $\mathcal{L}$ be a multiplicative lattice and let $\mathfrak{M}$ be an $\mathcal{L}$-module with greatest element M. Recall from [5] that for an element a of $\mathcal{L}$, $\mathrm{Rad}(a) = \bigvee\{x \in \mathcal{L} \mid x^n \le a$ for some positive integer $n\}$. For an element B of $\mathfrak{M}$, we define $\mathrm{Rad}(B)$ to be $\mathrm{Rad}(B : M)$, that is, $\mathrm{Rad}(B) = \bigvee\{x \in \mathcal{L} \mid x^n M \le B$ for some positive integer $n\}$. An element Q of $\mathfrak{M}$ is defined to be primary if for all $b \in \mathcal{L}$ and $C \in \mathfrak{M}$, $bC \le Q$ implies either $b \le \mathrm{Rad}(Q)$ or $C \le Q$.

LEMMA 1. *Let $\mathcal{L}$ be an r-lattice, let $\mathfrak{M}$ be an $\mathcal{L}$-module with greatest element M, and let A and B be elements of $\mathfrak{M}$. Then $\mathrm{Rad}(A \wedge B) = \mathrm{Rad}(A) \wedge \mathrm{Rad}(B)$.*

PROOF. We have

$$
\begin{aligned}
\mathrm{Rad}(A \wedge B) &= \mathrm{Rad}\big((A \wedge B) : M\big) = \mathrm{Rad}(A : M \wedge B : M) \\
&= \mathrm{Rad}(A : M) \wedge \mathrm{Rad}(B : M) = \mathrm{Rad}(A) \wedge \mathrm{Rad}(B),
\end{aligned}
\tag{1}
$$

where the third equality follows from [5, Lemma 2.2]. □

LEMMA 2. *Let $\mathcal{L}$ be a totally quasi-local lattice with maximal element m, let $\mathfrak{M}$ be an $\mathcal{L}$-module, and let Q be an element of $\mathfrak{M}$. If $\mathrm{Rad}(Q) = m$, then Q is primary.*

PROOF. Suppose $\mathrm{Rad}(Q) = m$. Also suppose that $b \in \mathcal{L}$ and $C \in \mathfrak{M}$ such that $bC \le Q$ and $C \not\le Q$. Then $b \ne I$, and since $\mathcal{L}$ is totally quasi-local, it follows that $b \le m = \mathrm{Rad}(Q)$. Thus, Q is primary. □

Let $\mathcal{L}$ be a totally quasi-local lattice with maximal element m and let $\mathfrak{M}$ be an $\mathcal{L}$-module. For an element Q of $\mathfrak{M}$, Q is said to be m-primary if $\mathrm{Rad}(Q) = m$.

LEMMA 3. *Let $\mathcal{L}$ be a local Noether lattice with maximal element m, let $\mathfrak{M}$ be an $\mathcal{L}$-module with greatest element M, and let Q be an element of $\mathfrak{M}$ different from M. Then Q is m-primary if and only if there exists a positive integer n such that $m^n M \le Q$.*

PROOF. Suppose Q is m-primary. Since $\mathrm{Rad}(Q : M) = m$, there exists a positive integer n such that $m^n \le Q : M$. Thus, $m^n M \le Q$. Conversely, suppose that there exists a positive integer n such that $m^n M \le Q$. Then $m = \mathrm{Rad}(m^n M) \le \mathrm{Rad}(Q) \le m$, and so Q is m-primary. □

Let $\mathcal{L}$ be a local Noether lattice with maximal element m and let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M. Define a metric d (the m-adic metric) on $\mathfrak{M}$ as follows: $d(A, B) = 0$ if $A \vee m^n M = B \vee m^n M$ for all nonnegative integers n, and otherwise, $d(A, B) = 2^{-s(A,B)}$, where $s(A, B) = \sup\{n \mid A \vee m^n M = B \vee m^n M\}$. This metric gives rise to the m-adic completions of $\mathcal{L}$ and $\mathfrak{M}$ (see [4]).

LEMMA 4. *Let $\mathcal{L}$ be a local Noether lattice with maximal element m, let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M, and let Q be an element of $\mathfrak{M}$. Then Q is m-primary if and only if $\mathfrak{M}/Q$ is finite dimensional.*

PROOF. Suppose that Q is m-primary. Then by Lemma 3, there exists a positive integer n such that $m^n M \leq Q$. Since $\mathfrak{M}/m^n M$ is finite dimensional [1, Corollary 5.2], if follows that $\mathfrak{M}/Q$ is finite dimensional.

On the other hand, suppose that Q is not m-primary. Then by Lemma 3, $m^n M \not\leq Q$ for all positive integers n. It follows that $\{Q \vee m^n M\}$ is a strictly decreasing sequence of elements of $\mathfrak{M}$ with meet Q, so $\mathfrak{M}/Q$ is not finite dimensional. $\square$

Let $\mathcal{L}$ be a local Noether lattice with maximal element m, let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M. We say that $\mathfrak{M}$ has small cofinite irreducibles if for every positive integer n, there exists a meet-irreducible element Q of $\mathfrak{M}$ such that $Q \leq m^n M$ and $\mathfrak{M}/Q$ is finite dimensional.

THEOREM 5. *Let $\mathcal{L}$ be a local Noether lattice with maximal element m and let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M. Then the following are equivalent:*

(1) $\mathfrak{M}$ has small cofinite irreducibles.

(2) For every positive integer n, there exists a meet-irreducible m-primary element Q of $\mathfrak{M}$ such that $Q \leq m^n M$.

(3) For every m-primary element Q' of $\mathfrak{M}$, there exists a meet irreducible m-primary element Q of $\mathfrak{M}$ such that $Q \leq Q'$.

(4) 0 is a closure point in the set of all meet-irreducible m-primary elements of $\mathfrak{M}$ in the m-adic topology on $\mathfrak{M}$.

PROOF. We begin by showing that (1) implies (2). Suppose $\mathfrak{M}$ has small cofinite irreducibles. Suppose also that n is a positive integer. Then there exists a meet-irreducible element Q of $\mathfrak{M}$ such that $Q \leq m^n M$ and $\mathfrak{M}/Q$ is finite dimensional. By Lemma 4, we have that Q is m-primary, so (2) holds.

We next show that (2) implies (4). Suppose that (2) holds and that $\epsilon > 0$. Choose n to be a positive integer satisfying $2^{-n} < \epsilon$. Using (2), there exists a meet-irreducible m-primary element Q of $\mathfrak{M}$ such that $Q \leq m^n M$. Thus $Q \vee m^n M = m^n M$, and so $d(Q, 0) \leq 2^{-n} < \epsilon$. Therefore, 0 is a closure point in the set of meet-irreducible m-primary elements of $\mathfrak{M}$ in the m-adic topology on $\mathfrak{M}$.

Now we show that (4) implies (3). Suppose that (4) holds and that Q' is an m-primary element of $\mathfrak{M}$. By Lemma 3, there exists a positive integer n such that $m^n M \leq Q'$. Since 0 is a closure point in the set of meet-irreducible m-primary elements of $\mathfrak{M}$ in the m-adic topology on $\mathfrak{M}$, there exists a meet-irreducible m-primary element Q of $\mathfrak{M}$ such that $d(Q, 0) \leq 2^{-n}$. Hence, $Q \vee m^n M = m^n M$, and so it follows that $Q \leq m^n M$. Thus the meet-irreducible m-primary element Q satisfies $Q \leq Q'$.

Finally, we show that (3) implies (1). Suppose that (3) holds and that n is a positive integer. Since $m^n M$ is m-primary, then by (3), there exists a meet-irreducible m-primary element Q of $\mathfrak{M}$ such that $Q \leq m^n M$. Also, by Lemma 4, $\mathfrak{M}/Q$ is finite dimensional. Thus, $\mathfrak{M}$ has small cofinite irreducibles. □

THEOREM 6. *Let $\mathcal{L}$ be a local Noether lattice with maximal element m and let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M. Then $\mathfrak{M}$ has small cofinite irreducibles if and only if there exists a decreasing sequence $\{Q_n\}$ of meet-irreducible m-primary elements of $\mathfrak{M}$ such that for each m-primary element Q' of $\mathfrak{M}$, there exists a positive integer n such that $Q_n \leq Q'$.*

PROOF. Suppose that m has small cofinite irreducibles. Since mM is an m-primary element of $\mathfrak{M}$, use (2) to pick Q_1 to be a meet-irreducible m-primary element of $\mathfrak{M}$ such that $Q_1 \leq mM$. For $n > 1$, recursively define Q_n as follows: choose Q_n to be a meet-irreducible m-primary element of $\mathfrak{M}$ such that $Q_n \leq Q_{n-1} \wedge m^n M$, which is possible by Lemma 1 since

$$\text{Rad} \, (Q_{n-1} \wedge m^n M) = \text{Rad} \, (Q_{n-1}) \wedge \text{Rad} \, (m^n M) = m, \tag{2}$$

and so $Q_{n-1} \wedge m^n M$ is an m-primary element of $\mathfrak{M}$. By construction, $\{Q_n\}$ is a decreasing sequence of meet-irreducible m-primary elements of $\mathfrak{M}$. Moreover, if Q' is an m-primary element of $\mathfrak{M}$, then by Lemma 3 there exists a positive integer n such that $m^n M \leq Q'$, and so $Q_n \leq Q'$.

Conversely, suppose that there exists a decreasing sequence $\{Q_n\}$ of meet-irreducible m-primary elements of $\mathfrak{M}$ such that for all m-primary elements Q' of $\mathfrak{M}$, there exists a positive integer n such that $Q_n \leq Q'$. We immediately have that (2) holds since for each positive integer n, $m^n M$ is an m-primary element of $\mathfrak{M}$. Thus, by Theorem 5, $\mathfrak{M}$ has small cofinite irreducibles, which completes the proof. □

Let $\mathcal{L}$ be a local Noether lattice with maximal element m and let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M. Following [2], $\mathcal{L}^*$ denotes the set of all formal sums $\sum_{i=1}^{\infty} a_i$ of elements of $\mathcal{L}$ such that

$$a_i = a_{i+1} \vee m^i \tag{3}$$

for all positive integers i. On $\mathcal{L}^*$, define

$$\sum_{i=1}^{\infty} a_i \leq \sum_{i=1}^{\infty} b_i \quad \text{if and only if } a_i \leq b_i \ \forall i,$$

$$\left(\sum_{i=1}^{\infty} a_i \right) \left(\sum_{i=1}^{\infty} b_i \right) = \sum_{i=1}^{\infty} (a_i b_i \vee m^i). \tag{4}$$

For an element a of $\mathcal{L}$, a^* denotes the element $\sum_{i=1}^{\infty} (a \vee m^i)$ of $\mathcal{L}^*$. Then $\mathcal{L}^*$ is a local Noether lattice with maximal element $m^* = \sum_{i=1}^{\infty} m$. It can be seen that $\mathcal{L}^*$ is a collection of representatives of equivalence classes of Cauchy sequences of $\mathcal{L}$ with the m-adic metric and in fact is the completion of $\mathcal{L}$ with this metric. Additional properties can be found in [2]. Similarly, $\mathfrak{M}^*$ denotes the set of all formal sums $\sum_{i=1}^{\infty} B_i$

of elements of $\mathfrak{M}$ such that

$$B_i = B_{i+1} \vee m^i M \tag{5}$$

for all positive integers i. On $\mathfrak{M}^*$, define

$$\sum_{i=1}^{\infty} B_i \leq \sum_{i=1}^{\infty} C_i \quad \text{if and only if } B_i \leq C_i \ \forall i,$$

$$\left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} a_i B_i \vee m^i M. \tag{6}$$

It is known [1] that $\mathfrak{M}^*$ is a Noetherian $\mathcal{L}^*$-module with greatest element $M^* = \sum_{i=1}^{\infty} M$. For an element B of $\mathfrak{M}$, B^* denotes the element $\sum_{i=1}^{\infty} B \vee m^i M$ of $\mathfrak{M}^*$. Also, if $B = \sum_{i=1}^{\infty} B_i$ is an element of $\mathfrak{M}^*$, then $C(B)$ denotes the element $\bigwedge_{i=1}^{\infty} B_i$ of $\mathfrak{M}$.

THEOREM 7. *Let $\mathcal{L}$ be a local Noether lattice with maximal element m and let $\mathfrak{M}$ be a Noetherian $\mathcal{L}$-module with greatest element M. Then the $\mathcal{L}$-module $\mathfrak{M}$ has small cofinite irreducibles if and only if the $\mathcal{L}^*$-module $\mathfrak{M}^*$ has small cofinite irreducibles.*

PROOF. For any positive integer i, $\mathfrak{M}/m^i M \simeq \mathfrak{M}^*/(m^*)^i M^*$. If for every positive integer n, $m^n M$ contains an irreducible m-primary element Q, then choose i so that $m^i M \leq Q$. Then the element of $\mathfrak{M}^*/(m^*)^i M^*$ corresponding to Q is irreducible and m^*-primary. The argument is reversible.　□

Let R be a local Noetherian ring with maximal ideal m and let M be a Noetherian R-module. Then the R-module M is said to have small cofinite irreducibles if for every positive integer n, there exists an irreducible submodule Q of M such that $Q \subseteq m^n M$ and M/Q has finite length. Let $\mathcal{L}(R)$ denote the lattice of ideals of R and let $\mathcal{L}(M)$ denote the lattice of R-submodules of M. Since the set of irreducible submodules of M is precisely the set of meet-irreducible elements of the $\mathcal{L}(R)$-submodule $\mathcal{L}(M)$, we immediately have the following theorem.

THEOREM 8. *Let R be a local Noetherian ring with maximal element m and let M be a Noetherian R-module. Then the R-module M has small cofinite irreducibles if and only if the $\mathcal{L}(R)$-module $\mathcal{L}(M)$ has small cofinite irreducibles.*

For a Noetherian module M over a local ring R with maximal ideal m, we let M^* and R^* denote the completions of M and R, respectively, in the m-adic topology.
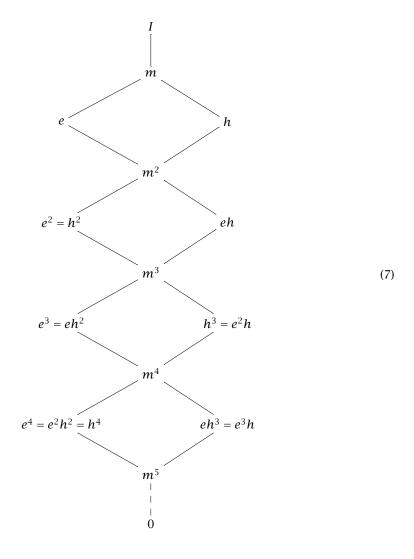
THEOREM 9. *Let R be a local Noetherian ring with maximal element m and let M be a Noetherian R-module. Then the following statements are equivalent:*
 (i) *The R-module M has small cofinite irreducibles.*
 (ii) *The R^*-module M^* has small cofinite irreducibles.*
 (iii) *The $\mathcal{L}(R)$-module $\mathcal{L}(M)$ has small cofinite irreducibles.*
 (iv) *The $\mathcal{L}(R^*)$-module $\mathcal{L}(M^*)$ has small cofinite irreducibles.*
 (v) *The $\mathcal{L}(R)^*$-module $\mathcal{L}(M)^*$ has small cofinite irreducibles.*

PROOF. The equivalence of (i) and (iii) follows from Theorem 8. So does the equivalence of (ii) and (iv). The equivalence of (iii) and (v) follows from Theorem 7. The equivalence of (iv) and (v) is established in [3].　□

In Theorem 8, we showed that the lattice of submodules of a module having small cofinite irreducibles is a lattice module having small cofinite irreducibles. We conclude this paper by giving an example of a module having small cofinite irreducibles which is not the lattice of submodules of any module.

Let $\mathcal{L}$ be the local Noether lattice with maximal element m in which the quotient m/m^2 has exactly two points, e and h. Further, assume each quotient m^n/m^{n+1} has exactly two points for each n, with $e^i h^j = e^r h^s$ if $i + j = r + s$ and j and s are both even or j and s are both odd.

$$
\begin{array}{c}
I \\
| \\
m \\
\diagdown \\
e \qquad\qquad h \\
\diagdown \\
m^2 \\
\diagup\diagdown \\
e^2 = h^2 \qquad\qquad eh \\
\diagdown \\
m^3 \\
\diagup\diagdown \\
e^3 = eh^2 \qquad\qquad h^3 = e^2 h \\
\diagdown \\
m^4 \\
\diagup\diagdown \\
e^4 = e^2 h^2 = h^4 \qquad\qquad eh^3 = e^3 h \\
\diagdown \\
m^5 \\
| \\
0
\end{array}
\qquad (7)
$$

It is clear that every power of m contains an irreducible m-primary element. If $\mathcal{L}$ is the lattice of submodules of any module, then every cyclic submodule $\neq m, I$ is contained in e or h. Then $m = e \cup h$ is a submodule, with $e \nsubseteq h$ and $h \nsubseteq e$, which is impossible.

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