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SEMICLASSICAL QUANTIZATION OF CIRCULAR BILLIARD IN HOMOGENEOUS MAGNETIC FIELD: BERRY-TABOR APPROACH

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Abstract. Semiclassical methods are accurate in general in leading order of $\hbar$, since they approximate quantum mechanics via canonical invariants. Often canonically noninvariant terms appear in the Schrödinger equation which are proportional to $\hbar^2$, therefore a discrepancy between different semiclassical trace formulas in order of $\hbar^2$ seems to be possible. We derive here the Berry-Tabor formula for a circular billiard in a homogeneous magnetic field. The formula derived for the semiclassical density of states surprisingly coincides with the results of Creagh-Littlejohn theory despite the presence of canonically noninvariant terms.

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1. Introduction. Semiclassical methods are part of an important aspect of quantum chaos, the understanding of the transition from classical to quantum mechanics. These methods can be viewed as generalizations of the Bohr-Sommerfeld quantization rules. The most famous one is Gutzwiller's trace formula for the level density of classically chaotic systems [9, 10, 11], where the old quantization rules do not apply. This formula relates the density of states to the actions, periods, and stability of classical periodic orbits. Since this method is applicable only when all the involved orbits are isolated in phase space, other methods were developed for systems where periodic orbits form continuous families. The Berry-Tabor formula [1, 2] has been developed for integrable systems while the Creagh-Littlejohn theory [4, 5] for systems with continuous symmetries.

In special cases the leading order term of the semiclassical quantization reproduces the exact quantum result [13], however, in general, quantum corrections to higher orders are of great interest in the study of the accuracy of the semiclassical approximations [6, 8, 12, 13, 14]. One reason why semiclassical methods are accurate in leading order of $\hbar$ only, is that they are approximating quantum mechanics via canonical invariants. Often, canonically noninvariant terms appear in the Schrödinger equation upon coordinate transformation which are proportional with $\hbar^2$. Consequently, a discrepancy between different semiclassical methods of order $\hbar^2$ might occur.

The goal of this paper is to derive the Berry-Tabor semiclassical trace formula for the density of states of the circular billiard in a homogeneous magnetic field and to compare the result with a similar trace formula [3] derived from the Creagh-Littlejohn theory. This system has the property that when the Schrödinger equation is transformed from Cartesian coordinates to polar coordinates, a $(\hbar^2/r)(\partial/\partial r)$ term appears

(r is the distance from the center, ϕ is the angle), which has no definite classical counterpart in the classical Hamilton function in polar coordinates. Thus this system is ideal for testing whether there is a discrepancy between different semiclassical level densities in orders of $\hbar^2$.

2. The Berry-Tabor level density of two-dimensional integrable systems. Generally, in d dimensions, a system is integrable if there are d independent constants of the motion. Usually this is the result of the fact that the Hamiltonian is separable, that is, in a suitably chosen coordinate system the Hamiltonian depends only on separate functions $\phi_i(q_i,p_i)$ of the coordinates and the conjugated momenta. This means that the dynamics can be viewed as a collection of independent one-dimensional dynamical systems. The function $\phi(q_i,p_i)$ plays the role of the Hamiltonian in each subsystem. The one-dimensional semiclassical quantization procedure can be carried out in each subsystem separately

$$I_i = \frac{1}{2\pi}\oint p_i\,dq_i = \hbar\left(n_i + \frac{\nu_i}{4}\right), \quad n_i = 0,1,2,\ldots, \tag{2.1}$$

where I_i is the action variable and ν_i is the Maslov index. The Maslov index is the sum of the Maslov indices of the turning points of the classical motion. Smooth or "soft" classical turning points (i.e., zeros of $p_i(q_i)$) contribute $+1$ to the Maslov index, while "hard" classical turning points (i.e., infinite potential walls) contribute $+2$ to the Maslov index.

The quantized energies can be recovered if we express the Hamiltonian in terms of I_i

$$\begin{aligned} E(n_1,n_2,\ldots,n_d) &= H(I_1,I_2,\ldots,I_d) \\ &= H\left(\hbar\left(n_1+\frac{\nu_1}{4}\right),\hbar\left(n_2+\frac{\nu_2}{4}\right),\ldots,\hbar\left(n_d+\frac{\nu_d}{4}\right)\right). \end{aligned} \tag{2.2}$$

The semiclassical density of states is the density of these energies,

$$d(E) = \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} \delta\left(E - E(n_1,n_2,\ldots,n_d)\right). \tag{2.3}$$

The density of states can be rewritten via the Poisson resummation technique

$$\begin{aligned} d(E) &= \int d^d I\,\delta\left(E - H(I_1,I_2,\ldots,I_d)\right)\prod_{i=1}^{d}\sum_{n_i=-\infty}^{+\infty}\delta\left(I_i - \hbar\left(n_i+\frac{\nu_i}{4}\right)\right) \\ &= \sum_{m_1,m_2,\ldots,m_d=-\infty}^{\infty}\int d^d I\,dt\,\frac{1}{2\pi\hbar^{d+1}}e^{(i/\hbar)(t(E-H(I_1,\ldots,I_d))+2\pi\sum_i m_i(I_i-\hbar\nu_i/4))}. \end{aligned} \tag{2.4}$$

Here, we used the Fourier expansion of the delta spike train. The term $m_i = 0$ ($i = 1,2,\ldots,d$) can be evaluated directly and yields the nonoscillatory average density of states. Other terms can be evaluated by the saddle point method when $\hbar \to 0$. The saddle point conditions select the periodic orbits of the system and the result of the integration is

$$d(E) = d_0(E) + \sum_p \sum_{r=1}^{+\infty} \frac{(2\pi)^{(d-1)/2}}{2^{(\chi_p-1)}\hbar^{(d+1)/2}} \frac{\cos\left(rS_p(E)/\hbar - (\pi/2)r\nu_p + (\pi/4)(d-1)\right)}{\sqrt{(rT_p)^{d-1}(-\det D_p)}}. \tag{2.5}$$

Here p is the index of the primitive periodic orbits, r is the number of repetitions, S_p is the classical action along the orbit, T_p is the time period of the orbit, ν_p is the Maslov-index. The quantity χ_p is the number of action variables of the periodic orbit whose saddle point value is zero ($I_k = 0$), since in this case the Gaussian saddle point integral is only one-sided and its contribution is half of the full Gaussian integral. The matrix D_p is related to the second derivative matrix

$$\det D = \det \begin{pmatrix} \dfrac{\partial^2 H(I_1,\dots,I_d)}{\partial I_i \partial I_j} & \dfrac{\partial H(I_1,\dots,I_d)}{\partial I_i} \\ \dfrac{\partial H(I_1,\dots,I_d)}{\partial I_j} & 0 \end{pmatrix}. \tag{2.6}$$

Equation (2.5) is the generic form of the semiclassical density of states in terms of periodic orbits, known as the Berry-Tabor formula.

In two dimensions, very often the Hamiltonian cannot be expressed with the action variables explicitly, only the implicit function

$$I_2 = g(I_1, H) \tag{2.7}$$

is available. In this case it is more useful to express the quantities in the Berry-Tabor trace formula in terms of the derivatives of g. With the simple transformations detailed in Appendix A, one can write down the period and the main determinant simply as

$$T = 2\pi m \frac{\partial g(I_1, E)}{\partial E}, \qquad \det D = \frac{(2\pi m)^3}{T^3} \frac{\partial^2 g}{\partial I_1^2}. \tag{2.8}$$

In the expressions above m is the number of cycles in the motion projected to the variable I_2 under one cycle of the orbit. The density of states in two dimensions is then

$$d(E) = d_0(E) + \sum_p \sum_{r=1}^{+\infty} \frac{\cos\left(rS_p(E)/\hbar - (\pi/2)r\nu_p + (\pi/4)(d-1)\right)}{2^{\chi_p}\pi(\hbar)^{3/2}\sqrt{-rm_p^3(\partial^2 g/\partial I_1^2)/T_p^2}}. \tag{2.9}$$

3. The circle billiard in homogeneous magnetic field. In this section, we apply the general theory outlined in Section 2 for the case of the circle billiard in a homogeneous magnetic field. In polar coordinates the Hamiltonian and the conjugated momenta are as follows:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{e^2B^2r^2}{4} - eBp_\phi\right), \quad p_r = m\dot{r}, \; p_\phi = mr^2\dot{\phi} + \frac{eBr^2}{2}. \tag{3.1}$$

The action integrals I_r and I_ϕ are generated via (2.1) as

$$I_\phi = \frac{1}{2\pi}\oint p_\phi\, d\phi = p_\phi, \tag{3.2}$$

$$\begin{aligned} I_r &= \frac{1}{2\pi}\oint p_r\, dr = \frac{1}{2\pi}\oint \left(2mE - \frac{p_\phi^2}{r^2} - \frac{e^2B^2r^2}{4} + eBp_\phi\right)^{1/2} dr \\ &= \frac{1}{2\pi}\Bigg[\sqrt{-I_\phi^2 + (2mEa^2 + 2\alpha I_\phi)z^2 - \alpha^2 z^4} \\ &\qquad - \frac{(mEa^2+\alpha I_\phi)}{\alpha}\arcsin\left(\frac{mEa^2+\alpha I_\phi - \alpha^2 z^2}{mEa^2\sqrt{1+(2\alpha I_\phi/mEa^2)}}\right) \\ &\qquad - I_\phi \arcsin\left(\frac{(mEa^2+\alpha I_\phi)z^2 - I_\phi^2}{z^2 mEa^2\sqrt{1+(2\alpha I_\phi/mEa^2)}}\right)\Bigg]_{r_2}^{r_1}, \end{aligned} \tag{3.3}$$

where we introduced the new variables $\alpha := eBa^2/2$ and $z := r/a$, where a denotes the radius of the billiard. In the first equation p_ϕ is a constant of the motion. At the second equation, to determine I_r we have to substitute into r_1 and r_2 the classical turning points. In the case of cyclotron orbits (i.e., orbits not hitting the wall) these are the zeros of the function $p_r(r)$

$$r_1 = \frac{\sqrt{2mE}a + \sqrt{2mEa^2 + 4\alpha p_\phi}}{2\alpha}, \qquad r_2 = \left|\frac{\sqrt{2mE}a - \sqrt{2mEa + 4\alpha p_\phi}}{2\alpha}\right|. \tag{3.4}$$

The integration in (3.3) within these boundaries gives

$$I_r = \frac{mEa^2}{2\alpha}. \tag{3.5}$$

For bouncing orbits the upper limit in (3.3) should be replaced by the wall of the billiard, resulting in

$$\begin{aligned} I_r = \frac{1}{2\pi}\Bigg[&\sqrt{-I_\phi^2 + 2mEa^2 + 2\alpha I_\phi - \alpha^2} \\ &- \frac{(mEa^2+\alpha I_\phi)}{\alpha}\arcsin\left(\frac{mEa^2+\alpha I_\phi-\alpha^2}{mEa^2\sqrt{1+(2\alpha I_\phi/mEa^2)}}\right) \\ &- I_\phi\arcsin\left(\frac{mEa^2+\alpha I_\phi - I_\phi^2}{mEa^2\sqrt{1+(2\alpha I_\phi/mEa^2)}}\right) + \frac{(mEa^2+\alpha I_\phi)}{\alpha}\frac{\pi}{2} - I_\phi\frac{\pi}{2}\Bigg]. \end{aligned} \tag{3.6}$$

We see that the right-hand side of (3.6) is the function $g(E, I_\phi)$ which connects the energy and I_ϕ to I_r. According to (2.8), (2.9) we need the

$$\frac{\partial^2 g}{\partial {I_\phi}^2} = \frac{1}{\pi}\frac{mEa^2 + I_\phi\alpha - \alpha^2}{(mEa^2 + 2I_\phi\alpha)\sqrt{-{I_\phi}^2 + 2mEa^2 + 2I_\phi\alpha - \alpha^2}}, \tag{3.7}$$

derivative of g for the semiclassical density of states.

The next step is the classification and examination of the periodic orbits. This is discussed in detail in [3], so here we only summarize their results. Every primitive bouncing periodic orbit can be indexed in the following way: $(q_p, w_p)^{\pm}$, where q_p denotes the number of corners (vertices), and w_p is the winding number, that is, it counts how often the orbit winds around the center. (Of course, $q_p \geq 2$ and $w_p \geq 1$.) The additional upper index is "+" if in the weak field limit, the orbit segments are bent toward the center of the billiard, and "−" if they are bent outward (see Figures B.1 and B.2). From now on in every formula the upper signs are for the "+" orbits, the lower signs for the "−" orbits. From simple geometry (outlined in Appendix B) the length L, the time T, the action S, and the action integral I_ϕ of the orbits $(q, w)^{\pm}$ are given by

$$\phi_p := \frac{\pi w_p}{q_p}, \qquad \psi_p := \arcsin\left(\frac{a}{R_c}\sin(\phi_p)\right), \tag{3.8}$$

$$L_p = q_p R_c 2\psi_p, \tag{3.9}$$

$$T_p = \frac{L_p}{v} = \frac{L_p}{2\sqrt{2mE}} = \frac{q_p R_c \psi_p}{\sqrt{2mE}}, \tag{3.10}$$

$$I_{\phi,p} = \pm\sqrt{2mE}\,a\cos(\phi_p \pm \psi_p) + \alpha, \tag{3.11}$$

$$S_p = \begin{cases} \sqrt{2mE}L_p - q_p B\left[\dfrac{a^2}{2}\sin(2\phi_p) + R_c^2\psi_p - \dfrac{R_c^2}{2}\sin(2\psi_p)\right] & (-,\ R_c > a), \\ \sqrt{2mE}L_p - q_p B\left[R_c^2(\pi - \psi_p) + \dfrac{R_c^2}{2}\sin(2\psi_p) - \dfrac{a^2}{2}\sin(2\phi_p)\right] & (-,\ R_c < a), \\ \sqrt{2mE}L_p + q_p B\left[\dfrac{a^2}{2}\sin(2\phi_p) - \left(R_c^2\psi_p - \dfrac{R_c^2}{2}\sin(2\psi_p)\right)\right] & (+), \end{cases} \tag{3.12}$$

where we introduced $R_c = \sqrt{2mE}/eB$, the cyclotron radius.

Now we are in the position to construct the semiclassical density of states for the $R_c \geq a$ regime, where only bouncing orbits exist. First, we substitute (3.11) into (3.7) resulting

$$\frac{\partial^2 g}{\partial I_\phi^2} = \frac{1}{\pi\sqrt{2mE}} F_{BT}(a, R_c, \phi_p, \psi_p), \tag{3.13}$$

where

$$F_{BT}(a, R_c, \phi_p, \psi_p) = \frac{1 \pm (a/R_c)\cos(\phi_p \pm \psi_p)}{a\sin(\phi_p + \psi_p)\left(1 \pm 2(a/R_c)\cos(\phi_p \pm \psi_p) + (a^2/R_c^2)\right)}. \tag{3.14}$$

We introduce the new variable $k = \sqrt{2mE}/\hbar$. For the sake of simplicity we use the units $\hbar = 1$, $m = 1/2$, and $e = 1$. Substituting (3.10), (3.12), and (3.14) into (2.9) gives

$$d(E) = d_0(E) + \sum_{p,r} A_{p,r}\cos\left(rS_p + \frac{3rq_p\pi}{2} + \frac{\pi}{4}\right), \tag{3.15}$$

where

$$A_{p,r} = \frac{R_c\psi_p}{2^{\chi_p}\sqrt{k\pi r q_p F_{BT}(a, R_c, \phi_p, \psi_p)}}. \tag{3.16}$$

To complete the derivation of the semiclassical density of states we now treat the case when $R_c \leq a$. Here an additional term appears due to the cyclotron orbits. This contribution cannot be obtained using (2.9) since the function g in this case does not depend on I_ϕ. We need to go back to the general form (2.4) of the density of states and carry out the integral with respect to I_ϕ directly instead of using the saddle point method. This is easy, since the integrand does not depend on the integration variable, so the result is the measure of the interval of allowed $I_{\phi^{-s}}$. The I_r and t integrals can be evaluated with the saddle point method, as usual. The detailed calculation of these integrals and the action are given in Appendix C. The cyclotron orbit contribution to the density of states is

$$d_{\text{cyc}}(E) = \frac{1}{2}(a-R_c)^2 \sum_{r=1}^{\infty} \cos(rk\pi R_c - r\pi). \tag{3.17}$$

4. Comparison with exact quantum mechanics. Our formulas (3.15), (3.16) can now be compared to the exact result. The exact eigenenergies are given by the zeros of the confluent hypergeometrical functions [7]. We developed an alternative way to determine the levels by writing the radial Schrödinger equation as an ordinary differential equation and solving it using a simple shooting method. We regularized the PO sum in (3.15) with Gaussian smoothing with a γ broadening factor as discussed in [3]. Figure 4.1 shows the quantum mechanical results and the semiclassical density of states using $\gamma = 0.25$ broadening factor at various magnetic field parameters.

There is a spike in the level density at each eigenenergy. This means that the semiclassical density of states obtained from the Berry-Tabor formula is in good agreement with the quantum mechanical energy levels in the whole parameter range. Note that at level crossings the spike is about twice as high as usual due to the double degeneracy of the levels.

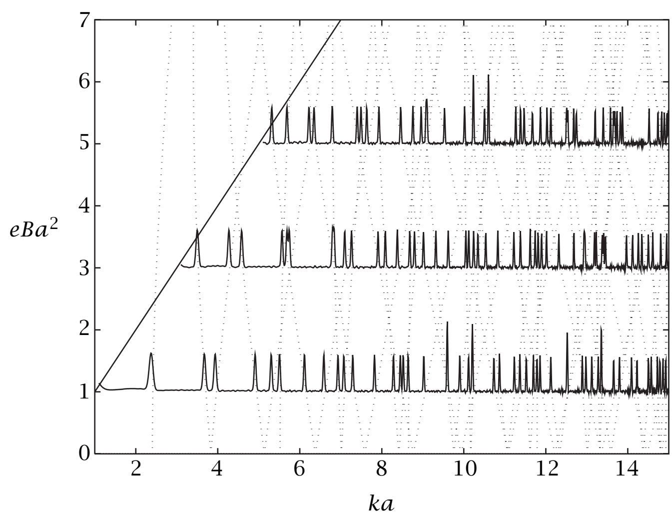

FIGURE 4.1. The semiclassical level density and the quantum mechanical eigenstates in the $R_c > a$ regime.

5. Comparison with Creagh-Littlejohn theory. Recently Blaschke and Brack [3] have derived a similar periodic orbit formula for the semiclassical density of states based on Creagh-Littlejohn theory of continuous symmetries. If we compare our formula with the Blaschke-Brack formula term by term, we see that the actions and Maslov indices of periodic orbits and the cyclotron orbit contributions are the same in both cases, while the amplitudes of orbits hitting the wall seem to be different in general.

The trace formula published in [3] and our trace formula (3.15), (3.16) differ only in function $F_{BT}(a,R_c,\phi,\psi)$. In their formula F_{BT} is replaced by F_{CL}

$$F_{CL}(a,R_c,\phi,\psi) := \frac{R_c\cos(\psi)}{a\sin(\phi+\psi)\left(R_c\cos(\psi)\pm a\cos(\phi)\right)}. \tag{5.1}$$

Both F_{BT} and F_{CL} contain a factor $1/a\sin(\phi+\psi)$. We now concentrate on how are the rest of the expressions related to each other. First, we define the functions f_{BT} and f_{CL} as

$$\begin{aligned} f_{BT}(a,R_c,\phi,\psi) &:= \frac{1\pm(a/R_c)\cos(\phi\pm\psi)}{1\pm 2(a/R_c)\cos(\phi\pm\psi)+a^2/R_c^2}, \\ f_{CL}(a,R_c,\phi,\psi) &:= \frac{R_c\cos(\psi)}{R_c\cos(\psi)\pm a\cos(\phi)} = \frac{\cos(\psi)}{\cos(\psi)\pm(a/R_c)\cos(\phi)}, \end{aligned} \tag{5.2}$$

(i.e., without the common factor in function F_{BT} and F_{CL}). Using (3.8) which connects the two angles ϕ and ψ the function f_{BT} can be rewritten as

$$\begin{aligned} f_{BT}(a,R_c,\phi,\psi) &= \frac{1\pm(a/R_c)\cos(\phi)\cos(\psi)\mp(a/R_c)\sin(\phi)\sin(\psi)}{1\pm 2(a/R_c)\cos(\phi)\cos(\psi)\mp 2\sin(\phi)\sin(\psi)+a^2/R_c^2} \\ &= \frac{1\pm(a/R_c)\cos(\phi)\cos(\psi)\mp\sin^2(\psi)}{1\pm 2(a/R_c)\cos(\phi)\cos(\psi)\mp 2(a^2/R_c^2)\sin^2(\phi)+a^2/R_c^2}, \end{aligned} \tag{5.3}$$

and using the simple trigonometrical identity $\sin^2(\psi) = 1-\cos^2(\psi)$, we find

$$\begin{aligned} f_{BT}(a,R_c,\phi,\psi) &= \frac{\pm(a/R_c)\cos(\phi)\cos(\psi)+\cos^2(\psi)}{1\pm 2(a/R_c)\cos(\phi)\cos(\psi)\mp(a^2/R_c^2)\sin^2(\phi)+(a^2/R_c^2)\cos^2(\phi)} \\ &= \frac{\cos(\psi)\left[\cos(\psi)\pm(a/R_c)\cos(\phi)\right]}{\cos^2(\psi)\pm 2(a/R_c)\cos(\phi)\cos(\psi)+(a^2/R_c^2)\cos^2(\phi)}. \end{aligned} \tag{5.4}$$

We can further simplify this by dividing out the common factor $\cos(\psi)\pm(a/R_c)\cos(\phi)$

$$f_{BT}(a,R_c,\phi,\psi) = \frac{\cos(\psi)}{\cos(\psi)\pm(a/R_c)\cos(\phi)} = f_{CL}(a,R_c,\phi,\psi), \tag{5.5}$$

so the two formulas derived via different semiclassical theories coincide. Consequently, the $\hbar$ expansion of the formulas also coincide, including the canonically noninvariant part.

6. Summary. We derived a semiclassical formula for the level density of a circular billiard in a homogeneous magnetic field using the Berry-Tabor formula. Unexpectedly the result in this case is the same as the semiclassical density of states derived from

Creagh-Littlejohn theory of continuous symmetries for the same system, despite of the presence of canonically noninvariant terms. This result is promising from the point of view of semiclassical theory since it indicates that in certain practically interesting cases different approaches can yield results of comparable precision.

Appendices

A. The main determinant in the Berry-Tabor formula. In this section, we express the quantities in the Berry-Tabor trace formula in terms of the derivatives of g. Taking the partial derivative of (2.7) with respect to I_1 yields

$$0 = \frac{\partial g(I_1,H)}{\partial I_1} + \frac{\partial g(I_1,H)}{\partial H}\frac{\partial H(I_1,I_2)}{\partial I_1}, \tag{A.1}$$

while the partial derivative of (2.7) with respect to I_2 gives

$$1 = \frac{\partial g(I_1,H)}{\partial H}\frac{\partial H(I_1,I_2)}{\partial I_2}. \tag{A.2}$$

The frequencies can be expressed from these equations

$$\omega_1 = \frac{\partial H(I_1,I_2)}{\partial I_1} = -\frac{\partial g(I_1,H)/\partial I_1}{\partial g(I_1,H)/\partial H}, \qquad \omega_2 = \frac{\partial H(I_1,I_2)}{\partial I_2} = \frac{1}{\partial g(I_1,H)/\partial H}. \tag{A.3}$$

Periodic orbits are recovered from $\omega_1 = 2\pi n/T$ and $\omega_2 = 2\pi m/T$. The action I_1 for a periodic orbit at the energy E can be obtained by solving the following equation:

$$\frac{\omega_1}{\omega_2} = \frac{n}{m} = \frac{n_p}{m_p} - \frac{\partial g(I_1,E)}{\partial I_1}, \tag{A.4}$$

where $m = rm_p$ and $n = rn_p$ corresponding to the primitive orbit. Then the period can be expressed simply as

$$T = 2\pi m\frac{\partial g(I_1,E)}{\partial E}. \tag{A.5}$$

The main determinant to be calculated is

$$\begin{aligned}\det D &= \begin{vmatrix} \frac{\partial^2 H(I_1,I_2)}{\partial I_1^2} & \frac{\partial^2 H(I_1,I_2)}{\partial I_1\partial I_2} & \frac{\partial H(I_1,I_2)}{\partial I_1} \\ \frac{\partial^2 H(I_1,I_2)}{\partial I_1\partial I_2} & \frac{\partial^2 H(I_1,I_2)}{\partial I_2^2} & \frac{\partial H(I_1,I_2)}{\partial I_2} \\ \frac{\partial H(I_1,I_2)}{\partial I_1} & \frac{\partial H(I_1,I_2)}{\partial I_2} & 0 \end{vmatrix} \\ &= \left(-\frac{\partial^2 H}{\partial I_1^2}\left(\frac{\partial H}{\partial I_2}\right)^2 + 2\frac{\partial^2 H}{\partial I_1\partial I_2}\frac{\partial H}{\partial I_1}\frac{\partial H}{\partial I_2} - \frac{\partial^2 H}{\partial I_2^2}\left(\frac{\partial H}{\partial I_1}\right)^2\right).\end{aligned} \tag{A.6}$$

Now, the second derivatives of H can be expressed with the second derivatives of g by taking further partial derivatives of (A.1) and (A.2) with respect to I_1 and I_2. Then

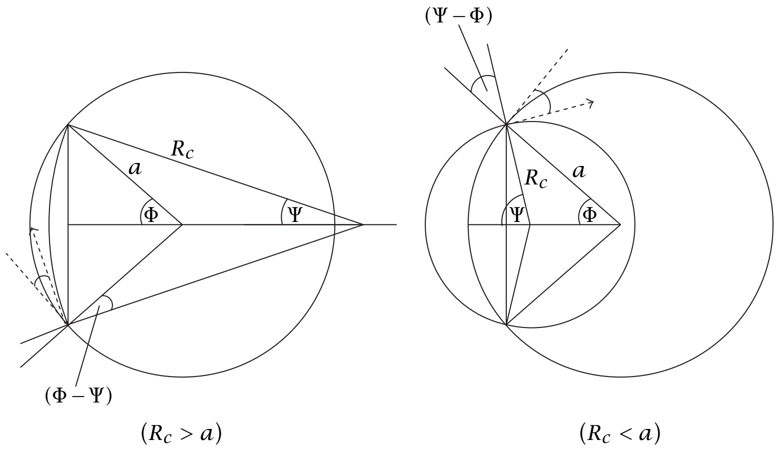

FIGURE B.1. The geometry of the "−" orbits.

we can express the second derivatives as

$$\frac{\partial^2 H}{\partial I_1^2} = \frac{1}{(\partial g/\partial H)^3}\left(2\frac{\partial^2 g}{\partial H\partial I_1}\frac{\partial g}{\partial I_1}\frac{\partial g}{\partial H} - \frac{\partial^2 g}{\partial H^2}\left(\frac{\partial g}{\partial I_1}\right)^2 - \frac{\partial^2 g}{\partial I_1^2}\left(\frac{\partial g}{\partial H}\right)^2\right),$$
$$\frac{\partial^2 H}{\partial I_1\partial I_2} = \frac{1}{(\partial g/\partial H)^3}\left(\frac{\partial^2 g}{\partial H^2}\frac{\partial g}{\partial I_1} - \frac{\partial^2 g}{\partial I_1\partial H}\frac{\partial g}{\partial H}\right), \tag{A.7}$$
$$\frac{\partial^2 H}{\partial I_2^2} = -\frac{1}{(\partial g/\partial H)^3}\frac{\partial^2 g}{\partial H^2}.$$

Using these expressions, the determinant becomes

$$\det D = \frac{1}{(\partial g/\partial H)^3}\frac{\partial^2 g}{\partial I_1^2} = \frac{(2\pi m)^3}{T^3}\frac{\partial^2 g}{\partial I_1^2}. \tag{A.8}$$

B. Geometry of the bouncing orbits. We examine an orbit which winds w times around the center and touches the wall q times. There are two angles which describe the arcs building up the orbit: ϕ and ψ

$$\phi = \frac{\pi w}{q}, \qquad \psi = \arcsin\left(\frac{a}{R_c}\sin(\phi)\right). \tag{B.1}$$

Since $I_\phi = p_\phi =$ constant, Figures B.1 and B.2 show that (3.1) can be written as

$$I_\phi = \pm\sqrt{2mE}\,a\cos(\phi\pm\psi) + \alpha. \tag{B.2}$$

The upper signs are for the "+" orbits, the lower signs are for the "−" orbits.

The length of the orbit is simply the length of the arcs multiplied by q and the time period of the orbit is the length divided by v, the velocity, as shown in (3.9) and (3.10). In nonzero magnetic fields the momentum of the free particle is replaced by $\underline{p} - e\underline{A}$, so the action integral becomes

$$S = \int(\underline{p} - e\underline{A})\,d\underline{q} = \int \underline{p}\,d\underline{q} - e\iint \underline{B}\,d\underline{F}, \tag{B.3}$$

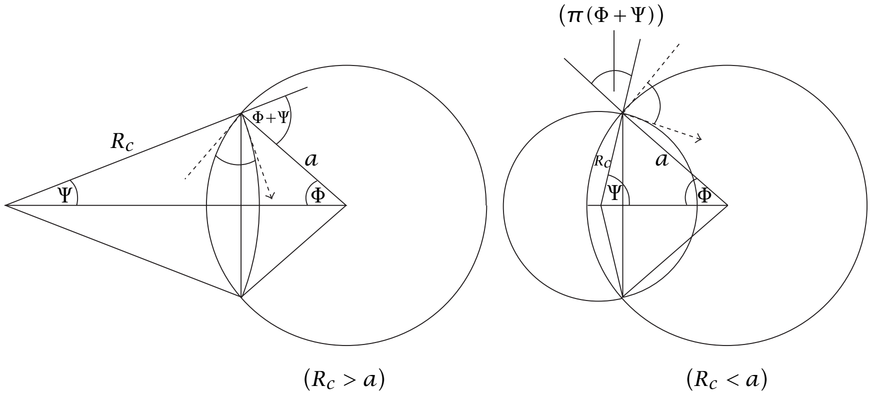

FIGURE B.2. The geometry of the “+” orbits.

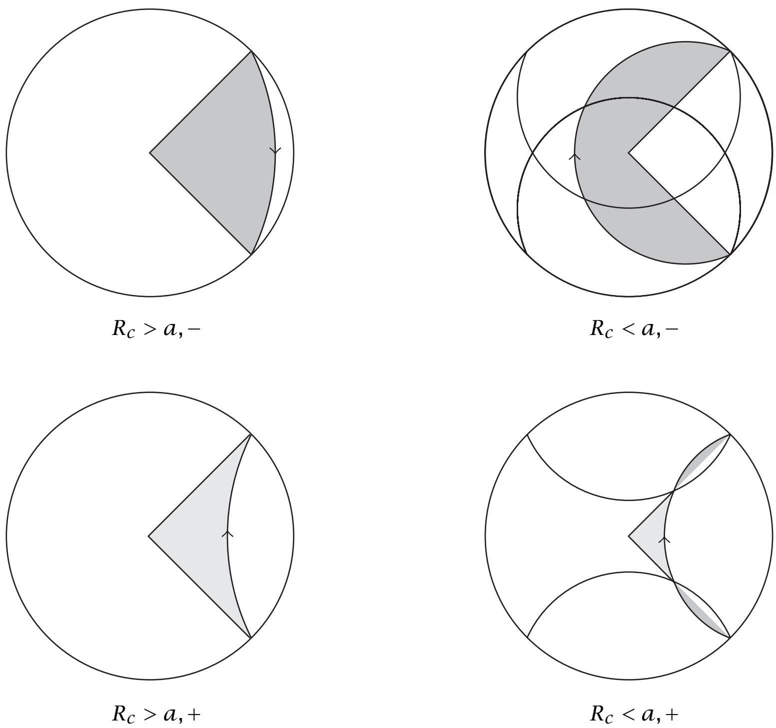

FIGURE B.3. The enclosed areas between two bounces.

where we transformed the second term to a surface-integral which is calculated by integrating the magnetic field on the enclosed area. The sign of this term depends on whether the orbit encloses clockwise or anticlockwise. With the help of Figure B.3 the action turns out to be (3.12).

C. The cyclotron orbits. In case of cyclotron orbits, the integration with respect to I_ϕ in (2.4) simply multiplies the rest of the expression by the measure of the interval

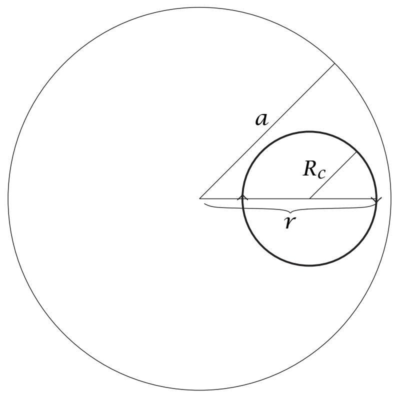

FIGURE C.1. The cyclotron orbit.

of possible I_ϕ values. According to (3.1) and (3.2)

$$I_\phi = p_\phi = mr^2\dot{\phi} + \frac{eBr^2}{2}, \tag{C.1}$$

which is constant throughout the motion since the Hamiltonian does not depend on ϕ. This constant value in our units is

$$I_\phi = -kr + \frac{Br^2}{2} = \frac{B}{2}\left(r - \frac{k}{B}\right)^2 - \frac{k^2}{2B}, \tag{C.2}$$

where r denotes the maximum distance between the center of the billiard and the electron throughout the motion (see Figure C.1). As a function of r this is a parabola with a minimum value of $-k^2/2B$ at $r = k/B$, whereas the possible maximum value of I_ϕ is (C.2) evaluated at $r = a$. This means that the integral with respect to I_ϕ is equivalent to the following multiplying factor in (2.4):

$$\frac{Ba^2}{2} - ka + \frac{k^2}{2B} = \frac{1}{2}\left(Ba^2 - 2ka + \frac{k^2}{B}\right). \tag{C.3}$$

The I_r and t integrals can be evaluated with the saddle point method just as in case of a one-dimensional system. The determinant of the second derivative matrix is

$$\det \mathbf{D} = \det \begin{pmatrix} -\dfrac{\partial^2 H}{\partial I_r^2} T & -\dfrac{\partial H}{\partial I_r} \\ -\dfrac{\partial H}{\partial I_r} & 0 \end{pmatrix} = -\left(\frac{\partial H}{\partial I_r}\right)^2. \tag{C.4}$$

According to (A.2), we find

$$\frac{\partial H}{\partial I_r} = \frac{1}{\partial g/\partial E} = \frac{2\pi m}{T} \longrightarrow \det \mathbf{D} = -\frac{4\pi^2 m^2}{T^2} = -\frac{4k^2}{R_c^2}, \tag{C.5}$$

where the time period of the cyclotron orbit is $T = \pi R_c m/k$. Thus the total amplitude standing in front of the oscillating factors in (2.4) (with $\chi_p = 0$) is

$$\frac{R_c}{2k}\left(Ba^2 - 2ka + \frac{k^2}{B}\right) = \frac{1}{2}(a^2 - 2aR_c + R_c^2) = \frac{a^2}{2}\left(1 - \frac{R_c}{a}\right)^2. \tag{C.6}$$

Finally, with the action being $S = k\pi R_c$ and with a Maslov-index $\nu = 2$ the expression (2.4) takes the following form:

$$d_{\text{cyc}}(E) = \frac{a^2}{2}\left(1-\frac{R_c}{a}\right)^2 \sum_{r=1}^{\infty} \cos\left(rk\pi R_c - r\pi\right). \tag{C.7}$$

Note that since formally in the argument of the cosine in (2.4) $d = 1$ since we used the saddle point method only in one action variable.

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