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ON IMAGINABLE T-FUZZY SUBALGEBRAS AND IMAGINABLE T-FUZZY CLOSED IDEALS IN BCH-ALGEBRAS

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ABSTRACT. We inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a t-norm T, we introduce the notion of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable T-fuzzy subalgebra and an imaginable T-fuzzy closed ideal. We discuss the direct product and T-product of T-fuzzy subalgebras. We show that the family of T-fuzzy closed ideals is a completely distributive lattice.

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1. Introduction. In 1983, Hu et al. introduced the notion of a BCH-algebra which is a generalization of a BCK/BCI-algebra (see [6, 7]). In [4], Chaudhry et al. stated ideals and filters in BCH-algebras, and studied their properties. For further properties on BCH-algebras, we refer to [2, 3, 5]. In [8], the first author considered the fuzzification of ideals and filters in BCH-algebras, and then described the relation among fuzzy subalgebras, fuzzy closed ideals and fuzzy filters in BCH-algebras. In this paper, we inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a t-norm T, we introduce the notion of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable T-fuzzy subalgebra and an imaginable T-fuzzy closed ideal. We discuss the direct product and T-product of T-fuzzy subalgebras. We show that the family of T-fuzzy closed ideals is a completely distributive lattice.

2. Preliminaries. By a *BCH-algebra* we mean an algebra $(X,*,0)$ of type (2,0) satisfying the following axioms:

(H1) $x*x=0$,

(H2) $x*y=0$ and $y*x=0$ imply $x=y$,

(H3) $(x*y)*z=(x*z)*y$,

for all $x,y,z\in X$.

In a BCH-algebra X, the following statements hold:

(P1) $x*0=x$.

(P2) $x * 0 = 0$ implies $x = 0$.
(P3) $0 * (x * y) = (0 * x) * (0 * y)$.

A nonempty subset A of a BCH-algebra X is called a *subalgebra* of X if $x * y \in A$ whenever $x, y \in A$. A nonempty subset A of a BCH-algebra X is called a *closed ideal* of X if

(i) $0 * x \in A$ for all $x \in A$,
(ii) $x * y \in A$ and $y \in A$ imply that $x \in A$.

In what follows, let X denote a BCH-algebra unless otherwise specified. A *fuzzy set* in X is a function $\mu : X \to [0,1]$. Let μ be a fuzzy set in X. For $\alpha \in [0,1]$, the set $U(\mu;\alpha) = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a *level set* of μ.

A fuzzy set μ in X is called a *fuzzy subalgebra* of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in X. \tag{2.1}$$

Definition 2.1 (see [1]). By a t*-norm* T on $[0,1]$, we mean a function $\mathrm{T} : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T1) $T(x,1) = x$,
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$,
(T3) $T(x,y) = T(y,x)$,
(T4) $T(x,T(y,z)) = T(T(x,y),z)$, for all $x,y,z \in [0,1]$.

In what follows, let T denote a t-norm on $[0,1]$ unless otherwise specified. Denote by Δ_T the set of elements $\alpha \in [0,1]$ such that $T(\alpha,\alpha) = \alpha$, that is,

$$\Delta_T := \{\alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha\}. \tag{2.2}$$

Note that every t-norm T has a useful property:

(P4) $T(\alpha,\beta) \leq \min(\alpha,\beta)$ for all $\alpha,\beta \in [0,1]$.

3. Fuzzy closed ideals

Definition 3.1 (see [8]). A fuzzy set μ in X is called a *fuzzy closed ideal* of X if

(F1) $\mu(0 * x) \geq \mu(x)$ for all $x \in X$,
(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Theorem 3.2. *Let D be a subset of X and let μ_D be a fuzzy set in X defined by*

$$\mu_D(x) = \begin{cases} \alpha_1 & \text{if } x \in D, \\ \alpha_2 & \text{if } x \notin D, \end{cases} \tag{3.1}$$

for all $x \in X$ and $\alpha_1 > \alpha_2$. Then μ_D is a fuzzy closed ideal of X if and only if D is a closed ideal of X.

Proof. Assume that μ_D is a fuzzy closed ideal of X. Let $x \in D$. Then, by (F1), we have $\mu(0 * x) \geq \mu(x) = \alpha_1$ and so $\mu(0 * x) = \alpha_1$. It follows that $0 * x \in D$. Let $x, y \in X$ be such that $x * y \in D$ and $y \in D$. Then $\mu_D(x * y) = \alpha_1 = \mu_D(y)$, and hence

$$\mu_D(x) \geq \min\{\mu_D(x * y), \mu_D(y)\} = \alpha_1. \tag{3.2}$$

Thus $\mu_D(x) = \alpha_1$, that is, $x \in D$. Therefore D is a closed ideal of X.

Conversely, suppose that D is a closed ideal of X. Let $x \in X$. If $x \in D$, then $0*x \in D$ and thus $\mu_D(0*x) = \alpha_1 = \mu_D(x)$. If $x \notin D$, then $\mu_D(x) = \alpha_2 \le \mu_D(0*x)$. Let $x, y \in X$. If $x*y \in D$ and $y \in D$, then $x \in D$. Hence

$$\mu_D(x) = \alpha_1 = \min\{\mu_D(x*y), \mu_D(y)\}. \tag{3.3}$$

If $x*y \notin D$ and $y \notin D$, then clearly $\mu_D(x) \ge \min\{\mu_D(x*y), \mu_D(y)\}$. If exactly one of $x*y$ and y belong to D, then exactly one of $\mu_D(x*y)$ and $\mu_D(y)$ is equal to α_2. Therefore, $\mu_D(x) \ge \alpha_2 = \min\{\mu_D(x*y), \mu_D(y)\}$. Consequently, μ_D is a fuzzy closed ideal of X. □

Using the notion of level sets, we give a characterization of a fuzzy closed ideal.

Theorem 3.3. *A fuzzy set μ in X is a fuzzy closed ideal of X if and only if the nonempty level set $U(\mu;\alpha)$ of μ is a closed ideal of X for all $\alpha \in [0,1]$.*

We then call $U(\mu;\alpha)$ a *level closed ideal* of μ.

Proof. Assume that μ is a fuzzy closed ideal of X and $U(\mu;\alpha) \ne \emptyset$ for all $\alpha \in [0,1]$. Let $x \in U(\mu;\alpha)$. Then $\mu(0*x) \ge \mu(x) \ge \alpha$, and so $0*x \in U(\mu;\alpha)$. Let $x, y \in X$ be such that $x*y \in U(\mu;\alpha)$ and $y \in U(\mu;\alpha)$. Then

$$\mu(x) \ge \min\{\mu(x*y), \mu(y)\} \ge \min\{\alpha, \alpha\} = \alpha, \tag{3.4}$$

and thus $x \in U(\mu;\alpha)$. Therefore $U(\mu;\alpha)$ is a closed ideal of X. Conversely, suppose that $U(\mu;\alpha) \ne \emptyset$ is a closed ideal of X. If $\mu(0*a) < \mu(a)$ for some $a \in X$, then $\mu(0*a) < \alpha_0 < \mu(a)$ by taking $\alpha_0 := 1/2(\mu(0*a)+\mu(a))$. It follows that $a \in U(\mu;\alpha_0)$ and $0*a \notin U(\mu;\alpha_0)$, which is a contradiction. Hence $\mu(0*x) \ge \mu(x)$ for all $x \in X$. Assume that there exist $x_0, y_0 \in X$ such that

$$\mu(x_0) < \min\{\mu(x_0*y_0), \mu(y_0)\}. \tag{3.5}$$

Taking $\beta_0 := 1/2(\mu(x_0)+\min\{\mu(x_0*y_0), \mu(y_0)\})$, we get $\mu(x_0) < \beta_0 < \mu(x_0*y_0)$ and $\mu(x_0) < \beta_0 < \mu(y_0)$. Thus $x_0*y_0 \in U(\mu;\beta_0)$ and $y_0 \in U(\mu;\beta_0)$, but $x_0 \notin U(\mu;\beta_0)$. This is impossible. Hence μ is a fuzzy closed ideal of X. □

Theorem 3.4. *Let μ be a fuzzy set in X and $\operatorname{Im}(\mu) = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$, where $\alpha_i < \alpha_j$ whenever $i > j$. Let $\{D_k \mid k = 0,1,2,\dots,n\}$ be a family of closed ideals of X such that*

(i) $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = X$,

(ii) $\mu(D_k^*) = \alpha_k$, *where* $D_k^* = D_k \setminus D_{k-1}$ *and* $D_{-1} = \emptyset$ *for* $k = 0,1,\dots,n$.

Then μ is a fuzzy closed ideal of X.

Proof. For any $x \in X$ there exists $k \in \{0,1,\dots,n\}$ such that $x \in D_k^*$. Since D_k is a closed ideal of X, it follows that $0*x \in D_k$. Thus $\mu(0*x) \ge \alpha_k = \mu(x)$. To prove that μ satisfies condition (F2), we discuss the following cases: if $x*y \in D_k^*$ and $y \in D_k^*$, then $x \in D_k$ because D_k is a closed ideal of X. Hence

$$\mu(x) \ge \alpha_k = \min\{\mu(x*y), \mu(y)\}. \tag{3.6}$$

If $x*y\notin D_k^*$ and $y\notin D_k^*$, then the following four cases arise:

(i) $x*y\in X\setminus D_k$ and $y\in X\setminus D_k$,

(ii) $x*y\in D_{k-1}$ and $y\in D_{k-1}$,

(iii) $x*y\in X\setminus D_k$ and $y\in D_{k-1}$,

(iv) $x*y\in D_{k-1}$ and $y\in X\setminus D_k$.

But, in either case, we know that $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}$. If $x*y\in D_k^*$ and $y\notin D_k^*$, then either $y\in D_{k-1}$ or $y\in X\setminus D_k$. It follows that either $x\in D_k$ or $x\in X\setminus D_k$. Thus $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}$. Similarly for the case $x*y\notin D_k^*$ and $y\in D_k^*$, we have the same result. This completes the proof. □

Theorem 3.5. *Let Λ be a subset of $[0,1]$ and let $\{D_\lambda\mid\lambda\in\Lambda\}$ be a collection of closed ideals of X such that*

(i) $X=\cup_{\lambda\in\Lambda}D_\lambda$,

(ii) *$\alpha>\beta$ if and only if $D_\alpha\subsetneq D_\beta$ for all $\alpha,\beta\in\Lambda$.*

Define a fuzzy set μ in X by $\mu(x)=\sup\{\lambda\in\Lambda\mid x\in D_\lambda\}$ for all $x\in X$. Then μ is a fuzzy closed ideal of X.

Proof. Let $x\in X$. Then there exists $\alpha_i\in\Lambda$ such that $x\in D_{\alpha_i}$. It follows that $0*x\in D_{\alpha_j}$ for some $\alpha_j\geq\alpha_i$. Hence

$$\mu(x)=\sup\{\alpha_k\in\Lambda\mid\alpha_k\leq\alpha_i\}\leq\sup\{\alpha_k\in\Lambda\mid\alpha_k\leq\alpha_j\}=\mu(0*x).\tag{3.7}$$

Let $x,y\in X$ be such that $\mu(x*y)=m$ and $\mu(y)=n$, where $m,n\in[0,1]$. Without loss of generality we may assume that $m\leq n$. To prove μ satisfies condition (F2), we consider the following three cases:

$$(1^\circ)\ \lambda\leq m,\qquad(2^\circ)\ m<\lambda\leq n,\qquad(3^\circ)\ \lambda>n.\tag{3.8}$$

Case (1°) implies that $x*y\in D_\lambda$ and $y\in D_\lambda$. It follows that $x\in D_\lambda$ so that

$$\mu(x)=\sup\{\lambda\in\Lambda\mid x\in D_\lambda\}\geq m=\min\{\mu(x*y),\mu(y)\}.\tag{3.9}$$

For the case (2°), we have $x*y\notin D_\lambda$ and $y\in D_\lambda$. Then either $x\in D_\lambda$ or $x\notin D_\lambda$. If $x\in D_\lambda$, then $\mu(x)=n\geq\min\{\mu(x*y),\mu(y)\}$. If $x\notin D_\lambda$, then $x\in D_\delta-D_\lambda$ for some $\delta<\lambda$, and so $\mu(x)>m=\min\{\mu(x*y),\mu(y)\}$. Finally, case (3°) implies $x*y\notin D_\lambda$ and $y\notin D_\lambda$. Thus we have that either $x\in D_\lambda$ or $x\notin D_\lambda$. If $x\in D_\lambda$ then obviously $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}$. If $x\notin D_\lambda$ then $x\in D_\epsilon-D_\lambda$ for some $\epsilon<\lambda$, and thus $\mu(x)\geq m=\min\{\mu(x*y),\mu(y)\}$. This completes the proof. □

Let D be a subset of X. The least closed ideal of X containing D is called the closed ideal *generated* by D, denoted by $\langle D\rangle$. Note that if C and D are subsets of X and $C\subseteq D$, then $\langle C\rangle\subseteq\langle D\rangle$. Let μ be a fuzzy set in X. The least fuzzy closed ideal of X containing μ is called a fuzzy closed ideal of X *generated* by μ, denoted by $\langle\mu\rangle$.

Lemma 3.6. *For a fuzzy set μ in X, then*

$$\mu(x)=\sup\{\alpha\in[0,1]\mid x\in U(\mu;\alpha)\},\quad\forall x\in X.\tag{3.10}$$

Proof. Let $\delta:=\sup\{\alpha\in[0,1]\mid x\in U(\mu;\alpha)\}$ and let $\varepsilon>0$ be given. Then $\delta-\varepsilon<\alpha$ for some $\alpha\in[0,1]$ such that $x\in U(\mu;\alpha)$, and so $\delta-\varepsilon<\mu(x)$. Since ε is arbitrary, it follows that $\mu(x) \geq \delta$. Now let $\mu(x) = \beta$. Then $x \in U(\mu;\beta)$ and hence $\beta \in \{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}$. Therefore

$$\mu(x) = \beta \leq \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} = \delta, \tag{3.11}$$

and consequently $\mu(x) = \delta$, as desired. □

Theorem 3.7. *Let μ be a fuzzy set in X. Then the fuzzy set μ^* in X defined by*

$$\mu^*(x) = \sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\} \tag{3.12}$$

for all $x \in X$ is the fuzzy closed ideal $\langle \mu \rangle$ generated by μ.

Proof. We first show that μ^* is a fuzzy closed ideal of X. For any $\gamma \in \mathrm{Im}(\mu^*)$, let $\gamma_n = \gamma - 1/n$ for any $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers, and let $x \in U(\mu^*;\gamma)$. Then $\mu^*(x) \geq \gamma$, and so

$$\sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\} \geq \gamma > \gamma_n, \tag{3.13}$$

for all $n \in \mathbf{N}$. Hence there exists $\beta \in [0,1]$ such that $\beta > \gamma_n$ and $x \in \langle U(\mu;\beta)\rangle$. It follows that $U(\mu;\beta) \subseteq U(\mu;\gamma_n)$ so that $x \in \langle U(\mu;\beta)\rangle \subseteq \langle U(\mu;\gamma_n)\rangle$ for all $n \in \mathbf{N}$. Consequently, $x \in \cap_{n\in\mathbf{N}}\langle U(\mu;\gamma_n)\rangle$. On the other hand, if $x \in \cap_{n\in\mathbf{N}}\langle U(\mu;\gamma_n)\rangle$, then $\gamma_n \in \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$ for any $n \in \mathbf{N}$. Therefore

$$\gamma - \frac{1}{n} = \gamma_n \leq \sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\} = \mu^*(x), \tag{3.14}$$

for all $n \in \mathbf{N}$. Since n is an arbitrary positive integer, it follows that $\gamma \leq \mu^*(x)$ so that $x \in U(\mu^*;\gamma)$. Hence $U(\mu^*;\gamma) = \cap_{n\in\mathbf{N}}\langle U(\mu;\gamma_n)\rangle$, which is a closed ideal of X. Using Theorem 3.3, we know that μ^* is a fuzzy closed ideal of X. We now prove that μ^* contains μ. For any $x \in X$, let $\beta \in \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$. Then $x \in U(\mu;\beta)$ and so $x \in \langle U(\mu;\beta)\rangle$. Thus we get $\beta \in \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}$, and so

$$\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\} \subseteq \{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}. \tag{3.15}$$

It follows from Lemma 3.6 that

$$\begin{aligned}\mu(x) &= \sup\{\alpha \in [0,1] \mid x \in U(\mu;\alpha)\}\\ &\leq \sup\{\alpha \in [0,1] \mid x \in \langle U(\mu;\alpha)\rangle\}\\ &= \mu^*(x).\end{aligned} \tag{3.16}$$

Hence $\mu \subseteq \mu^*$. Finally let ν be a fuzzy closed ideal of X containing μ and let $x \in X$. If $\mu^*(x) = 0$, then clearly $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = \gamma \neq 0$. Then $x \in U(\mu^*;\gamma) = \cap_{n\in\mathbf{N}}\langle U(\mu;\gamma_n)\rangle$, that is, $x \in U(\mu;\gamma_n)$ for all $n \in \mathbf{N}$. It follows that $\nu(x) \geq \mu(x) \geq \gamma_n = \gamma - 1/n$ for all $n \in \mathbf{N}$ so that $\nu(x) \geq \gamma = \mu^*(x)$ since n is arbitrary. This shows that $\mu^* \subseteq \mu$, completing the proof. □

Definition 3.8. A fuzzy closed ideal μ of X is said to be *n-valued* if $\mathrm{Im}(\mu)$ is a finite set of n elements. When no specific n is intended, we call μ a *finite-valued fuzzy closed ideal.*

Theorem 3.9. *Let μ be a fuzzy closed ideal of X. Then μ is finite valued if and only if there exists a finite-valued fuzzy set ν in X which generates μ. In this case, the range sets of μ and ν are identical.*

Proof. If $\mu : X \to [0,1]$ is a finite-valued fuzzy closed ideal of X, then we may choose $\nu = \mu$. Conversely, assume that $\nu : X \to [0,1]$ is a finite-valued fuzzy set. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be distinct elements of $\nu(X)$ such that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, and let $C_i = \nu^{-1}(\alpha_i)$ for $i = 1,2,\dots,n$. Clearly, $\cup_{i=1}^{j} C_i \subseteq \cup_{i=1}^{k} C_i$ whenever $j < k \le n$. Hence if we let $D_j = \langle \cup_{i=1}^{j} C_i \rangle$, then we have the following chain:

$$D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n = X. \tag{3.17}$$

Define a fuzzy set $\mu : X \to [0,1]$ as follows:

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x \in D_1, \\ \alpha_j & \text{if } x \in D_j \setminus D_{j-1}. \end{cases} \tag{3.18}$$

We claim that μ is a fuzzy closed ideal of X generated by ν. Clearly $\mu(0 * x) \ge \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then there exist i and j in $\{1,2,\dots,n\}$ such that $x * y \in D_i$ and $y \in D_j$. Without loss of generality, we may assume that i and j are the smallest integers such that $i \ge j$, $x * y \in D_i$, and $y \in D_j$. Since D_i is a closed ideal of X, it follows from $D_j \subseteq D_i$ that $x \in D_i$. Hence $\mu(x) \ge \alpha_i = \min\{\mu(x * y), \mu(y)\}$, and so μ is a fuzzy closed ideal of X. If $\nu(x) = \alpha_j$ for every $x \in X$, then $x \in C_j$ and thus $x \in D_j$. But we have $\mu(x) \ge \alpha_j = \nu(x)$. Therefore μ contains ν. Let $\delta : X \to [0,1]$ be a fuzzy closed ideal of X containing ν. Then $U(\nu;\alpha_j) \subseteq U(\delta;\alpha_j)$ for every j. Hence $U(\delta;\alpha_j)$, being a closed ideal, contains the closed ideal generated by $U(\nu;\alpha_j) = \cup_{i=1}^{j} C_i$. Consequently, $D_j \subseteq U(\delta;\alpha_j)$. It follows that μ is contained in δ and that μ is generated by ν. Finally, note that $|\mathrm{Im}(\mu)| = n = |\mathrm{Im}(\nu)|$. This completes the proof. □

Theorem 3.10. *Let $D_1 \supseteq D_2 \supseteq \cdots$ be a descending chain of closed ideals of X which terminates at finite step. For a fuzzy closed ideal μ of X, if a sequence of elements of $\mathrm{Im}(\mu)$ is strictly increasing, then μ is finite valued.*

Proof. Suppose that μ is infinite valued. Let $\{\alpha_n\}$ be a strictly increasing sequence of elements of $\mathrm{Im}(\mu)$. Then $0 \le \alpha_1 < \alpha_2 < \cdots \le 1$. Note that $U(\mu;\alpha_t)$ is a closed ideal of X for $t = 1,2,3,\dots$. Let $x \in U(\mu;\alpha_t)$ for $t = 2,3,\dots$. Then $\mu(x) \ge \alpha_t > \alpha_{t-1}$, which implies that $x \in U(\mu;\alpha_{t-1})$. Hence $U(\mu;\alpha_t) \subseteq U(\mu;\alpha_{t-1})$ for $t = 2,3,\dots$. Since $\alpha_{t-1} \in \mathrm{Im}(\mu)$, there exists $x_{t-1} \in X$ such that $\mu(x_{t-1}) = \alpha_{t-1}$. It follows that $x_{t-1} \in U(\mu;\alpha_{t-1})$, but $x_{t-1} \notin U(\mu;\alpha_t)$. Thus $U(\mu;\alpha_t) \subsetneq U(\mu;\alpha_{t-1})$, and so we obtain a strictly descending chain $U(\mu;\alpha_1) \supsetneq U(\mu;\alpha_2) \supsetneq \cdots$ of closed ideals of X which is not terminating. This is impossible and the proof is complete. □

Now we consider the converse of Theorem 3.10.

Theorem 3.11. *Let μ be a finite-valued fuzzy closed ideal of X. Then every descending chain of closed ideals of X terminates at finite step.*

Proof. Suppose there exists a strictly descending chain $D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \cdots$ of closed ideals of X which does not terminate at finite step. Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} \dfrac{n}{n+1} & \text{if } x \in D_n \setminus D_{n+1},\ n = 0,1,2,\ldots, \\ 1 & \text{if } x \in \cap_{n=0}^{\infty} D_n, \end{cases} \tag{3.19}$$

where D_0 stands for X. Clearly, $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x * y \in D_n \setminus D_{n+1}$ and $y \in D_k \setminus D_{k+1}$ for $n = 0,1,2,\ldots$; $k = 0,1,2,\ldots$. Without loss of generality, we may assume that $n \leq k$. Then clearly $y \in D_n$, and so $x \in D_n$ because D_n is a closed ideal of X. Hence

$$\mu(x) \geq \frac{n}{n+1} = \min\{\mu(x * y), \mu(y)\}. \tag{3.20}$$

If $x * y \in \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then $x \in \cap_{n=0}^{\infty} D_n$. Thus $\mu(x) = 1 = \min\{\mu(x * y), \mu(y)\}$. If $x * y \notin \cap_{n=0}^{\infty} D_n$ and $y \in \cap_{n=0}^{\infty} D_n$, then there exists a positive integer k such that $x * y \in D_k \setminus D_{k+1}$. It follows that $x \in D_k$ so that

$$\mu(x) \geq \frac{k}{k+1} = \min\{\mu(x * y), \mu(y)\}. \tag{3.21}$$

Finally suppose that $x * y \in \cap_{n=0}^{\infty} D_n$ and $y \notin \cap_{n=0}^{\infty} D_n$. Then $y \in D_r \setminus D_{r+1}$ for some positive integer r. It follows that $x \in D_r$, and hence

$$\mu(x) \geq \frac{r}{r+1} = \min\{\mu(x * y), \mu(y)\}. \tag{3.22}$$

Consequently, we conclude that μ is a fuzzy closed ideal of X and μ has an infinite number of different values. This is a contradiction, and the proof is complete. □

Theorem 3.12. *The following are equivalent:*

(i) *Every ascending chain of closed ideals of X terminates at finite step.*

(ii) *The set of values of any fuzzy closed ideal of X is a well-ordered subset of* $[0,1]$.

Proof. (i)⇒(ii). Let μ be a fuzzy closed ideal of X. Suppose that the set of values of μ is not a well-ordered subset of $[0,1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\mu(x_n) = \alpha_n$. It follows that

$$U(\mu;\alpha_1) \subsetneq U(\mu;\alpha_2) \subsetneq U(\mu;\alpha_3) \subsetneq \cdots \tag{3.23}$$

is a strictly ascending chain of closed ideals of X. This is impossible.

(ii)⇒(i). Assume that there exists a strictly ascending chain

$$D_1 \subsetneq D_2 \subsetneq D_3 \subsetneq \cdots \tag{3.24}$$

of closed ideals of X. Note that $D := \cup_{n \in \mathbf{N}} D_n$ is a closed ideal of X. Define a fuzzy set μ in X by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin D_n, \\ \dfrac{1}{k} & \text{where } k = \min\{n \in \mathbf{N} \mid x \in D_n\}. \end{cases} \tag{3.25}$$

We claim that μ is a fuzzy closed ideal of X. Let $x \in X$. If $x \notin D_n$, then obviously $\mu(0*x) \geq 0 = \mu(x)$. If $x \in D_n \setminus D_{n-1}$ for $n = 2,3,\ldots$, then $0*x \in D_n$. Hence $\mu(0*x) \geq 1/n = \mu(x)$. Let $x, y \in X$. If $x*y \in D_n \setminus D_{n-1}$ and $y \in D_n \setminus D_{n-1}$ for $n = 2,3,\ldots$, then $x \in D_n$. It follows that

$$\mu(x) \geq \frac{1}{n} = \min\{\mu(x*y), \mu(y)\}. \tag{3.26}$$

Suppose that $x*y \in D_n$ and $y \in D_n \setminus D_m$ for all $m < n$. Then $x \in D_n$, and so $\mu(x) \geq 1/n \geq 1/m+1 \geq \mu(y)$. Hence $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$. Similarly for the case $x*y \in D_n \setminus D_m$ and $y \in D_n$, we get $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$. Therefore μ is a fuzzy closed ideal of X. Since the chain (3.24) is not terminating, μ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy closed ideal is well ordered. This completes the proof. □

4. T-fuzzy subalgebras and T-fuzzy closed ideals

Definition 4.1. A fuzzy set μ in X is said to satisfy *imaginable property* if $\mathrm{Im}(\mu) \subseteq \Delta_T$.

Definition 4.2. A fuzzy set μ in X is called a *fuzzy subalgebra* of X with respect to a t-norm T (briefly, T*-fuzzy subalgebra* of X) if $\mu(x*y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$. A T-fuzzy subalgebra of X is said to be *imaginable* if it satisfies the imaginable property.

Example 4.3. Let T_m be a t-norm defined by $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0,1]$ and let $X = \{0, a, b, c, d\}$ be a BCH-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

(1) Define a fuzzy set $\mu : X \to [0,1]$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if } x \in \{0, d\}, \\ 0.09 & \text{otherwise}. \end{cases} \tag{4.1}$$

Then μ is a T_m-fuzzy subalgebra of X, which is not imaginable.

(2) Let ν be a fuzzy set in X defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \{0, d\}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.2}$$

Then ν is an imaginable T_m-fuzzy subalgebra of X.

Proposition 4.4. *Let A be a subalgebra of X and let μ be a fuzzy set in X defined by*

$$\mu(x) := \begin{cases} \alpha_1 & \text{if } x \in A, \\ \alpha_2 & \text{otherwise,} \end{cases} \tag{4.3}$$

for all $x \in X$, where $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$. Then μ is a T_m-fuzzy subalgebra of X. In particular, if $\alpha_1 = 1$ and $\alpha_2 = 0$ then μ is an imaginable T_m-fuzzy subalgebra of X, where T_m is the t-norm in Example 4.3.

Proof. Let $x, y \in X$. If $x \in A$ and $y \in A$ then

$$\begin{aligned} T_m(\mu(x),\mu(y)) &= T_m(\alpha_1,\alpha_1) = \max(2\alpha_1 - 1, 0) \\ &= \begin{cases} 2\alpha_1 - 1 & \text{if } \alpha_1 \geq \frac{1}{2} \\ 0 & \text{if } \alpha_1 < \frac{1}{2} \end{cases} \\ &\leq \alpha_1 = \mu(x * y). \end{aligned} \tag{4.4}$$

If $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$) then

$$\begin{aligned} T_m(\mu(x),\mu(y)) &= T_m(\alpha_1,\alpha_2) = \max(\alpha_1 + \alpha_2 - 1, 0) \\ &= \begin{cases} \alpha_1 + \alpha_2 - 1 & \text{if } \alpha_1 + \alpha_2 \geq 1 \\ 0 & \text{otherwise} \end{cases} \\ &\leq \alpha_2 \leq \mu(x * y). \end{aligned} \tag{4.5}$$

If $x, y \notin A$ then

$$\begin{aligned} T_m(\mu(x),\mu(y)) &= T_m(2\alpha_2 - 1, 0) \\ &= \begin{cases} 2\alpha_2 - 1 & \text{if } \alpha_2 \geq \frac{1}{2} \\ 0 & \text{if } \alpha_2 < \frac{1}{2} \end{cases} \\ &\leq \alpha_2 \leq \mu(x * y). \end{aligned} \tag{4.6}$$

Hence μ is a T_m-fuzzy subalgebra of X. Assume that $\alpha_1 = 1$ and $\alpha_2 = 0$. Then

$$\begin{aligned} T_m(\alpha_1,\alpha_1) &= \max(\alpha_1 + \alpha_1 - 1, 0) = 1 = \alpha_1, \\ T_m(\alpha_2,\alpha_2) &= \max(\alpha_2 + \alpha_2 - 1, 0) = 0 = \alpha_2. \end{aligned} \tag{4.7}$$

Thus $\alpha_1, \alpha_2 \in \Delta_{T_m}$, that is, $\text{Im}(\mu) \subseteq \Delta_{T_m}$ and so μ is imaginable. This completes the proof. □

Proposition 4.5. *If μ is an imaginable T-fuzzy subalgebra of X, then $\mu(0 * x) \geq \mu(x)$ for all $x \in X$.*

Proof. For any $x \in X$ we have

$$\begin{aligned} \mu(0*x) &\geq T(\mu(0),\mu(x)) \\ &= T(\mu(x*x),\mu(x)) \quad \text{[by (H1)]} \\ &\geq T(T(\mu(x),\mu(x)),\mu(x)) \quad \text{[by (T2) and (T3)]} \\ &= \mu(x), \quad \text{[since } \mu \text{ satisfies the imaginable property].} \end{aligned} \tag{4.8}$$

This completes the proof. □

Theorem 4.6. *Let μ be a T-fuzzy subalgebra of X and let $\alpha \in [0,1]$ be such that $T(\alpha,\alpha) = \alpha$. Then $U(\mu;\alpha)$ is either empty or a subalgebra of X, and moreover $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proof. Let $x, y \in U(\mu;\alpha)$. Then

$$\mu(x*y) \geq T(\mu(x),\mu(y)) \geq T(\alpha,\alpha) = \alpha, \tag{4.9}$$

which implies that $x*y \in U(\mu;\alpha)$. Hence $U(\mu;\alpha)$ is a subalgebra of X. Since $x*x = 0$ for all $x \in X$, we have $\mu(0) = \mu(x*x) \geq T(\mu(x),\mu(x)) = \mu(x)$ for all $x \in X$. □

Since $T(1,1) = 1$, we have the following corollary.

Corollary 4.7. *If μ is a T-fuzzy subalgebra of X, then $U(\mu;1)$ is either empty or a subalgebra of X.*

Theorem 4.8. *Let μ be a T-fuzzy subalgebra of X. If there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} T(\mu(x_n),\mu(x_n)) = 1$, then $\mu(0) = 1$.*

Proof. Let $x \in X$. Then $\mu(0) = \mu(x*x) \geq T(\mu(x),\mu(x))$. Therefore $\mu(0) \geq T(\mu(x_n),\mu(x_n))$ for each $n \in \mathbf{N}$. Since $1 \geq \mu(0) \geq \lim_{n\to\infty} T(\mu(x_n),\mu(x_n)) = 1$, it follows that $\mu(0) = 1$, this completes the proof. □

Let $f : X \to Y$ be a mapping of BCH-algebras. For a fuzzy set μ in Y, the *inverse image* of μ under f, denoted by $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in X$.

Theorem 4.9. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. If μ is a T-fuzzy subalgebra of Y, then $f^{-1}(\mu)$ is a T-fuzzy subalgebra of X.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}(\mu)(x*y) &= \mu(f(x*y)) = \mu(f(x)*f(y)) \\ &\geq T(\mu(f(x)),\mu(f(y))) \\ &= T(f^{-1}(\mu)(x),f^{-1}(\mu)(y)). \end{aligned} \tag{4.10}$$

This completes the proof. □

If μ is a fuzzy set in X and f is a mapping defined on X. The fuzzy set $f(\mu)$ in $f(X)$ defined by $f(\mu)(y) = \sup\{\mu(x) \mid x \in f^{-1}(y)\}$ for all $y \in f(X)$ is called the *image* of μ under f. A fuzzy set μ in X is said to have *sup property* if, for every subset $T \subseteq X$, there exists $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t) \mid t \in T\}$.

Theorem 4.10. *An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.*

Proof. Let $f : X \to Y$ be an onto homomorphism of BCH-algebras and let μ be a fuzzy subalgebra of X with sup property. Given $u, v \in Y$, let $x_0 \in f^{-1}(u)$ and $y_0 \in f^{-1}(v)$ be such that

$$\mu(x_0) = \sup\{\mu(t) \mid t \in f^{-1}(u)\}, \qquad \mu(y_0) = \sup\{\mu(t) \mid t \in f^{-1}(v)\}, \tag{4.11}$$

respectively. Then

$$\begin{aligned} f(\mu)(u * v) &= \sup\{\mu(z) \mid z \in f^{-1}(u * v)\} \\ &\geq \min\{\mu(x_0), \mu(y_0)\} \\ &= \min\{\sup\{\mu(t) \mid t \in f^{-1}(u)\}, \sup\{\mu(t) \mid t \in f^{-1}(v)\}\} \\ &= \min\{f(\mu)(u), f(\mu)(v)\}. \end{aligned} \tag{4.12}$$

Hence $f(\mu)$ is a fuzzy subalgebra of Y. □

Theorem 4.10 can be strengthened in the following way. To do this we need the following definition.

Definition 4.11. A t-norm T on $[0,1]$ is called a *continuous t-norm* if T is a continuous function from $[0,1] \times [0,1]$ to $[0,1]$ with respect to the usual topology.

Note that the function "min" is a continuous t-norm.

Theorem 4.12. *Let T be a continuous t-norm and let $f : X \to Y$ be an onto homomorphism of BCH-algebras. If μ is a T-fuzzy subalgebra of X, then $f(\mu)$ is a T-fuzzy subalgebra of Y.*

Proof. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, and $A_{12} = f^{-1}(y_1 * y_2)$, where $y_1, y_2 \in Y$. Consider the set

$$A_1 * A_2 := \{x \in X \mid x = a_1 * a_2 \text{ for some } a_1 \in A_1,\ a_2 \in A_2\}. \tag{4.13}$$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ and so

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2, \tag{4.14}$$

that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$. Thus $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f(\mu)(y_1 * y_2) &= \sup\{\mu(x) \mid x \in f^{-1}(y_1 * y_2)\} = \sup\{\mu(x) \mid x \in A_{12}\} \\ &\geq \sup\{\mu(x) \mid x \in A_1 * A_2\} \\ &\geq \sup\{\mu(x_1 * x_2) \mid x_1 \in A_1, x_2 \in A_2\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1,\ x_2 \in A_2\}. \end{aligned} \tag{4.15}$$

Since T is continuous, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that if $\sup\{\mu(x_1) \mid x_1 \in A_1\} - x_1^* \leq \delta$ and $\sup\{\mu(x_2) \mid x_2 \in A_2\} - x_2^* \leq \delta$ then

$$T(\sup\{\mu(x_1) \mid x_1 \in A_1\}, \sup\{\mu(x_2) \mid x_2 \in A_2\}) - T(x_1^*, x_2^*) \leq \varepsilon. \tag{4.16}$$

Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\sup\{\mu(x_1) \mid x_1 \in A_1\} - \mu(a_1) \le \delta$ and $\sup\{\mu(x_2) \mid x_2 \in A_2\} - \mu(a_2) \le \delta$. Then

$$T(\sup\{\mu(x_1) \mid x_1 \in A_1\},\ \sup\{\mu(x_2) \mid x_2 \in A_2\}) - T(\mu(a_1), \mu(a_2)) \le \varepsilon. \tag{4.17}$$

Consequently

$$\begin{aligned} f(\mu)(y_1 * y_2) &\ge \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2\} \\ &\ge T(\sup\{\mu(x_1) \mid x_1 \in A_1\},\ \sup\{\mu(x_2) \mid x_2 \in A_2\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)), \end{aligned} \tag{4.18}$$

which shows that $f(\mu)$ is a T-fuzzy subalgebra of Y. □

Lemma 4.13 (see [1]). *For all $\alpha, \beta, \gamma, \delta \in [0,1]$,*

$$T(T(\alpha,\beta), T(\gamma,\delta)) = T(T(\alpha,\gamma), T(\beta,\delta)). \tag{4.19}$$

Theorem 4.14. *Let $X = X_1 \times X_2$ be the direct product BCH-algebra of BCH-algebras X_1 and X_2. If μ_1 (resp., μ_2) is a T-fuzzy subalgebra of X_1 (resp., X_2), then $\mu = \mu_1 \times \mu_2$ is a T-fuzzy subalgebra of X defined by*

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)), \tag{4.20}$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $X = X_1 \times X_2$. Then

$$\begin{aligned} \mu(x * y) &= \mu((x_1, x_2) * (y_1, y_2)) = \mu(x_1 * y_1, x_2 * y_2) \\ &= T(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)) \\ &\ge T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) \\ &= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2))) \\ &= T(\mu(x_1, x_2), \mu(x_2, y_2)) \\ &= T(\mu(x), \mu(y)). \end{aligned} \tag{4.21}$$

Hence μ is a T-fuzzy subalgebra of X. □

We will generalize the idea to the product of n T-fuzzy subalgebras. We first need to generalize the domain of T to $\prod_{i=1}^{n}[0,1]$ as follows:

Definition 4.15 (see [1]). The function $T_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by

$$T_n(\alpha_1, \alpha_2, \dots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)), \tag{4.22}$$

for all $1 \le i \le n$, where $n \ge 2$, $T_2 = T$, and $T_1 = \text{id}$ (identity).

Lemma 4.16 (see [1]). *For every $\alpha_i, \beta_i \in [0,1]$ where $1 \le i \le n$ and $n \ge 2$,*

$$T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \dots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \dots, \alpha_n), T_n(\beta_1, \beta_2, \dots, \beta_n)). \tag{4.23}$$

Theorem 4.17. *Let $\{X_i\}_{i=1}^n$ be the finite collection of BCH-algebras and $X = \prod_{i=1}^n X_i$ the direct product BCH-algebra of $\{X_i\}$. Let μ_i be a T-fuzzy subalgebra of X_i, where $1 \le i \le n$. Then $\mu = \prod_{i=1}^n \mu_i$ defined by*

$$\begin{aligned} \mu(x_1,x_2,\dots,x_n) &= \left(\prod_{i=1}^n \mu_i\right)(x_1,x_2,\dots,x_n) \\ &= T_n(\mu_1(x_1),\mu_2(x_2),\dots,\mu_n(x_n)), \end{aligned} \tag{4.24}$$

is a T-fuzzy subalgebra of the BCH-algebra X.

Proof. Let $x = (x_1,x_2,\dots,x_n)$ and $y = (y_1,y_2,\dots,y_n)$ be any elements of $X = \prod_{i=1}^n X_i$. Then

$$\begin{aligned} \mu(x*y) &= \mu(x_1*y_1,x_2*y_2,\dots,x_n*y_n) \\ &= T_n(\mu_1(x_1*y_1),\mu_2(x_2*y_2),\dots,\mu_n(x_n*y_n)) \\ &\ge T_n(T(\mu_1(x_1),\mu_1(y_1)),T(\mu_2(x_2),\mu_2(y_2)),\dots,T(\mu_n(x_n),\mu_n(y_n))) \\ &= T(T_n(\mu_1(x_1),\mu_2(x_2),\dots,\mu_n(x_n)),T_n(\mu_1(y_1),\mu_2(y_2),\dots,\mu_n(y_n))) \\ &= T(\mu(x_1,x_2,\dots,x_n),\mu(y_1,y_2,\dots,y_n)) \\ &= T(\mu(x),\mu(y)). \end{aligned} \tag{4.25}$$

Hence μ is a T-fuzzy subalgebra of X. □

Definition 4.18. Let μ and ν be fuzzy sets in X. Then the *T-product* of μ and ν, written $[\mu\cdot\nu]_T$, is defined by $[\mu\cdot\nu]_T(x) = T(\mu(x),\nu(x))$ for all $x \in X$.

Theorem 4.19. *Let μ and ν be T-fuzzy subalgebras of X. If T^* is a t-norm which dominates T, that is,*

$$T^*(T(\alpha,\beta),T(\gamma,\delta)) \ge T(T^*(\alpha,\gamma),T^*(\beta,\delta)), \tag{4.26}$$

for all $\alpha,\beta,\gamma,\delta \in [0,1]$, then the T^*-product of μ and ν, $[\mu\cdot\nu]_{T^*}$, is a T-fuzzy subalgebra of X.

Proof. For any $x,y \in X$ we have

$$\begin{aligned} [\mu\cdot\nu]_{T^*}(x*y) &= T^*(\mu(x*y),\nu(x*y)) \\ &\ge T^*(T(\mu(x),\mu(y)),T(\nu(x),\nu(y))) \\ &\ge T(T^*(\mu(x),\nu(x)),T^*(\mu(y),\nu(y))) \\ &= T([\mu\cdot\nu]_{T^*}(x),[\mu\cdot\nu]_{T^*}(y)). \end{aligned} \tag{4.27}$$

Hence $[\mu\cdot\nu]_{T^*}$ is a T-fuzzy subalgebra of X. □

Let $f : X \to Y$ be an onto homomorphism of BCH-algebras. Let T and T^* be t-norms such that T^* dominates T. If μ and ν are T-fuzzy subalgebras of Y, then the T^*-product of μ and ν, $[\mu\cdot\nu]_{T^*}$, is a T-fuzzy subalgebra of Y. Since every onto homomorphic inverse image of a T-fuzzy subalgebra is a T-fuzzy subalgebra, the inverse images $f^{-1}(\mu)$, $f^{-1}(\nu)$, and $f^{-1}([\mu\cdot\nu]_{T^*})$ are T-fuzzy subalgebras of X. The next theorem provides that the relation between $f^{-1}([\mu\cdot\nu]_{T^*})$ and the T^*-product $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 4.20. *Let $f: X\to Y$ be an onto homomorphism of BCH-algebras. Let T^* be a t-norm such that T^* dominates T. Let μ and ν be T-fuzzy subalgebras of Y. If $[\mu\cdot\nu]_{T^*}$ is the T^*-product of μ and ν and $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}$ is the T^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then*

$$f^{-1}([\mu\cdot\nu]_{T^*}) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}. \tag{4.28}$$

Proof. For any $x\in X$ we get

$$\begin{aligned} f^{-1}([\mu\cdot\nu]_{T^*})(x) &= [\mu\cdot\nu]_{T^*}(f(x)) \\ &= T^*(\mu(f(x)),\nu(f(x))) \\ &= T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x)) \\ &= [f^{-1}(\mu)\cdot f^{-1}(\nu)]_{T^*}(x), \end{aligned} \tag{4.29}$$

This completes the proof. □

Definition 4.21. A fuzzy set μ in X is called a *fuzzy closed ideal* of X under a t-norm T (briefly, *T-fuzzy closed ideal* of X) if

(F1) $\mu(0*x)\geq\mu(x)$ for all $x\in X$,

(F3) $\mu(x)\geq T(\mu(x*y),\mu(y))$ for all $x,y\in X$.

A T-fuzzy closed ideal of X is said to be *imaginable* if it satisfies the imaginable property.

Example 4.22. Let T_m be a t-norm in Example 4.3. Consider a BCH-algebra $X=\{0,a,b,c\}$ with Cayley table as follows:

$*$	0	a	b	c
0	0	c	0	c
a	a	0	c	b
b	b	c	0	a
c	c	0	c	0

(1) Define a fuzzy set $\mu: X\to[0,1]$ by $\mu(0)=\mu(c)=0.8$ and $\mu(a)=\mu(b)=0.3$. Then μ is a T_m-fuzzy closed ideal of X which is not imaginable.

(2) Let ν be a fuzzy set in X defined by

$$\nu(x)=\begin{cases}1 & \text{if } x\in\{0,c\},\\ 0 & \text{otherwise.}\end{cases} \tag{4.30}$$

Then ν is an imaginable T_m-fuzzy closed ideal of X.

Theorem 4.23. *Every imaginable T-fuzzy subalgebra satisfying (F3) is an imaginable T-fuzzy closed ideal.*

Proof. Using Proposition 4.5, it is straightforward. □

Proposition 4.24. *If μ is an imaginable T-fuzzy closed ideal of X, then $\mu(0) \geq \mu(x)$ for all $x \in X$.*

Proof. Using (F1), (F3), and (T2), we have

$$\mu(0) \geq T(\mu(0 * x), \mu(x)) \geq T(\mu(x), \mu(x)) = \mu(x) \tag{4.31}$$

for all $x \in X$, completing the proof. □

Theorem 4.25. *Every T-fuzzy closed ideal is a T-fuzzy subalgebra.*

Proof. Let μ be a T-fuzzy closed ideal of X and let $x, y \in X$. Then

$$\begin{aligned} \mu(x * y) &\geq T(\mu((x * y) * x), \mu(x)) \quad \text{[by (F3)]} \\ &= T(\mu((x * x) * y), \mu(x)) \quad \text{[by (H3)]} \\ &= T(\mu(0 * y), \mu(x)) \quad \text{[by (H1)]} \\ &\geq T(\mu(x), \mu(y)) \quad \text{[by (F1), (T2), and (T3)].} \end{aligned} \tag{4.32}$$

Hence μ is a T-fuzzy subalgebra of X. □

The converse of Theorem 4.25 may not be true. For example, the T_m-fuzzy subalgebra μ in Example 4.3(1) is not a T_m-fuzzy closed ideal of X since

$$\mu(a) = 0.09 < 0.9 = T_m(\mu(a * d), \mu(d)). \tag{4.33}$$

We give a condition for a T-fuzzy subalgebra to be a T-fuzzy closed ideal.

Theorem 4.26. *Let μ be a T-fuzzy subalgebra of X. If μ satisfies the imaginable property and the inequality*

$$\mu(x * y) \leq \mu(y * x) \quad \forall x, y \in X, \tag{4.34}$$

then μ is a T-fuzzy closed ideal of X.

Proof. Let μ be an imaginable T-fuzzy subalgebra of X which satisfies the inequality

$$\mu(x * y) \leq \mu(y * x) \quad \forall x, y \in X. \tag{4.35}$$

It follows from Proposition 4.5 that $\mu(0 * x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in X$. Then

$$\begin{aligned} \mu(x) &= \mu(x * 0) \geq \mu(0 * x) = \mu((y * y) * x) \\ &= \mu((y * x) * y) \geq T(\mu(y * x), \mu(y)) \geq T(\mu(x * y), \mu(y)). \end{aligned} \tag{4.36}$$

Hence μ is a T-fuzzy closed ideal of X. □

Proposition 4.27. *Let T_m be a t-norm in Example 4.3. Let D be a closed ideal of X and let μ be a fuzzy set in X defined by*

$$\mu(x) = \begin{cases} \alpha_1 & \text{if } x \in D, \\ \alpha_2 & \text{otherwise,} \end{cases} \tag{4.37}$$

for all $x \in X$.

(i) *If* $\alpha_1 = 1$ *and* $\alpha_2 = 0$*, then* μ *is an imaginable* T_m*-fuzzy closed ideal of* X*.*

(ii) *If* $\alpha_1, \alpha_2 \in (0,1)$ *and* $\alpha_1 > \alpha_2$*, then* μ *is a* T_m*-fuzzy closed ideal of* X *which is not imaginable.*

Proof. (i) If $x \in D$, then $0 * x \in D$ and so $\mu(0*x) = 1 = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \le \mu(0*x)$. Now obviously if $x \in D$, then

$$\mu(x) = 1 \ge T_m(\mu(x*y), \mu(y)), \tag{4.38}$$

for all $y \in X$. Assume that $x \notin D$. Then $x * y \notin D$ or $y \notin D$, that is, $\mu(x*y) = 0$ or $\mu(y) = 0$. It follows that

$$T_m(\mu(x*y), \mu(y)) = 0 = \mu(x). \tag{4.39}$$

Hence $\mu(x) \ge T_m(\mu(x*y), \mu(y))$ for all $x, y \in X$. Clearly $\mathrm{Im}(\mu) \subseteq \Delta_{T_m}$.

(ii) Similar to (i), we know that μ is a T_m-fuzzy closed ideal of X. Taking $\alpha_1 = 0.7$, then

$$T_m(\alpha_1, \alpha_1) = T_m(0.7, 0.7) = \max(0.7 + 0.7 - 1, 0) = 0.4 \ne \alpha_1. \tag{4.40}$$

Hence $\alpha_1 \notin \Delta_{T_m}$, that is, $\mathrm{Im}(\mu) \nsubseteq \Delta_{T_m}$, and so μ is not imaginable. □

Proposition 4.28. *Let* μ *be an imaginable* T*-fuzzy closed ideal of* X*. If* μ *satisfies the inequality* $\mu(x) \ge \mu(0*x)$ *for all* $x \in X$*, then it satisfies the equality* $\mu(x*y) = \mu(y*x)$ *for all* $x, y \in X$*.*

Proof. Let μ be an imaginable T-fuzzy closed ideal of X satisfying the inequality $\mu(x) \ge \mu(0*x)$ for all $x \in X$. For every $x, y \in X$, we have

$$\begin{aligned}
\mu(y*x) &\ge \mu(0*(y*x)) \quad \text{[by assumption]} \\
&\ge T(\mu((0*(y*x))*(x*y)), \mu(x*y)) \quad \text{[by (F3)]} \\
&= T(\mu(((0*y)*(0*x))*(x*y)), \mu(x*y)) \quad \text{[by (P3)]} \\
&= T(\mu(((0*y)*(x*y))*(0*x)), \mu(x*y)) \quad \text{[by (H3)]} \\
&= T(\mu(((0*(x*y))*y)*(0*x)), \mu(x*y)) \quad \text{[by (H3)]} \\
&= T(\mu((((0*x)*(0*y))*y)*(0*x)), \mu(x*y)) \quad \text{[by (P3)]} \\
&= T(\mu((((0*x)*(0*y))*(0*x))*y), \mu(x*y)) \quad \text{[by (H3)]} \\
&= T(\mu((((0*x)*(0*x))*(0*y))*y), \mu(x*y)) \quad \text{[by (H3)]} \\
&= T(\mu((0*(0*y))*y), \mu(x*y)) \quad \text{[by (H1)]} \\
&= T(\mu(0), \mu(x*y)) \quad \text{[by (H3) and (H1)]} \\
&= T(\mu((x*y)*(x*y)), \mu(x*y)) \quad \text{[by (H1)]} \\
&\ge T(T(\mu(x*y), \mu(x*y)), \mu(x*y)) \quad \text{[by Proposition 4.24 and (T2)]} \\
&= \mu(x*y) \quad \text{[since } \mu \text{ is imaginable].}
\end{aligned} \tag{4.41}$$

Similarly we have $\mu(x*y) \ge \mu(y*x)$ for all $x, y \in X$, completing the proof. □

Theorem 4.29. *Every imaginable T-fuzzy closed ideal is a fuzzy closed ideal.*

Proof. Let μ be an imaginable T-fuzzy closed ideal of X. Then

$$\mu(x) \geq T(\mu(x * y), \mu(y)) \quad \forall x, y \in X. \tag{4.42}$$

Since μ is imaginable, we have

$$\begin{aligned}\min(\mu(x*y),\mu(y)) &= T(\min(\mu(x*y),\mu(y)),\min(\mu(x*y),\mu(y)))\\ &\leq T(\mu(x*y),\mu(y))\\ &\leq \min(\mu(x*y),\mu(y)).\end{aligned} \tag{4.43}$$

It follows that $\mu(x) \geq T(\mu(x*y),\mu(y)) = \min(\mu(x*y),\mu(y))$ so that μ is a fuzzy closed ideal of X. □

Combining Theorems 3.3, 4.29, we have the following corollary.

Corollary 4.30. *If μ is an imaginable T-fuzzy closed ideal of X, then the nonempty level set of μ is a closed ideal of X.*

Noticing that the fuzzy set μ in Example 4.22(1) is a fuzzy closed ideal of X, we know from Example 4.22(1) that there exists a t-norm such that the converse of Theorem 4.29 may not be true.

Proposition 4.31. *Every imaginable T-fuzzy closed ideal is order reversing.*

Proof. Let μ be an imaginable T-fuzzy closed ideal of X and let $x, y \in X$ be such that $x \leq y$. Using (P4), (T2), Theorem 4.29, Proposition 4.24, and the definition of a fuzzy closed ideal, we get

$$\begin{aligned}\mu(x) &\geq \min\{\mu(x*y),\mu(y)\} \geq T(\mu(x*y),\mu(y))\\ &= T(\mu(0),\mu(y)) \geq T(\mu(y),\mu(y)) = \mu(y).\end{aligned} \tag{4.44}$$

This completes the proof. □

Proposition 4.32. *Let μ be a T-fuzzy closed ideal of X, where T is a diagonal t-norm on $[0,1]$, that is, $T(\alpha,\alpha) = \alpha$ for all $\alpha \in [0,1]$. If $(x*a)*b = 0$ for all $a, b, x \in X$, then $\mu(x) \geq T(\mu(a),\mu(b))$.*

Proof. Let $a, b, x \in X$ be such that $(x*a)*b = 0$. Then

$$\begin{aligned}\mu(x) &\geq T(\mu(x*a),\mu(a))\\ &\geq T(T(\mu((x*a)*b),\mu(b)),\mu(a))\\ &= T(T(\mu(0),\mu(b)),\mu(a))\\ &\geq T(T(\mu(b),\mu(b)),\mu(a))\\ &= T(\mu(a),\mu(b)),\end{aligned} \tag{4.45}$$

completing the proof. □

Corollary 4.33. *Let μ be a T-fuzzy closed ideal of X, where T is a diagonal t-norm on $[0,1]$. If $(\cdots((x*a_1)*a_2)*\cdots)*a_n = 0$ for all $x, a_1, a_2, \ldots, a_n \in X$, then*

$$\mu(x) \geq T_n(\mu(a_1), \mu(a_2), \ldots, \mu(a_n)). \tag{4.46}$$

Proof. Using induction on n, the proof is straightforward. □

Theorem 4.34. *There exists a t-norm T such that every closed ideal of X can be realized as a level closed ideal of a T-fuzzy closed ideal of X.*

Proof. Let D be a closed ideal of X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D, \\ 0 & \text{otherwise,} \end{cases} \tag{4.47}$$

where $\alpha \in (0,1)$ is fixed. It is clear that $U(\mu;\alpha) = D$. We will prove that μ is a T_m-fuzzy closed ideal of X, where T_m is a t-norm in Example 4.3. If $x \in D$, then $0 * x \in D$ and so $\mu(0*x) = \alpha = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \leq \mu(0*x)$. Let $x, y \in X$. If $x \in D$, then $\mu(x) = \alpha \geq T_m(\mu(x*y), \mu(y))$. If $x \notin D$, then $x * y \notin D$ or $y \notin D$. It follows that $\mu(x) = 0 = T_m(\mu(x*y), \mu(y))$. This completes the proof. □

For a family $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of fuzzy sets in X, define the join $\vee_{\alpha\in\Lambda}\mu_\alpha$ and the meet $\wedge_{\alpha\in\Lambda}\mu_\alpha$ as follows:

$$(\vee_{\alpha\in\Lambda}\mu_\alpha)(x) = \sup\{\mu_\alpha(x) \mid \alpha \in \Lambda\}, \qquad (\wedge_{\alpha\in\Lambda}\mu_\alpha)(x) = \inf\{\mu_\alpha(x) \mid \alpha \in \Lambda\}, \tag{4.48}$$

for all $x \in X$, where Λ is any index set.

Theorem 4.35. *The family of T-fuzzy closed ideals in X is a completely distributive lattice with respect to meet "$\wedge$" and the join "$\vee$".*

Proof. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\vee_{\alpha\in\Lambda}\mu_\alpha$ and $\wedge_{\alpha\in\Lambda}\mu_\alpha$ are T-fuzzy closed ideals of X for a family of T-fuzzy closed ideals $\{\mu_\alpha \mid \alpha \in \Lambda\}$. For any $x \in X$, we have

$$\begin{aligned} (\vee_{\alpha\in\Lambda}\mu_\alpha)(0*x) &= \sup\{\mu_\alpha(0*x) \mid \alpha \in \Lambda\} \\ &\geq \sup\{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\ &= (\vee_{\alpha\in\Lambda}\mu_\alpha)(x), \\ (\wedge_{\alpha\in\Lambda}\mu_\alpha)(0*x) &= \inf\{\mu_\alpha(0*x) \mid \alpha \in \Lambda\} \\ &\geq \inf\{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\ &= (\wedge_{\alpha\in\Lambda}\mu_\alpha)(x). \end{aligned} \tag{4.49}$$

Let $x, y \in X$. Then

$$\begin{aligned} (\vee_{\alpha\in\Lambda}\mu_\alpha)(x) &= \sup\{\mu_\alpha(x) \mid \alpha \in \Lambda\} \\ &\geq \sup\{T(\mu_\alpha(x*y), \mu_\alpha(y)) \mid \alpha \in \Lambda\} \\ &\geq T(\sup\{\mu_\alpha(x*y) \mid \alpha \in \Lambda\}, \sup\{\mu_\alpha(y) \mid \alpha \in \Lambda\}) \\ &= T((\vee_{\alpha\in\Lambda}\mu_\alpha)(x*y), (\vee_{\alpha\in\Lambda}\mu_\alpha)(y)), \end{aligned}$$

$$
\begin{aligned}
(\wedge_{\alpha\in\Lambda}\mu_\alpha)(x) &= \inf\{\mu_\alpha(x) \mid \alpha\in\Lambda\} \\
&\geq \inf\{T(\mu_\alpha(x*y),\mu_\alpha(y)) \mid \alpha\in\Lambda\} \\
&\geq T(\inf\{\mu_\alpha(x*y) \mid \alpha\in\Lambda\},\inf\{\mu_\alpha(y) \mid \alpha\in\Lambda\}) \\
&= T((\wedge_{\alpha\in\Lambda}\mu_\alpha)(x*y),(\wedge_{\alpha\in\Lambda}\mu_\alpha)(y)).
\end{aligned}
\tag{4.50}
$$

Hence $\vee_{\alpha\in\Lambda}\mu_\alpha$ and $\wedge_{\alpha\in\Lambda}\mu_\alpha$ are T-fuzzy closed ideals of X, completing the proof. □

5. Conclusions and future works. We inquired into further properties on fuzzy closed ideals in BCH-algebras, and using a t-norm T, we introduced the notion of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy closed ideals, and obtained some related results. Moreover, we discussed the direct product and T-product of T-fuzzy subalgebras. We finally showed that the family of T-fuzzy closed ideals is a completely distributive lattice. These ideas enable us to define the notion of (imaginable) T-fuzzy filters in BCH-algebras, and to discuss the direct products and T-products of T-fuzzy filters. It also gives us possible problems to discuss relations among T-fuzzy subalgebras, T-fuzzy closed ideals and T-fuzzy filters, and to construct the normalizations. We may also use these ideas to introduce the notion of interval-valued fuzzy subalgebras/closed ideals.

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