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GLOBAL ATTRACTIVITY IN A GENOTYPE SELECTION MODEL

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We obtain a sufficient condition for the global attractivity of the genotype selection model $y_{n+1} = y_n e^{\beta_n(1-2y_{n-k})}/(1-y_n+y_n e^{\beta_n(1-2y_{n-k})})$, $n \in \mathbb{N}$. Our results improve the results established by Grove et al. (1994) and Kocić and Ladas (1993).

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1. Introduction. Let $\mathbb{Z}$ denote the set of all integers. For $a,b \in \mathbb{Z}$, define $\mathbb{N}(a) = \{a, a+1, \dots\}$, $\mathbb{N} = \mathbb{N}(0)$, and $\mathbb{N}(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$.

Consider the following nonlinear delay difference equation:

$$y_{n+1} = \frac{y_n e^{\beta_n(1-2y_{n-k})}}{1-y_n+y_n e^{\beta_n(1-2y_{n-k})}}, \quad n \in \mathbb{N}, \tag{1.1}$$

where $k \in \mathbb{N}$ and $\{\beta_n\}$ is a sequence of positive real numbers.

When $k = 0$ and $\beta_n \equiv \beta$ for all $n \in \mathbb{N}$, (1.1) was introduced by May [2, pages 513–560] as an example of a map generated by a simple model for frequency-dependent natural selection. The local stability of the equilibrium $\bar{y} = 1/2$ of (1.1) was investigated by May [2]. In [1] (see also [3]), Grove further investigated the stability of the equilibrium $\bar{y} = 1/2$ of (1.1) and proved that when $\beta_n \equiv \beta$, the equilibrium $\bar{y} = 1/2$ of (1.1) is locally asymptotically stable if $0 < \beta < 4\cos(k\pi/(2k+1))$ and is unstable if $0 < \beta < 4\cos(k\pi/(2k+1))$. Furthermore, if

$$0 < \beta \le \frac{2}{k}, \quad k \in \mathbb{N}(1). \tag{1.2}$$

Then this equilibrium is a global attractor of all solution $\{y_n\}$ of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \dots, y_0 \in (0,1)$.

On the basis of computer observations, the authors of [1] also observe that condition (1.2) is probably far from sharp when $k \in \mathbb{N}(2)$. Therefore, it is highly desirable to improve condition (1.2).

The purpose of this paper is to obtain new sufficient conditions for the global attractivity of the equilibrium $\bar{y} = 1/2$ of (1.1). Our main result is the following theorem.

Theorem 1.1. *Assume that $\{\beta_n\}$ is a positive sequence which satisfies*

$$\sum_{i=n-k}^{n} \beta_i \le 3 + \frac{1}{k+1}, \tag{1.3}$$

for all large n, and

$$\sum_{i=0}^{\infty} \beta_i = \infty. \tag{1.4}$$

Then every solution $\{y_n\}$ of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \ldots, y_0 \in (0,1)$ will tend to $\bar{y} = 1/2$.

Corollary 1.2. *Assume that $\beta_n \equiv \beta$ for all $n \in \mathbb{N}$ and*

$$\beta \le \frac{3}{k+1} + \frac{1}{(k+1)^2}. \tag{1.5}$$

Then every solution $\{y_n\}$ of (1.1) with initial conditions $y_{-k}, y_{-k+1}, \ldots, y_0 \in (0,1)$ will tend to $\bar{y} = 1/2$.

It is easy to see that when $k \in \mathbb{N}(2)$, (1.5) is an improvement on (1.2).

By a solution of (1.1), we mean a sequence $\{y_n\}$ that is defined for $n \in \mathbb{N}(-k)$ and that satisfies (1.1) for $n \in \mathbb{N}$. If $a_{-k}, a_{-k+1}, \ldots, a_0$ are $k+1$ given constants, then (1.1) has a unique solution satisfying the initial conditions

$$x_i = a_i \quad \text{for } i \in \mathbb{N}(-k, 0). \tag{1.6}$$

For the sake of convenience, throughout, we use the convention

$$\sum_{n=i}^{j} r_n \equiv 0, \quad \text{whenever } j \le i-1. \tag{1.7}$$

2. Proof of Theorem 1.1. Let $\{y_n\}$ be a solution of (1.1) with initial conditions y_{-k}, $y_{-k+1}, \ldots, y_0 \in (0,1)$. Then clearly, $y_n \in (0,1)$ for all $n \in \mathbb{N}(-k)$. By introducing the substitution

$$x_n = \ln \frac{y_n}{1-y_n}, \quad n \in \mathbb{N}(-k), \tag{2.1}$$

we obtain

$$\Delta x_n + r_n f(x_{n-k}) = 0, \quad n \in \mathbb{N}, \tag{2.2}$$

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in (-\infty, \infty), \tag{2.3}$$

where

$$\Delta x_n = x_{n+1} - x_n, \qquad r_n = \frac{1}{2}\beta_n, \qquad f(x) = 2 - \frac{4}{e^x+1}. \tag{2.4}$$

It is easy to see that

$$f(0) = 0, \qquad x f(x) > 0 \quad \forall x \in \mathbb{R}, \tag{2.5}$$

$$f'(x) = \frac{4e^x}{(e^x+1)^2} \quad \forall x \in \mathbb{R}. \tag{2.6}$$

Thus, f is increasing, we also have

$$f'(x) < \frac{4e^x}{(2\sqrt{e^x})^2} = 1 \quad \text{for } x \neq 0, \tag{2.7}$$

which implies that

$$|f(x)| < |x| \quad \text{for } x \neq 0. \tag{2.8}$$

Define h as follows

$$h(x) = \max\{f(x), -f(-x)\} \quad \text{for } x > 0. \tag{2.9}$$

We have from (2.5), (2.8), and the increasing property of f that $h(x)$ is increasing in $[0,\infty)$, and

$$|f(x)| \leq h(|x|) < |x| \quad \text{for } x \neq 0. \tag{2.10}$$

We will now prove that

$$\lim_{n\to\infty} x_n = 0. \tag{2.11}$$

There are two cases to consider.

Case 1. The sequence $\{x_n\}$ is eventually nonnegative or eventually nonpositive. We assume that $\{x_n\}$ is eventually nonnegative, then there exists an integer $n_0 \in \mathbb{N}(k)$ such that $x_{n-k} \geq 0$ for all $n \in \mathbb{N}(n_0)$. By (2.2), we have $\Delta x_n \leq 0$ for all $n \in \mathbb{N}(n_0)$ and there exists $a \geq 0$ such that

$$\lim_{n\to\infty} x_n = a. \tag{2.12}$$

If $a > 0$, by the increasing property of f, it follows that

$$\Delta x_n \leq -r_n f(a) \quad \forall n \in \mathbb{N}(n_0+k). \tag{2.13}$$

Summing (2.13) from n_0+k to $n-1$ and using (1.4), we have

$$x_n - x_{n_0+k} \leq -f(a) \sum_{i=n_0+k}^{n-1} r_i \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty, \tag{2.14}$$

which contradicts (2.12). The case when $\{x_n\}$ is eventually nonpositive can be dealt with similarly.

Case 2. The sequence $\{x_n\}$ is oscillatory. By (1.3) and (2.4), then there exists an integer $n^* \in \mathbb{N}(2k)$ such that

$$\sum_{i=n-k}^{n} r_i \leq \alpha = \frac{3}{2} + \frac{1}{2(k+1)}, \quad n \in \mathbb{N}(n^*-2k), \tag{2.15}$$

$$x_{n^*-1} x_{n^*} \leq 0, \quad x_{n^*} \neq 0. \tag{2.16}$$

By virtue of the choice of n^*, there exists a real number $\lambda \in [0,1)$ such that

$$x_{n^*-1} + \lambda(x_{n^*} - x_{n^*-1}) = 0. \tag{2.17}$$

Let l be a positive constant such that

$$\max_{n\in\mathbb{N}(n^*-2k-1,n^*-1)} |x_n| \le l. \tag{2.18}$$

By (2.2), (2.10), (2.18), and the increasing property of h, we have

$$|\Delta x_n| \le r_n h(l), \quad n\in\mathbb{N}(n^*-1,n^*+k-1). \tag{2.19}$$

Which, together with (2.17), implies that

$$\begin{aligned} |x_{n-k}| &= |x_{n-k}-x_{n^*-1}-\lambda(x_{n^*}-x_{n^*-1})| \\ &= \left| -\sum_{j=n-k}^{n^*-2}\Delta x_j - \lambda\Delta x_{n^*-1}\right| \\ &\le h(l)\left(\sum_{j=n-k}^{n^*-2} r_j+\lambda r_{n^*-1}\right), \quad n\in\mathbb{N}(n^*-1,n^*+k-1). \end{aligned} \tag{2.20}$$

In view of (2.2), (2.10), and (2.20), we obtain

$$|\Delta x_n| \le r_n h(l)\left(\sum_{j=n-k}^{n^*-2} r_j+\lambda r_{n^*-1}\right), \quad n\in\mathbb{N}(n^*-1,n^*+k-1). \tag{2.21}$$

Now we show that

$$|x_n| \le h(l) \quad \forall n\in\mathbb{N}(n^*,n^*+k). \tag{2.22}$$

There are two possible cases to consider.

Case 1. Suppose that $d=\sum_{i=n^*}^{n^*+k-1} r_i+(1-\lambda)r_{n^*-1}\le 1$. By (2.15), (2.17), and (2.21) we have for $n\in\mathbb{N}(n^*,n^*+k)$

$$\begin{aligned} |x_n| &= |x_n-x_{n^*-1}-\lambda(x_{n^*}-x_{n^*-1})| \\ &= \left|\sum_{i=n^*}^{n-1}\Delta x_i+(1-\lambda)\Delta x_{n^*-1}\right| \\ &\le \sum_{i=n^*}^{n^*+k-1} r_i h(l)\left(\sum_{j=i-k}^{n^*-2} r_j+\lambda r_{n^*-1}\right)+(1-\lambda)r_{n^*-1}h(l)\left(\sum_{j=n^*-k-1}^{n^*-2} r_j+\lambda r_{n^*-1}\right) \\ &= h(l)\sum_{i=n^*}^{n^*+k-1} r_i\left[\sum_{j=i-k}^{i} r_j-\sum_{j=n^*}^{i} r_j-(1-\lambda)r_{n^*-1}\right] \\ &\quad + h(l)(1-\lambda)r_{n^*-1}\left[\sum_{j=n^*-k-1}^{n^*-1} r_j-(1-\lambda)r_{n^*-1}\right] \end{aligned}$$

$$
\begin{aligned}
&\le h(l)\left[\alpha d-\sum_{i=n^*}^{n^*+k-1} r_i\sum_{j=n^*}^{i} r_j-(1-\lambda)r_{n^*-1}d\right]\\
&=h(l)\left[\alpha d-\frac{1}{2}\left(\sum_{i=n^*}^{n^*+k-1} r_i\right)^2-\frac{1}{2}\sum_{i=n^*}^{n^*+k-1} r_i^2-(1-\lambda)r_{n^*-1}d\right]\\
&=h(l)\left[\alpha d-\frac{1}{2}d^2-\frac{1}{2}\left(\sum_{i=n^*}^{n^*+k-1} r_i^2+(1-\lambda)^2r_{n^*-1}^2\right)\right].
\end{aligned}
\tag{2.23}
$$

Since

$$
\sum_{i=n^*}^{n^*+k-1} r_i^2+(1-\lambda)^2r_{n^*-1}^2\ge\frac{1}{k+1}\left(\sum_{i=n^*}^{n^*+k-1} r_i+(1-\lambda)r_{n^*-1}\right)^2=\frac{d^2}{k+1}. \tag{2.24}
$$

We obtain

$$
\begin{aligned}
|x_n|&\le h(l)\left[\alpha d-\left(\frac{1}{2}+\frac{1}{2(k+1)}\right)d^2\right]\\
&\le h(l)\left[\alpha-\left(\frac{1}{2}+\frac{1}{2(k+1)}\right)\right]\\
&=h(l).
\end{aligned}
\tag{2.25}
$$

Case 2. Suppose that $d=\sum_{i=n^*}^{n^*+k-1} r_i+(1-\lambda)r_{n^*-1}>1$. In this case, there exists an integer $m\in\mathbb{N}(n^*,n^*+k)$ such that

$$
\sum_{i=m}^{n^*+k-1} r_i\le 1,\qquad \sum_{i=m-1}^{n^*+k-1} r_i>1. \tag{2.26}
$$

Therefore, there is an $\eta\in(0,1]$ such that

$$
\sum_{i=m}^{n^*+k-1} r_i+(1-\eta)r_{m-1}=1. \tag{2.27}
$$

By (2.15), (2.17), (2.19), and (2.21), we have for $n\in\mathbb{N}(n^*,n^*+k)$

$$
\begin{aligned}
|x_n|&=|x_n-x_{n^*-1}-\lambda\Delta x_{n^*-1}|\\
&=\left|\sum_{i=n^*}^{n-1}\Delta x_i+(1-\lambda)\Delta x_{n^*-1}\right|\\
&=\sum_{j=n^*}^{n^*+k-1}|\Delta x_j|+(1-\lambda)|\Delta x_{n^*-1}|
\end{aligned}
$$

$$
\begin{aligned}
&= (1-\lambda)|\Delta x_{n^*-1}| + \sum_{j=n^*}^{m-2}|\Delta x_j| + \eta|\Delta x_{m-1}| + (1-\eta)|\Delta x_{m-1}| + \sum_{j=m}^{n^*+k-1}|\Delta x_j| \\
&\le h(l)\left((1-\lambda)r_{n^*-1} + \sum_{j=n^*}^{m-2} r_j + \eta r_{m-1}\right) + h(l)(1-\eta)r_{m-1}\left(\sum_{j=m-1-k}^{n^*-2} r_j + \lambda r_{n^*-1}\right) \\
&\quad + h(l)\sum_{j=m}^{n^*+k-1} r_j\left(\sum_{i=j-k}^{n^*-2} r_i + \lambda r_{n^*-1}\right) \\
&= h(l)\left[(1-\lambda)r_{n^*-1} + \sum_{j=n^*}^{m-1} r_j - (1-\eta)r_{m-1}\right] \\
&\quad + h(l)(1-\eta)r_{m-1}\left[\sum_{j=m-1-k}^{m-1} r_j - \sum_{j=n^*}^{m-1} r_j - (1-\lambda)r_{n^*-1}\right] \\
&\quad + h(l)\sum_{j=m}^{n^*+k-1} r_j\left[\sum_{i=j-k}^{j} r_i - \sum_{i=m}^{j} r_i - \sum_{i=n^*}^{m-1} r_i - (1-\lambda)r_{n^*-1}\right] \\
&\le h(l)\left[\alpha - (1-\eta)r_{m-1} - \sum_{j=m}^{n^*+k-1} r_j \sum_{i=m}^{j} r_i\right] \\
&= h(l)\left[\alpha - (1-\eta)r_{m-1} - \frac{1}{2}\left(\sum_{j=m}^{n^*+k-1} r_j\right)^2 - \frac{1}{2}\sum_{j=m}^{n^*+k-1} r_j^2\right] \\
&= h(l)\left[\alpha - (1-\eta)r_{m-1} - \frac{1}{2}\left(1-(1-\eta)r_{m-1}\right)^2 - \frac{1}{2}\sum_{j=m}^{n^*+k-1} r_j^2\right] \\
&= h(l)\left[\alpha - \frac{1}{2} - \frac{1}{2}\left(\sum_{j=m}^{n^*+k-1} r_j^2 + (1-\eta)^2 r_{m-1}^2\right)\right].
\end{aligned}
\tag{2.28}
$$

Since

$$
\sum_{j=m}^{n^*+k-1} r_j^2 + (1-\eta)^2 r_{m-1}^2 \ge \frac{1}{n^*-m+k+1}\left(\sum_{j=m}^{n^*+k-1} r_j + (1-\eta)r_{m-1}\right)^2 \ge \frac{1}{k+1}. \tag{2.29}
$$

We obtain

$$
|x_n| \le h(l)\left(\alpha - \frac{1}{2} - \frac{1}{2(k+1)}\right) = h(l). \tag{2.30}
$$

Furthermore, we can prove that

$$
|x_n| \le h(l) \quad \forall n \in \mathbb{N}(n^*). \tag{2.31}
$$

Assume, for the sake of contradiction, that (2.31) is not true. Then there exists $m_1 \in \mathbb{N}(n^*+k+1)$ such that $|x_{m_1}| > h(l)$ and $|x_n| \le h(l)$ for $n \in \mathbb{N}(n^*, m_1-1)$. Set

$$m_2 = \max\{n \in \mathbb{N}(n^*, m_1) : x_{n-1}x_n \le 0, x_n \ne 0\}. \tag{2.32}$$

In case $m_1 \le m_2 + k$. From (2.10), we have

$$\max_{n\in\mathbb{N}(m_2-2k-1, m_2-1)} |x_n| \le h(l) < l. \tag{2.33}$$

By a similar method to the proof of (2.22), we obtain

$$|x_n| \le h(l) \quad \forall n \in \mathbb{N}(m_2, m_2+k) \tag{2.34}$$

which contradicts the definition of m_1. In case $m_1 - 1 \ge m_2 + k$, it follows from the choice of m_1 and m_2 that

$$x_n > 0 \quad \text{or} \quad x_n < 0 \quad \forall n \in \mathbb{N}(m_2, m_1). \tag{2.35}$$

Assume that $x_n > 0$ for all $n \in \mathbb{N}(m_2, m_1)$. (In case $x_n < 0$, the proof is similar.) From (2.2) we have

$$\Delta x_n \le 0 \quad \text{for } n \in \mathbb{N}(m_1-1, m_1+k) \tag{2.36}$$

which implies that

$$x_{m_1} \le x_{m_1-1} \le h(l). \tag{2.37}$$

This contradicts the definition of m_1. Thus (2.31) holds.

From the argument above, we can establish a sequence $\{n_i\}$ of positive integers with $n_1 = n^*$, $n_{i+1} - n_i > 2k$ such that

$$x_{n_i-1}x_{n_i} \le 0, \quad x_{n_i} \ne 0, \tag{2.38}$$

and a sequence $\{z_i\}$ with $z_1 = l$, $z_{i+1} = h(z_i)$ such that

$$\max_{n\in\mathbb{N}(n_i-2k-1, n_i-1)} |x_n| \le z_i, \quad |x_n| \le z_{i+1} \quad \forall n \in \mathbb{N}(n_i). \tag{2.39}$$

By (2.10), we obtain

$$\lim_{i\to\infty} z_i = 0 \tag{2.40}$$

which, together with (2.39), implies that $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

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