Internat. J. Math. & Math. Sci.
Vol. 2 No. 2 (1979) 239-250

REMAINDERS OF POWER SERIES

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(Received February 21, 1978 and in Revised form March 20, 1979)

ABSTRACT. Suppose $\Sigma_{n=0}^{\infty} a_n z^n$ has radius of convergence R and $\sigma_N(z) = |\Sigma_{n=N}^{\infty} a_n z^n|$. Suppose $|z_1| < |z_2| < R$, and T is either z_2 or a neighborhood of z_2. Put $S = \{N|\ \sigma_N(z_1) > \sigma_N(z) \text{ for } z \in T\}$. Two questions are asked: (a) can S be cofinite? (b) can S be infinite? This paper provides some answers to these questions. The answer to (a) is no, even if $T = z_2$. The answer to (b) is no, for $T = z_2$ if $\lim a_n = a \neq 0$. Examples show (b) is possible if $T = z_2$ and for T a neighborhood of z_2.

KEY WORDS AND PHRASES. Power-Series, Remainders, Radius of Convergence.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 30A10.

1. INTRODUCTION.

This paper originated in a question of approximation by power series raised in Query 51 in the American Mathematical Society Notices [1]. (The query originated in considerations of analytically continuing a polynomial series from the interval [-1,1] to the region of convergence of the series.) Suppose $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ has radius of convergence R and $\sigma_N(z) = |\Sigma_{n=N}^{\infty} a_n z^n|$. Suppose $|z_1| < |z_2| < R$ and T is either z_2 or a neighborhood of z_2. Put $S = \{n | \sigma_n(z_1) > \sigma_n(z) \text{ for } z \in T\}$. S is cofinite if its complement is finite. Two questions are asked:

(a) can S be cofinite?

(b) can S be infinite?

One might expect the answer to both questions to be no since one expects the approximation to f by partial sums of its power series to be worse, closer to the circle of convergence.

This paper provides some answers to these questions. Section 2 shows (a) is impossible for any T. Section 3 shows (b) is impossible if $T = z_2$ and $\lim a_n = a \neq 0$. Section 4 shows (b) is possible for $T = z_2$ and Section 5 shows (b) is possible for T a neighborhood of z_2.

Section 5 suggests the conjecture that if T is a neighborhood of z_2, then S must be "thin." The S which appears in Section 5 is lacunary.

These questions can also be raised about other series of orthonormal polynomials with elliptic domains of convergence. (cf. Szegö [5], pp. 309-10).

2. S CANNOT BE COFINITE.

The following theorem was suggested by P. Lax [3].

THEOREM 1. If $\lim |a_n|^{1/n} = 1/R < \infty$, $0 < |z_1|$, $|z_2| < R$ and $0 < \delta < |z_2|/|z_1|$, then the set $S = \{n \mid |\Sigma_{k=n}^{\infty} a_k z_2{}^k| < \delta^n |\Sigma_{k=n}^{\infty} a_k z_1{}^k|\}$ cannot be cofinite.

PROOF. Suppose S contains a nonempty tail set τ; i.e. $n \in \tau$ implies $n+1 \in \tau$. Then for $n \in \tau$,

$$\sigma_n(z_1) \geq \sigma_{n+1}(z_1) - |a_n||z_1|^n \geq \delta^{-(n+1)}\ \sigma_{n+1}(z_2) - |a_n||z_1|^n$$

$$\geq \delta^{-(n+1)}[|a_n||z_2|^n - \sigma_n(z_2)] - |a_n||z_1|^n$$

$$\geq |a_n|\ [\delta^{-(n+1)}|z_2|^n - |z_1|^n] - \delta^{-1}\ \sigma_n(z_1)\ .$$

Hence

$$(1+\delta^{-1})\ \sigma_n(z_1) \geq |a_n|\ [\delta^{-(n+1)}|z_2|^n - |z_1|^n]\ .$$

Suppose $1/R \neq 0$. Choose $\varepsilon > 0$ so that $(R^{-1} + \varepsilon)|z_1| < 1$ and choose n τ so large that $|a_k|^{1/k} < (1/R + \varepsilon)$ for $k \geq n$. Also choose n so that $|a_n|^{1/n} > 1/R - \varepsilon$. Then

$$\frac{[(R^{-1}+\varepsilon)|z_1|]^n}{1-(R^{-1}+\varepsilon)|z\ \ |} > \Sigma_{k=n}^{\infty} |a_k||z_1|^k \geq \sigma_n(z_1)$$

$$\geq \frac{|a_n|}{1+\delta^{-1}}\ [\delta^{-(n+1)}|z_2|^n - |z_1|^n]$$

$$\geq \frac{(R^{-1}-\varepsilon)^n}{1+\delta^{-1}} [\delta^{-(n+1)}|z_2|^n - |z_1|^n]$$

$$= \frac{(R^{-1}-\varepsilon)^n}{1+\delta^{-1}} \left(\frac{|z_2|}{\delta}\right)^n \left[\delta^{-1} - \left(\frac{\delta|z_1|}{z_2}\right)^n\right] .$$

Now in addition to the other conditions on n, choose n large enough so that

$$\left(\frac{\delta|z_1|}{|z_2|}\right)^n < \delta^{-1} .$$

Then, since

$$\frac{(R^{-1}+\varepsilon)|z_1|}{[1-(R^{-1}+\varepsilon)|z_1|]^{1/n}} \geq \frac{R^{-1}-\varepsilon}{(1+\delta^{-1})^{1/n}} \frac{|z_2|}{\delta} \left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^n\right]^{1/n} ,$$

one obtains upon letting $\varepsilon \to 0$ and $n \to \infty$:

$$|z_1| \geq \frac{|z_2|}{\delta} ,$$

contradicting $\delta < |z_2|/|z_1|$.

Suppose $R^{-1} = 0$. Then $|a_n|^{1/n}$ converges to zero. If we add zero to the set, $\{|a_n|^{1/n}|n \geq 1\}$ the new set is closed and bounded and thus compact with the largest element $|a_{n_1}|^{1/n_1}$. Deleting $|a_1|$, $|a_2|^{1/2}, \ldots, |a_{n_1}|^{1/n_1}$, there is a largest element $|a_{n_2}|^{1/n_2}$ in the remaining set and so forth. Thus we obtain a sequence n_i, $i = 1,2,\ldots$, with $|a_{n_i}|^{1/n_i} = \varepsilon_i \neq 0$ and $|a_n|^{1/n} \leq \varepsilon_i$ for $n \geq n_i$. Also $\lim_{i\to\infty} \varepsilon_i = 0$. Thus for i large enough that $\varepsilon_i|z_1| < 1$:

$$\frac{[\varepsilon_i|z_1|]^{n_i}}{1-\varepsilon_i|z_1|} \geq \Sigma_{k=n_i}^{\infty} |a_k||z_1|^k \geq \sigma_{n_i}(z_1)$$

$$\geq \frac{|a_{n_i}|}{1+\delta^{-1}}\left[\delta^{-(n_i+1)} |z_2|^{n_i} - |z_1|^{n_i}\right]$$

$$= \frac{|\ {}_{n_i}|}{1+\delta^{-1}} \frac{|z_2|}{\delta}^{n_i}\left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^{n_i}\right].$$

Now choose n_i so that $(\delta|z_1|/|z_2|)^{n_i} < \delta^{-1}$. Then

$$\frac{\varepsilon_i|z_1|}{(1-\varepsilon_i|z_1|)^{1/n_i}} \geq \frac{|a_{n_i}|^{1/n_i}}{(1+\delta^{-1})^{1/n_i}} \frac{|z_2|}{\delta}\left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^{n_i}\right]^{1/n_i}$$

or

$$\frac{|z_1|}{(1-\varepsilon_i|z_1|)^{1/n_i}} \geq \frac{|z_2|}{\delta(1+\delta^{-1})^{1/n_i}}\left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^{n_i}\right]^{1/n_i} .$$

Letting $\varepsilon_i \to 0$ and $n_i \to \infty$, one obtains

$$|z_1| \geq \frac{|z_2|}{\delta} ,$$

contradicting $\delta < |z_2|/|z_1|$. This completes the proof of Theorem 1.

The following observation about general series was made by a referee. Let $\Sigma_0^{\infty} A_\mu$ be convergent. If $\Sigma_0^{\infty} \mu|b_\mu| < \infty$, then

$$S = \left\{N \,\Big|\, \left|\sum_{\mu \geq N} A_\mu\right| < \left|\sum_{\mu \geq N} A_\mu b_\mu\right|\right\}$$

is not cofinite. For let $R_n = \Sigma_{\mu \geq n} A_\mu$. Then $A_\mu = R_n - R_{n+1}$. If S were cofinite, then for $n \geq n_o$,

$$|A_\mu| \leq |R_n| + |R_{n+1}| \leq 2 \; \Sigma_{\mu \geq n} |A_\mu| \; |b_\mu|$$

or

$$\sum_{\mu \geq N} |A_\mu| \leq 2 \sum_{\mu \geq N} \sum_{\mu \geq n} |A_\mu| \; |b_\mu| \leq 2 \sum_{\mu \geq N} \mu |A_\mu| \; |b_\mu| < \infty .$$

If N_o is selected so large that $\mu |b_\mu| < 1/2$, then for $N > N_o$,

$$\sum_{\mu \geq N} |A_\mu| < 2 \, \frac{1}{2} \sum_{\mu \geq N} |A_\mu| = \sum_{\mu \geq N} |A_\mu| ,$$

which is a contradiction. If one puts

$$A_\mu = a_\mu \, z_2^\mu \; , \; b_\mu = \left(\frac{z_1}{z_2} \right)^\mu ,$$

then under the hypothesis of Theorem 1, one obtains the weaker result that the set

$$S = \left\{ n \;\middle|\; \left| \sum_{k=n}^{\infty} a_k z_2{}^k \right| < \left| \sum_{k=n}^{\infty} a_k z_1{}^k \right| \right\}$$

cannot be cofinite.

3. CASE OF $\mathrm{LIM}_{N \to \infty} A_N = A \neq 0$.

In this section it is shown that (b) is impossible for even a single point if $\lim_{n \to \infty} a_n = a \neq 0$. The proof is as follows. For $\varepsilon > 0$, N large enough, and $|z| < R = 1$

$$\sigma_N(z) = \left|\sum_{n=N}^{\infty} a_n z^n\right| = \left|a \sum_{n=N}^{\infty} z^n + \sum_{n=N}^{\infty} (a_n - a)z^n\right|$$

$$\leq |a| \frac{|z|^n}{|1-z|} + \varepsilon \frac{|z|^n}{1-|z|} .$$

Also

$$|a| \frac{|z|^N}{|1-z|} = \left|a\sum_{n=N}^{\infty} z^n\right| = \left|\sum_{n=N}^{\infty} a_n z^n + \sum_{n=N}^{\infty} (a-a_n)z^n\right|$$

$$\leq \sigma_n(z) + \varepsilon \frac{|z|^N}{1-|z|} .$$

Thus

$$|a| \frac{|z|^N}{|1-z|} - \varepsilon \frac{|z|^N}{1-|z|} \leq \sigma_N(z) \leq |a| \frac{|z|^N}{|1-z|} + \varepsilon \frac{|z|^N}{1-|z|} \quad . \tag{1}$$

Suppose $\sigma_N(z_2) < \sigma_N(z_1)$ for infinitely many N. Then (1) gives

$$|a| \frac{|z_2|^N}{|1-z_2|} - \varepsilon \frac{|z_2|^N}{1-|z_2|} \leq \sigma_N(z_2) < \sigma_N(z_1)$$

$$\leq |a| \frac{|z_1|^N}{|1-z_1|} + \varepsilon \frac{|z_1|^N}{1-|z_1|}$$

for infinitely many N. Taking Nth roots, letting $N \to \infty$, and $\varepsilon \to 0$, yields

$$|z_2| \leq |z_1| ,$$

a contradiction of $|z_1| < |z_2|$.

4. FOR $T = \{z_2\}$, (b) IS POSSIBLE.

The following example shows (b) is possible if $T = \{z_2\}$. Let

$$F(z) = (1-2z)(1-z^2)^{-1}$$

$$= 1-2z + z^2 - 2z^3 + z^4 - 2z^5 + \ldots .$$

One has:

$$\sigma_{2k}(z) = |z^{2k} - 2z^{2k+1} + z^{2k+2} - 2z^{2k+3} + \ldots |$$

$$= |z|^{2k}\ |1 - 2z + z^2 - 2z^3 + \ldots |$$

$$= |z|^{2k}\ |1 - 2z||1 - z^2|^{-1}$$

and thus $\sigma_{2k}(1/2) = 0$. So for any $z_1 \neq 1/2$ and $0 < |z_1| < 1$, $\sigma_{2k}(z_1) > \sigma_{2k}(1/2)$.

Note that for an ε-neighborhood of 1/2: $N = \{z \,|\, |z - 1/2| < \varepsilon\}$, $0 < \varepsilon < 1/2$ and for any z_1 with $|z_1| < 1/2 - \varepsilon$, $\sigma_{2k}(z_1)$ converges to zero faster than $\sigma_{2k}(z)$ at any point z in N except 1/2. So we cannot extend the result to a neighborhood of 1/2.

5. CASE OF T A NEIGHBORHOOD OF z_2.

THEOREM 2. For each R, $0 < R \leq \infty$, there exist points z_1 and z_2 with $|z_1| < |z_2| < R$ and a power series $\Sigma_{n=0}^{\infty} a_n z^n$ with radius of convergence R such that for infinitely many values of N, $\sigma_N(z_1)/3 \geq \sigma_N(z)$ for all z in some neighborhood of z_2.

PROOF. Suppose R = 1. Put $n_k = 4^k$ and $P_k(z) = (1/b_k)\ z^{n_{2k-1}}\ (z - 1/2)^{n_{2k}}$, where $b_k = \max_{0 \leq j \leq n_{2k}} \left\{ \binom{n_{2k}}{j} 2^{-j} \right\}$. The power series $\Sigma_{k=1}^{\infty} P_k(z) = \Sigma_{n=0}^{\infty} a_n z^n$ will be shown to satisfy the Theorem for R = 1 with $z_1 = -1/4$ and $z_2 = 1/2$. Note that

$$n_{2k} + n_{2k-1} < n_{2k+1} \tag{2}$$

and

$$n_{2k-1}\,(\log 4/\log 3 + 1) < n_{2k} \tag{3}$$

for all k. (2) implies that each a_n is either zero or appears exactly once as a coefficient in the expansion of some $P_k(z)$. Let j_k be the integer for which $\max_{0\le j\le n_{2k}} \left\{ \binom{n_{2k}}{j} 2^{-j} \right\}$ is obtained. Then

$$|a_{j+n_{2k-1}}|^{1/(j+n_{2k-1})} = \left(\frac{\binom{n_{2k}}{j} 2^{-j}}{\binom{n_{2k}}{j_k} 2^{-j_k}} \right)^{1/(j+n_{2k-1})} . \quad (0 \le j \le n_{2k})$$

This is less than or equal to one for all j and equal to one for $j = j_k$, which implies the radius of convergence is one.

For all z with $|z - 1/2| < 1/4$:

$$\begin{aligned} |P_{k+1}(z)| &= \frac{1}{b_{k+1}} |z|^{n_{2k+1}} |z - 1/2|^{n_{2k+2}} \\ &< \frac{1}{b_k} |z|^{n_{2k-1}} |z-1/2|^{n_{2k}} |z-1/2|^{n_{2k+2} - n_{2k}} \\ &\le |P_k(z)|\,(1/4)^{n_{2k+2} - n_{2k}} \\ &\le (1/4)\,|P_k(z)| . \end{aligned} \quad (4$$

Next, for $|z - 1/2| < 1/4$,

$$\frac{|P_k(z)|}{|P_k(-1/4)|} = |z|^{n_{2k-1}} |z-1/2|^{n_{2k}} 4^{n_{2k-1}} (4/3)^{n_{2k}}$$

$$< 4^{-n_{2k}} \, 4^{n_{2k-1}} (4/3)^{n_{2k}} \qquad (5)$$

$$= 4^{n_{2k-1}} \, 3^{-n_{2k}} < 1/4$$

by (3). Hence, for $|z - 1/2) < 1/4$,

$$\sigma_{n_{2k-1}}(z) = \left| \sum_{j=n_{2k-1}}^{\infty} a_j z^j \right| \leq \sum_{j=k}^{\infty} |P_j(z)|$$

$$\leq \left(\sum_{j=k}^{\infty} 4^{k-j} \right) |P_k(z)| \quad \text{by} \quad (4)$$

$$= (4/3)|P_k(z)| < (1/3)|P_k(-1/4)| \quad \text{by} \quad (5)$$

$$\leq (1/3) \left| \sum_{j=k}^{\infty} b_j^{-1} \, (-1/4)^{n_{2j-1}} \, (-3/4)^{n_{2j}} \right|$$

$$= (1/3) \, \sigma_{n_{2k-1}}(-1/4) \, ,$$

since all n_j's are even. This shows that the assertion holds for $z_1 = -1/4$ and $z_2 = 1/2$.

For the case $0 < R < \infty$, use the power series $\Sigma_{n=0}^{\infty} a_n (z/R)^n$. Then the result holds for $z_1 = -R/2$, $z_2 = R/2$, and the neighborhood $|z - R/2| < R/4$.

For the case $R = \infty$, let

$$b_k = (n_{2k-1})^{n_{2k-1}} \binom{n_{2k}}{j_k} 2^{-j_k} \, .$$

For $0 \leq j \leq n_{2k}$:

$$|a_{j+n_{2k-1}}|^{1/(j+n_{2k-1})} = \left(\frac{\binom{n_{2k}}{j} 2^{-j}}{(n_{2k-1})^{n_{2k-1}} \binom{n_{2k}}{j_k} 2^{-j_k}} \right)^{1/(j+n_{2k-1})}$$

$$\leq (n_{2k-1})^{-n_{2k-1}/(j+n_{2k-1})}$$

$$\leq (n_{2k})^{-n_{2k-1}/(n_{2k} + n_{2k-1})}$$

$$= (n_{2k})^{-1/5} \to 0 \ .$$

as $k \to \infty$ and hence $\overline{\lim}\ |a_n|^{1/n} = 0$. The rest of the proof follows the case $R = 1$.

6. AVERAGE REMAINDER

Suppose $\Sigma\, a_n z^n$ has a radius of convergence R. It follows from results in Pólya and Szegö [4, Part III, problems 307-310] that the geometric mean:

$$G^N(r) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\, \sigma_N(re^{i\theta})d\theta \right) \quad , \ (r < R)$$

and the pth mean, $p > 0$:

$$I_p^N\,(r) = \frac{1}{2\pi}\int_0^{2\pi} \sigma_N^p(re^{i\theta})d\,\theta \ , \ (r < R)$$

are both monotone increasing functions of r for each N and $\log G^N(r)$ and $\log I_p^N(r)$ are convex functions of $\log r$. Thus in the geometric mean sense and pth mean sense, $\sigma_N(z)$ become larger as one approaches the circle of convergence.

ACKNOWLEDGMENT

The authors thank Dr. Leon Heller of the Los Alamos Scientific Laboratory, who proposed this problem, for many discussions and contributions.

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