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STEADY VORTEX FLOWS OBTAINED FROM A CONSTRAINED VARIATIONAL PROBLEM

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We prove the existence of steady two-dimensional ideal vortex flows occupying the first quadrant and containing a bounded vortex; this is done by solving a constrained variational problem. Kinetic energy is maximized subject to the vorticity, being a rearrangement of a prescribed function and subject to a linear constraint.

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1. Introduction. In this paper, we prove the existence of steady two-dimensional ideal vortex flows occupying the first quadrant, Π_+, containing a bounded vortex. This is done by solving a constrained variational problem. Such a flow will be described by a stream function $\hat{\psi}:\Pi_+ \to \mathbb{R}$. At infinity we will have $\hat{\psi} \to -\lambda x_1 x_2$ which is the stream function for an irrotational flow with velocity field $-\lambda(x_1,x_2)$, where λ is not known a priori. The vorticity is given by $-\Delta\hat{\psi}$, where Δ is the Laplacian, and $-\Delta\hat{\psi}$ vanishes outside a bounded region. It will be shown that $\hat{\psi}$ satisfies the following semilinear partial differential equation:

$$-\Delta\hat{\psi} = \phi \circ \hat{\psi}, \tag{1.1}$$

almost everywhere in Π_+ for ϕ an increasing function, unknown a priori. In our result the vorticity function $\zeta(= -\Delta\hat{\psi})$ is a rearrangement of a prescribed nonnegative, nontrivial function ζ_0 having bounded support, and the *impulse*, $\mathfrak{I}$, given by

$$\mathfrak{I}(\zeta) := \int_{\Pi_+} x_1 x_2 \zeta, \tag{1.2}$$

is a prescribed positive number. We prove that the variational problem, $P(I)$ (see Section 2), is solvable provided that I is sufficiently large. Since the domain of interest Π_+ is unbounded, we first consider the problem over bounded sets, $\Pi_+(\xi,\eta)$, where Burton's theory, related to constrained variational problems, can be applied. We then show that the maximizers are the same for all sufficiently large $\Pi_+(\xi,\eta)$.

Problems of this kind have been investigated by many authors; in particular we cite Badiani [1], Burton [2], Burton and Emamizadeh [3], Elcrat and Miller [7], Emamizadeh [8, 9, 10, 11], Nycander [14] for theoretical results and Elcrat et al. [5, 6] for numerical.

2. Notation, definitions, and statement of the results. Henceforth p denotes a real number in $(2,\infty)$. The first quadrant is denoted Π_+. Generic points in $\mathbb{R}^2$ are denoted by x, y, and so forth. Thus, for example, $x = (x_1,x_2)$. For $x \in \mathbb{R}^2$, $\overline{x}$, $\underline{x}$, and $\underline{\overline{x}}$ denote the reflections of x about the x_1-axis, x_2-axis, and the origin, respectively. For positive η and ξ we set

$$\Pi_+(\eta) := \{x \in \Pi_+ \mid x_1x_2 < \eta\},$$
$$\Pi_+(\xi,\eta) := \{x \in \Pi_+ \mid x_1x_2 < \eta,\ \max\{x_1,x_2\} < \xi\}. \tag{2.1}$$

For $A \subset \mathbb{R}^2$, $|A|$ denotes the two-dimensional Lebesgue measure of A.

For a measurable function ζ, the strong support of ζ is defined by

$$\operatorname{supp}(\zeta) = \{x \in \operatorname{dom}(\zeta) \mid \zeta(x) > 0\}. \tag{2.2}$$

To define the rearrangement class needed for our variational problem, we fix a non-negative, nontrivial function $\zeta_0 \in L^p(\mathbb{R}^2)$ which vanishes outside a bounded set. In addition, we assume that

$$|\operatorname{supp}(\zeta_0)| = \pi a^2, \tag{2.3}$$

for some $a > 0$. We say that ζ is a rearrangement of ζ_0 if and only if

$$|\{x \mid \zeta(x) \geq \alpha\}| = |\{x \mid \zeta_0(x) \geq \alpha\}|, \tag{2.4}$$

for every positive α. The set of rearrangements of ζ_0 which vanish outside bounded subsets of Π_+ is denoted by $\mathcal{F}$. The set of functions $\zeta \in \mathcal{F}$ that satisfy $\mathfrak{I}(\zeta) = I$, for some $I > 0$, is denoted by $\mathcal{F}(I)$; and the set of functions in $\mathcal{F}(I)$ that vanish outside $\Pi_+(\xi,\eta)$ is denoted by $\mathcal{F}(\xi,\eta,I)$; to ensure that $\mathcal{F}(\xi,\eta,I) \neq \emptyset$, we present the following definition: let $I_1 := \mathfrak{I}(\zeta_0^*)$, where ζ_0^* is the Schwarz-symmetrisation of ζ_0, and assume that $I > I_1$; we say that $\Pi_+(\xi,\eta)$ satisfies the hypothesis $\mathcal{H}(I)$ if the following two conditions hold:

$$\xi \geq \eta^{1/2}, \tag{2.5}$$

$$\eta \geq 4\max\{a^2, l(I)\}, \tag{2.6}$$

where $l(I) := (I - I_1)/\|\zeta_0\|_1$. Now it is immediate that if $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$, for $I > I_1$, then $\mathcal{F}(\xi,\eta,I) \neq \emptyset$. Indeed if we set $t = l(I)^{1/2}$, then $(\zeta_0^*)_t(x) := \zeta_0^*(x_1 - t, x_2 - t)$ belongs to $\mathcal{F}(\xi,\eta,I)$.

The Green's function for $-\Delta$ on Π_+ with homogeneous Dirichlet boundary conditions is denoted by G_+, hence

$$G_+(x,y) = \frac{1}{2\pi}\log\frac{|x-\overline{y}|\,|x-\underline{y}|}{|x-y|\,|x-\underline{\overline{y}}|}. \tag{2.7}$$

Next we define the integral operator K_+

$$K_+\zeta(x) = \int_{\Pi_+} G_+(x,y)\zeta(y)\,dy, \tag{2.8}$$

for measurable functions ζ on $\mathbb{R}^2$, whenever the integral exists. The *Kinetic energy* is defined by

$$\Psi(\zeta) = \int_{\Pi_+} \zeta K_+\zeta, \tag{2.9}$$

whenever the integral exists.

In this paper, we are concerned with constrained variational problems which are defined as follows. For $I > I_1$,

$$P(I): \sup_{\zeta \in \mathcal{F}(I)} \Psi(\zeta); \tag{2.10}$$

and the corresponding solution set is denoted by $\Sigma(I)$. If $I > I_1$ and $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$, then we define the truncated variational problem

$$P(\xi,\eta,I): \sup_{\zeta \in \mathcal{F}(\xi,\eta,I)} \Psi(\zeta), \tag{2.11}$$

with the solution set $\Sigma(\xi,\eta,I)$.

We are now in a position to state our main result.

Theorem 2.1. *There exists $I_0 > 0$ such that if $I > I_0$ then $P(I)$ has a solution, that is, $\Sigma(I) \neq \emptyset$; if ζ is a solution and $\psi := K_+\zeta$ then the following semilinear elliptic partial differential equation holds*

$$-\Delta\psi = \phi \circ (\psi - \lambda x_1 x_2), \quad \text{a.e. in } \Pi_+, \tag{2.12}$$

where ϕ is an increasing function and $\lambda > 0$, both unknown a priori. Furthermore, I_0 can be chosen to ensure that the vortex core, the strong support of ζ, avoids $\partial\Pi_+$.

3. Preliminary results.

We present some lemmas that are used in the proof of Theorem 2.1. We begin by stating a lemma from Burton's theory, see for example, Burton and McLeod [4].

Lemma 3.1. *Let Ω be a nonempty open set in $\mathbb{R}^n$. Let $1 \le p < \infty$ and p^* denote the conjugate exponent of p. For $\zeta \in L^p(\mu)$ let $\mathcal{F}(\Omega)$ denote the set of rearrangements of ζ on Ω. Let*

$$\mathcal{L} := \sum_{1 \le |\alpha| \le m} \mathcal{A}^\alpha(x)\mathcal{D}^\alpha \tag{3.1}$$

be an mth-order linear partial differential operator, whose coefficients $\mathcal{A}^\alpha$ are finite-valued measurable functions on Ω, having no 0th-order term, and suppose that there exists a compact, symmetric, positive linear operator $K : L^p(\Omega) \to L^{p^*}(\Omega)$ such that if $\zeta \in L^p(\Omega)$, then $K\zeta \in L^{p^*}(\Omega) \cap W^{m,1}_{\text{loc}}(\Omega)$ and $\mathcal{L}K\zeta = \zeta$ almost everywhere in Ω. Define

$$\hat{\Psi}(\zeta) := \int_\Omega \zeta K\zeta, \quad \zeta \in L^p(\Omega). \tag{3.2}$$

Let $w \in L^{p^*}(\Omega) \cap W^{m,1}_{\text{loc}}(\Omega)$ be such that $\mathcal{L}w$ is essentially constant, and define

$$\mathcal{T}(\zeta) := \int_\Omega w\zeta, \quad \zeta \in L^p(\Omega). \tag{3.3}$$

Let $b \in \mathbb{R}$. Then

(i) *If $b \in \mathcal{T}(\mathcal{F}(\Omega))$ then*

$$\sup \hat{\Psi}(\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)) = \sup \hat{\Psi}(\mathcal{T}^{-1}(b) \cap \overline{\mathcal{F}(\Omega)^w}), \tag{3.4}$$

and the supremum is attained by at least one element of $\mathcal{T}^{-1}(b) \cap \mathcal{F}(\Omega)$.

(ii) *If b is, relatively, interior to $\mathcal{T}(\mathcal{F}(\Omega))$, and if $\overline{\zeta}$ is a maximizer for Ψ relative to $\mathcal{T}^{-1}(b)\cap\mathcal{F}(\Omega)$, then there exist scalar λ and an increasing function ϕ such that*

$$\overline{\zeta}=\phi\circ\left(K\overline{\zeta}+\lambda w\right),\quad \text{a.e. in } \Omega. \tag{3.5}$$

Remark 3.2. It is clear that if $I>I_1$ and $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$ then, by Lemma 3.1(i), $\Sigma(\xi,\eta,I)\neq\varnothing$.

Before stating the next result we give the following definition: for $I>I_1$,

$$\sigma(I):=\inf\left\{\Psi(\zeta)\mid \zeta\in\Sigma(\xi,\eta,I),\ \text{for some } \Pi_+(\xi,\eta) \text{ satisfying } \mathcal{H}(I)\right\}. \tag{3.6}$$

We point out that $\sigma(I)=\Psi(\hat{\zeta})$ for some $\hat{\zeta}\in\Sigma(\xi_0,\eta_0,I)$, where $\Pi_+(\xi_0,\eta_0)$ is the minimal region that satisfies $\mathcal{H}(I)$.

Lemma 3.3. *Let σ be as defined in (3.6), then*

$$\lim_{I\to\infty}\sigma(I)=\infty. \tag{3.7}$$

Proof. Let $I>I_1$ and set $t=l(I)^{1/2}$. If $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$, then $(\zeta_0^*)_t\in\mathcal{F}(\xi,\eta,I)$ and therefore, according to the last remark, we have

$$\sigma(I)\geq\Psi\left((\zeta_0^*)_t\right). \tag{3.8}$$

Now applying same method as in Burton [2, Lemma 12], we obtain $\Psi((\zeta_0^*)_t)\geq k\log t$, for all sufficiently large t, hence large I. Thus our claim is done. □

Let $I>I_1$ and $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$. We set

$$\begin{aligned} M(\xi,\eta,I):=\big\{(\zeta,\phi,\lambda)\mid \zeta\in\Sigma(\xi,\eta,I)&\text{ for some } \phi,\lambda\in\mathbb{R}\\ \text{such that } \zeta=\phi\circ\left(K_+\zeta-\lambda x_1x_2\right)&\text{ a.e. in } \Pi_+(\xi,\eta)\big\}. \end{aligned} \tag{3.9}$$

Note that under the conditions imposed on ξ, η, I and in view of Lemma 3.1(ii) the set $M(\xi,\eta,I)$ is nonempty. The following two inequalities are standard, see Burton [2]

$$\left|K_+\zeta(x)\right|\leq N\min\{x_1,x_2\}, \tag{3.10}$$

$$\left|\nabla K_+\zeta(x)\right|\leq N, \tag{3.11}$$

for every $x\in\Pi_+$ and every $\zeta\in\mathcal{F}$, where N is a universal constant.

Lemma 3.4. *For $I>I_1$ we define*

$$\begin{aligned} \Lambda(I):=\sup\big\{\lambda\mid(\zeta,\phi,\lambda)\in M(\xi,\eta,I)&\text{ for some } \zeta,\phi\\ \text{and some } \Pi_+(\xi,\eta)&\text{ satisfying } \mathcal{H}(I)\big\}. \end{aligned} \tag{3.12}$$

Then, $\limsup_{I\to\infty}\Lambda(I)\leq 0$.

Proof. Assume that the assertion of the lemma is not true and seek a contradiction. Hence, to this end we suppose that there exists $\beta\in(0,\infty]$ such that $\limsup_{I\to\infty}\Lambda(I)=\beta$. Hence there exists $\Lambda>0$ such that the set

$$S:=\{I\mid\Lambda(I)>\Lambda\} \tag{3.13}$$

is unbounded. Consider $I\in S$, then from the definition of $\Lambda(I)$, there exists $(\zeta,\phi,\lambda)\in$

$M(\xi,\eta,I)$ such that $\Pi_+(\xi,\eta)$ satisfies $\mathscr{H}(I)$ and $\Lambda(I) \geq \lambda > \Lambda > 0$. Observe that by taking I sufficiently large we can ensure the existence of ξ_1 such that $\Pi_+(\xi,\eta) \supseteq \Pi_+(\xi_1,a)$ and $|\Pi_+(\xi_1,a)| \geq \pi a^2 = |\operatorname{supp}(\zeta)|$. Now define the set

$$U := \{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq -\lambda a\}. \tag{3.14}$$

Then, $\Pi_+(\xi_1,a) \subseteq U$ and $|U| \geq |\operatorname{supp}(\zeta)|$. Since ζ is essentially an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi,\eta)$ we deduce that $\operatorname{supp}(\zeta) \subseteq U$.

Next we show that there exists a constant $C > 0$, independent of $I \in S$, such that for $x \in \operatorname{supp}(\zeta)$ we have $x_1 x_2 \leq C$. From (3.10) we observe that for a sufficiently large $k > 0$

$$K_+\zeta(x) \leq \frac{\Lambda}{2} x_1 x_2, \tag{3.15}$$

for all $\zeta \in \mathscr{F}$ and all x for which $\min\{x_1,x_2\} \geq k$. We next define

$$\begin{aligned} S_1 &:= \{x \in \Pi_+ \mid \min\{x_1,x_2\} \geq k\}, \\ S_2 &:= \{x \in \Pi_+ \mid \min\{x_1,x_2\} < k,\ x_1 < \alpha,\ x_2 < \alpha\}, \\ S_3 &:= \{x \in \Pi_+ \mid \min\{x_1,x_2\} < k,\ \max\{x_1,x_2\} \geq \alpha\}, \end{aligned} \tag{3.16}$$

where $\alpha := \max\{2N/\lambda, k\}$. First consider $x \in \operatorname{supp}(\zeta) \cap S_1$; then

$$-\lambda a \leq K_+\zeta(x) - \lambda x_1 x_2 \leq \frac{\Lambda}{2} x_1 x_2 - \lambda x_1 x_2 < -\frac{\lambda}{2} x_1 x_2, \tag{3.17}$$

where the first inequality follows from $\operatorname{supp}(\zeta) \subseteq U$ and the second one from (3.15); whence $x_1 x_2 < 2a$. Next, consider $x \in \operatorname{supp}(\zeta) \cap S_2$; then we have

$$x_1 x_2 < \alpha^2 \leq \left(\max\left\{\frac{2N}{\Lambda}, k\right\}\right)^2, \tag{3.18}$$

since $\lambda > \Lambda$. Finally, consider $x \in \operatorname{supp}(\zeta) \cap S_3$; then an application of (3.10) yields that

$$\begin{aligned} -\lambda a &\leq K_+\zeta(x) - \lambda x_1 x_2 \\ &\leq N \min\{x_1,x_2\} - \lambda x_1 x_2 \\ &= \frac{N}{\alpha} \alpha \min\{x_1,x_2\} - \lambda x_1 x_2 \\ &\leq \frac{N}{\alpha} x_1 x_2 - \lambda x_1 x_2 \\ &\leq N \frac{\lambda}{2N} x_1 x_2 - \lambda x_1 x_2 \\ &= -\frac{\lambda}{2} x_1 x_2, \end{aligned} \tag{3.19}$$

hence $x_1 x_2 \leq 2a$. Therefore, from above argument, it is clear that a constant $C > 0$, as required, exists. This, in turn, implies that

$$I = \mathfrak{I}(\zeta) := \int_{\Pi_+} x_1 x_2 \zeta \leq C \|\zeta_0\|_1. \tag{3.20}$$

Thus S is bounded, which is a contradiction. Hence, the proof of Lemma 3.4. □

Lemma 3.5. *For $I > I_1$ we define*

$$A(I) := \inf\Big\{ \operatorname*{ess\,inf}_{x\in \operatorname{supp}(\zeta)} (K_+\zeta(x) - \lambda x_1 x_2) \mid (\zeta,\phi,\lambda) \in M(\xi,\eta,I) \\ \text{for some } \Pi_+(\xi,\eta) \text{ and some } \phi \Big\}, \tag{3.21}$$

where $\Pi_+(\xi,\eta)$ is to satisfy $\mathcal{H}(I)$. Then, $\liminf_{I\to\infty} A(I) \geq 0$.

Proof. Fix $\epsilon > 0$. By definition of $A(I)$ there exists $\Pi_+(\xi,\eta)$, satisfying $\mathcal{H}(I)$, and $(\zeta,\phi,\lambda) \in M(\xi,\eta,I)$ such that

$$A(I) + \epsilon \geq \operatorname*{ess\,inf}_{x\in \operatorname{supp}(\zeta)} (K_+\zeta(x) - \lambda x_1 x_2). \tag{3.22}$$

Note that by increasing I, the size of $\Pi_+(\xi,\eta)$ increases as well, hence there is no loss of generality if we assume that $\Pi_+(\xi,\eta)$ contains the square $D := [0,2a]\times[0,2a]$, since I will eventually tend to infinity. For $x \in D$ we have

$$K_+\zeta(x) - \lambda x_1 x_2 \geq -4a^2\Lambda(I)^+, \tag{3.23}$$

where $\Lambda(I)^+$ denotes the positive part of $\Lambda(I)$, since $K_+\zeta$ is nonnegative. From this, we infer that

$$D \subseteq \{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq -4a^2\Lambda(I)^+\}. \tag{3.24}$$

Hence

$$\big|\{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq -4a^2\Lambda(I)^+\}\big| > |\operatorname{supp}(\zeta)|, \tag{3.25}$$

since $4a^2 > |\operatorname{supp}(\zeta)|$. Since ζ is essentially an increasing function of $K_+\zeta - \lambda x_1 x_2$ on $\Pi_+(\xi,\eta)$, we then deduce that

$$\operatorname{supp}(\zeta) \subseteq \{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1 x_2 \geq -4a^2\Lambda(I)^+\}, \tag{3.26}$$

hence, by applying (3.22), we obtain $A(I) + \epsilon \geq -4a^2\Lambda(I)^+$. Therefore, from Lemma 3.4 we have

$$\liminf_{I\to\infty} A(I) + \epsilon \geq 0. \tag{3.27}$$

Since ϵ was arbitrary, we derive the desired conclusion. □

The next two results can be proved similarly to Burton [2, Lemmas 8 and 9]; they bear some resemblance to Pohazaev-type identities proved in Friedman and Turkington [12] for 3-dimensional vortex rings. We add that, contrary to Burton [2], we can give a direct proof, using the weak divergence theorem (see, e.g., Grisvard [13]) for Lemma 3.6 below without referring to any density theorems.

Lemma 3.6. *Let $2 < p < \infty$, let $\zeta \in L^p(\Pi_+)$ have bounded support, and let $\psi := K_+\zeta$. Then*

$$\int_{\Pi_+} (x\cdot\nabla\psi)\zeta = 0. \tag{3.28}$$

Lemma 3.7. *Let $2 < p < \infty$, let $\zeta \in L^p(\Pi_+)$ be nonnegative, nontrivial and vanish outside the square $D(\xi) := [0,\xi] \times [0,\xi]$, for some $\xi > 0$. Let $\lambda \in \mathbb{R}$, and let $\psi := K_+\zeta - \lambda x_1 x_2$. Suppose that $\zeta = \phi \circ \psi$ almost everywhere in $D(\xi)$ for some increasing function ϕ, and that ϕ has a nonnegative indefinite integral F. Then*

$$2\int_{D(\xi)} F\circ\psi - 2\lambda\int_{D(\xi)} x_1x_2\zeta = \int_{\partial D(\xi)} (F\circ\psi)(x\cdot\vec{n}), \tag{3.29}$$

where $\vec{n}$ is the outward unit normal, and consequently

$$\int_{D(\xi)} F\circ\psi \geq \lambda\int_{D(\xi)} x_1x_2\zeta. \tag{3.30}$$

If additionally $F(s) = 0$ for some $s \leq \beta$, then

$$\int_{D(\xi)} \zeta K_+\zeta \geq 2\lambda\int_{D(\xi)} x_1x_2\zeta + \beta\|\zeta\|_1. \tag{3.31}$$

Lemma 3.8. *For $I > I_1$ we define*

$$\begin{aligned}\mu(I) := \inf\Big\{ &\sup_{x\in\Pi_+(\xi,\eta)} (K_+\zeta(x) - \lambda x_1x_2) \mid (\zeta,\phi,\lambda) \in M(\xi,\eta,I) \\ &\text{for some } \Pi_+(\xi,\eta) \text{ satisfying } \mathcal{H}(I), \text{ and some } \phi\Big\}.\end{aligned} \tag{3.32}$$

Then $\lim_{I\to\infty}\mu(I) = \infty$.

Proof. It clearly suffices to show that

$$\liminf_{I\to\infty}\mu(I) = \infty. \tag{3.33}$$

Let $I > I_1$ and consider $(\zeta,\phi,\lambda) \in M(\xi,\eta,I)$ for some $\Pi_+(\xi,\eta)$ satisfying $\mathcal{H}(I)$. Since $K_+\zeta(x) - \lambda x_1x_2 \geq \Lambda(I)$ for almost every $x \in \operatorname{supp}(\zeta)$, we may assume that $\phi(s) = 0$ for $-\infty < s < A(I)$. Now write

$$F(s) = \int_{-\infty}^{s} \phi, \tag{3.34}$$

for all s in the domain of ϕ. Now, by Lemma 3.7, we have

$$\begin{aligned}\int_{\Pi_+} \zeta(K_+\zeta - \lambda x_1x_2) &= 2\Psi(\zeta) - \lambda I \\ &= \frac{1}{2}(2\Psi(\zeta) - 2\lambda I - A(I)\|\zeta\|_1) + \Psi(\zeta) + \frac{1}{2}A(I)\|\zeta\|_1 \\ &\geq \Psi(\zeta) + \frac{1}{2}A(I)\|\zeta\|_1 \\ &\geq \sigma(I) + \frac{1}{2}A(I)\|\zeta\|_1.\end{aligned} \tag{3.35}$$

Hence

$$\sup_{\Pi_+(\xi,\eta)} (K_+\zeta(x) - \lambda x_1x_2) \geq \frac{\sigma(I)}{\|\zeta\|_1} + \frac{1}{2}A(I). \tag{3.36}$$

Therefore

$$\mu(I) \geq \frac{\sigma(I)}{\|\zeta\|_1} + \frac{1}{2}A(I). \tag{3.37}$$

Thus by applying Lemmas 3.4 and 3.5 we obtain (3.33). □

Lemma 3.9. *There exists $I_2 > I_1$ such that*

$$A(I) \geq aN, \quad I \geq I_2. \tag{3.38}$$

Proof. By Lemma 3.7 there exists $I_2 > I_1$ such that

$$\mu(I) \geq 7aN, \quad I \geq I_2; \tag{3.39}$$

moreover by taking I_2 sufficiently large we can ensure that if $I \geq I_2$, then any $\Pi_+(\xi,\eta)$ satisfying $\mathscr{H}(I)$, also satisfies

$$\left|\Pi_+(\xi,\eta)\setminus\Pi_+\left(\xi,\frac{\eta}{2}\right)\right| \geq \pi a^2. \tag{3.40}$$

To see it, observe that in general we have

$$|\Pi_+(\xi,\eta)| = \eta\left(1+\log\frac{\xi^2}{\eta}\right), \tag{3.41}$$

for any $\Pi_+(\xi,\eta)$ satisfying (2.5); therefore

$$\left|\Pi_+(\xi,\eta)\setminus\Pi_+\left(\xi,\frac{\eta}{2}\right)\right| \geq \frac{1}{2}(1-\log 2)\eta. \tag{3.42}$$

Hence, in view of (2.6), for sufficiently large I we derive (3.40). Now, fix $I \geq I_2$ and consider $(\zeta,\phi,\lambda) \in M(\xi,\eta,I)$ for some $\Pi_+(\xi,\eta)$ satisfying $\mathscr{H}(I)$. Since $K_+\zeta - \lambda x_1x_2 \in C(\overline{\Pi_+(\xi,\eta)})$, it attains its maximum at, say, $z \in \overline{\Pi_+(\xi,\eta)}$. Now from the definition of $\mu(I)$ and (3.10) we infer that

$$\mu(I) \leq K_+\zeta(z) - \lambda z_1z_2 \leq N\min\{z_1,z_2\} - \lambda z_1z_2; \tag{3.43}$$

and applying (3.39), we obtain

$$7aN \leq N\min\{z_1,z_2\} - \lambda z_1z_2. \tag{3.44}$$

Clearly, if $\lambda \geq 0$ we obtain $\min\{z_1,z_2\} \geq 7a$. If $\lambda < 0$, then

$$7aN \leq N\min\{z_1,z_2\} - \lambda\eta, \tag{3.45}$$

or

$$N\min\{z_1,z_2\} \geq 7aN + \lambda\eta. \tag{3.46}$$

Now we consider two cases.

Case 1. When $\lambda\eta \geq -2aN$, then $N\min\{z_1,z_2\} \geq 5aN$, hence $\min\{z_1,z_2\} \geq 5a$. Therefore, when $\lambda \geq 0$, or when $\lambda < 0$, and $\lambda\eta \geq -2aN$ we find that $\min\{z_1,z_2\} \geq 5a$. Thus $\Pi_+(\xi,\eta)$ must contain at least a quadrant of $B_{4a}(z)$, denoted by Q. For $x \in Q$, by the mean value inequality, we have

$$\begin{aligned} K_+\zeta(x) - \lambda x_1x_2 &\geq K_+\zeta(x) \\ &\geq K_+\zeta(z) - 4aN \\ &= K_+\zeta(z) - \lambda z_1z_2 - 4aN + \lambda z_1z_2 \\ &\geq \mu(I) - 4aN + \lambda z_1z_2 \\ &\geq \mu(I) - 4aN + \lambda\eta \\ &\geq 7aN - 4aN - 2aN \\ &= aN. \end{aligned} \tag{3.47}$$

This means that

$$Q \subseteq \{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1x_2 \geq aN\}. \tag{3.48}$$

Case 2. When $\lambda\eta < -2aN$, then for $x \in \Pi_+(\xi,\eta) \setminus \Pi_+(\xi,\eta/2)$ we have

$$\begin{gathered} K_+\zeta(x) - \lambda x_1x_2 \geq -\lambda x_1x_2 > -\frac{\lambda\eta}{2}; \\ \Pi_+(\xi,\eta) \setminus \Pi_+\left(\xi,\frac{\eta}{2}\right) \subset \{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1x_2 \geq aN\}. \end{gathered} \tag{3.49}$$

From (3.40) and the fact that $|Q| = 4\pi a^2$, we infer that

$$\left|\{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1x_2 \geq aN\}\right| \geq \left|\operatorname{supp}(\zeta)\right|. \tag{3.50}$$

Since ζ is an increasing function of $K_+\zeta - \lambda x_1x_2$ on $\Pi_+(\xi,\eta)$, we derive

$$\operatorname{supp}(\zeta) \subseteq \{x \in \Pi_+(\xi,\eta) \mid K_+\zeta(x) - \lambda x_1x_2 \geq aN\}, \tag{3.51}$$

modulo a set of zero measure, from which we obtain (3.38). □

Lemma 3.10. *Let $b > 0$, let $2 < p < \infty$ and $0 < \gamma < 1$. Then there exist positive constants M_1, M_2, and M_3 such that*

$$\begin{aligned} K_+\zeta(x) \leq{}& M_1(x_1x_2)^{-1}\mathfrak{I}(\zeta) + M_2(x_1x_2)^{-1}\mathfrak{I}(\zeta)\log\frac{25x_1x_2}{4|x|} \\ &+ M_3(x_1x_2)^{-\gamma}\mathfrak{I}(\zeta)^{\gamma}\|\zeta\|_p^{1-\gamma}, \end{aligned} \tag{3.52}$$

for every $x \in \Pi_+$ such that $\min\{x_1x_2\} \geq b/2$ and every nonnegative $\zeta \in L^p(\Pi_+)$ that vanishes outside a set of measure πb^2.

Proof. Fix $x \in \Pi_+$ such that $\nu := \min\{x_1x_2\} \geq b/2$. For $y \in \Pi_+$ we define

$$\alpha := |x - \overline{y}|, \qquad \beta := |x - \underline{y}|, \qquad \rho := |x - y|, \qquad \delta := |x - \underline{\overline{y}}|. \tag{3.53}$$

Thus

$$\begin{aligned} K_+\zeta(x) &= \frac{1}{2\pi}\int_{\Pi_+}\log\frac{\alpha\beta}{\rho\delta}\zeta(y)\,dy \\ &= \frac{1}{2\pi}\int_{B_{\nu/2}(x)}\log\frac{\alpha\beta}{\rho\delta}\zeta(y)\,dy \\ &\quad + \frac{1}{2\pi}\int_{\Pi_+\setminus B_{\nu/2}(x)}\log\frac{\alpha\beta}{\rho\delta}\zeta(y)\,dy, \end{aligned} \tag{3.54}$$

where $B_{\nu/2}(x)$ denotes the ball centered at x with radius ν. From the identity

$$\alpha^2\beta^2 = \rho^2\delta^2 + 16x_1x_2y_1y_2, \tag{3.55}$$

we obtain

$$\begin{aligned} \int_{\Pi_+\setminus B_{\nu/2}(x)}\log\frac{\alpha\beta}{\rho\delta}\zeta(y)\,dy &= \frac{1}{2}\int_{\Pi_+\setminus B_{\nu/2}(x)}\log\left(1+\frac{16x_1x_2y_1y_2}{\rho^2\delta^2}\right)\zeta(y)\,dy \\ &\le 8x_1x_2\int_{\Pi_+\setminus B_{\nu/2}(x)}\frac{y_1y_2}{\rho^2\delta^2}\zeta(y)\,dy \\ &\le \frac{32x_1x_2}{\nu^2|x|^2}\int_{\Pi_+\setminus B_{\nu/2}(x)}y_1y_2\zeta(y)\,dy \\ &\le 32(x_1x_2)^{-1}\mathfrak{I}(\zeta), \end{aligned} \tag{3.56}$$

where the first inequality follows from the fact that $\log(1+x)\le x$, for $x\ge 0$. To estimate $\int_{B_{\nu/2}(x)}\log(\alpha\beta\rho^{-1}\delta^{-1})\zeta(y)\,dy$, we note that for $y\in B_{\nu/2}(x)$ we have

$$\alpha \le |x-\overline{x}| + |\overline{x}-\overline{y}| = 2x_2+\rho < \frac{5}{2}x_2. \tag{3.57}$$

Similarly, $\beta < 5/2x_1$. Therefore

$$\begin{aligned} \int_{B_{\nu/2}(x)}\log\frac{\alpha\beta}{\rho\delta}\zeta(y)\,dy &\le \int_{B_{\nu/2}(x)}\log\frac{25x_1x_2}{4\rho|x|}\zeta(y)\,dy \\ &= \log\frac{25x_1x_2}{4|x|}\int_{B_{\nu/2}(x)}\zeta(y)\,dy \\ &\quad + \int_{B_{\nu/2}(x)}\log\frac{1}{\rho}\zeta(y)\,dy. \end{aligned} \tag{3.58}$$

Observe that for $y\in B_{\nu/2}(x)$ we have $y_1y_2\ge x_1x_2/4$, hence

$$\int_{B_{\nu/2}(x)}\zeta(y)\,dy \le 4(x_1x_2)^{-1}\int_{B_{\nu/2}(x)}y_1y_2\zeta(y)\,dy \le 4(x_1x_2)^{-1}\mathfrak{I}(\zeta). \tag{3.59}$$

On the other hand, if we let $\hat{\zeta}$ denote the Schwarz-symmetrisation of $\overline{\zeta} := \zeta\chi_{B_{\nu/2}(x)}$, where $\chi_{B_{\nu/2}(x)}$ is the characteristic function of $B_{\nu/2}(x)$ in Π_+, about x; then by a standard inequality (see, e.g., [3]) and Hölder's inequality we obtain

$$\begin{aligned}\int_{B_{\nu/2}(x)} \log\frac{1}{\rho}\zeta(y)dy &\le \int_{B_{\nu/2}(x)} \log\frac{1}{\rho}\hat{\zeta}(y)dy \\ &\le \left(\int_{B_{\hat{b}}(x)} \left|\log\frac{1}{\rho}\right|^{\tau} dy\right)^{1/\tau} \|\hat{\zeta}\|_{\epsilon},\end{aligned} \tag{3.60}$$

where $\hat{b} := |\operatorname{supp}(\zeta\chi_{B_{\nu/2}(x)})|$ $(\le b)$, $\epsilon := p/(1+p\gamma-\gamma)$ and τ is the conjugate exponent of ϵ. It is elementary to show that

$$\int_{B_{\hat{b}}(x)} \left|\log\frac{1}{\rho}\right|^{\tau} dy \le C, \tag{3.61}$$

where C is a constant independent of x. Next observe that $\epsilon = \epsilon\gamma + (1-\epsilon\gamma)p$ and $\epsilon\gamma < 1$, hence applying the standard interpolation inequality yields

$$\|\hat{\zeta}\|_{\epsilon}^{\epsilon} \le \|\hat{\zeta}\|_1^{\epsilon\gamma}\|\hat{\zeta}\|_p^{(1-\epsilon\gamma)p}, \tag{3.62}$$

or

$$\|\hat{\zeta}\|_{\epsilon} \le \|\hat{\zeta}\|_1^{\gamma}\|\hat{\zeta}\|_p^{(1-\epsilon\gamma)p/\epsilon} = \|\hat{\zeta}\|_1^{\gamma}\|\hat{\zeta}\|_p^{1-\gamma}. \tag{3.63}$$

Therefore, we obtain

$$\|\hat{\zeta}\|_{\epsilon} \le 4^{\gamma}(x_1x_2)^{-\gamma}\mathfrak{I}(\zeta)^{\gamma}\|\zeta\|_p^{1-\gamma}. \tag{3.64}$$

Finally from (3.56), (3.58), (3.60), (3.61), and (3.64) we derive (3.52). □

By a simple modification of Burton [2, Lemma 1] we get the following lemma.

Lemma 3.11. *Let ζ be a nonnegative measurable function on Π_+, let $t > 0$. Let ζ_t be the function, defined on Π_+, obtained by translating ζ along the diagonal of Π_+, $\operatorname{diag}(\Pi_+)$, $\sqrt{2}t$ units, that is,*

$$\zeta_t(x_1,x_2) := \begin{cases}\zeta(x_1-t,x_2-t), & x_1 \ge t,\ x_2 \ge t\\ 0, & 0 < x_1 < t,\ 0 < x_2 < t.\end{cases} \tag{3.65}$$

Then

$$\int_{\Pi_+} \zeta_t K_+ \zeta_t \ge \int_{\Pi_+} \zeta K_+ \zeta. \tag{3.66}$$

Lemma 3.12. *Let $2 < p < \infty$ and $\zeta \in L^p(\Pi_+)$ be a nonnegative, nontrivial function which vanishes outside $\Pi_+(h)$ for some $h > 0$. Then*

$$K_+\zeta(x) \le \frac{4hx_1x_2}{\pi|x_1^2 - x_2^2|}\|\zeta\|_1 + N\min\{x_1,x_2\}, \tag{3.67}$$

provided that $x \in \Pi_+ \setminus \operatorname{diag}(\Pi_+)$.

Proof. Fix $x \in \Pi_+ \setminus \operatorname{diag}(\Pi_+)$ and define

$$U(x) := \left\{ y \in \Pi_+ \mid |(y_1^2 - y_2^2) - (x_1^2 - x_2^2)| < |x_1^2 - x_2^2|^{1/2} \right\}. \tag{3.68}$$

Next we decompose ζ as follows: $\zeta := \zeta_1 + \zeta_2$, where

$$\zeta_1(y) := \begin{cases} \zeta(y), & y \in \Pi_+(h) \cap U(x), \\ 0, & \text{otherwise}. \end{cases} \tag{3.69}$$

Again by setting $\alpha := |x - \overline{y}|$, $\beta := |x - \underline{y}|$, $\rho := |x - y|$, $\delta := |x - \underline{\overline{y}}|$, we obtain

$$\begin{aligned} K_+\zeta_2(x) &= \frac{1}{4\pi} \int_{\Pi_+} \log \frac{\alpha^2\beta^2}{\rho^2\delta^2} \zeta_2(y)\,dy \\ &= \frac{1}{4\pi} \int_{\Pi_+} \log\left(1 + \frac{16x_1x_2y_1y_2}{\rho^2\delta^2}\right) \zeta_2(y)\,dy \\ &\le \frac{4hx_1x_2}{\pi} \int_{\Pi_+ \setminus U(x)} \frac{1}{\rho^2\delta^2} \zeta_2(y)\,dy. \end{aligned} \tag{3.70}$$

In view of the following identity:

$$\rho^2\delta^2 = \left((y_1^2 - y_2^2) - (x_1^2 - x_2^2)\right)^2 + 4(x_1x_2 - y_1y_2)^2, \tag{3.71}$$

we infer that if $y \in \Pi_+ \setminus U(x)$, then $\rho^2\delta^2 > |x_1^2 - x_2^2|$. This, in conjunction with (3.70), yields

$$K_+\zeta_2(x) \le \frac{4hx_1x_2}{\pi|x_1^2 - x_2^2|} \|\zeta\|_1. \tag{3.72}$$

Finally, recalling (2.5) we obtain

$$K_+\zeta_1(x) \le N \min\{x_1, x_2\}. \tag{3.73}$$

Since $K_+\zeta(x) = K_+\zeta_1(x) + K_+\zeta_2(x)$, (3.67) follows from (3.72) and (3.73). □

Remark 3.13. Under the hypotheses of Lemma 3.12 with b replaced by a and an additional assumption, namely, $\mathfrak{I}(\zeta) \ge 1$ we can show the existence of a positive constant P such that

$$K_+\zeta(x) \le P(x_1x_2)^{-\gamma}\mathfrak{I}(\zeta), \tag{3.74}$$

provided that $\min\{x_1, x_2\} \ge a/2$ and $\zeta \in \mathcal{F}$. Clearly, the truth of (3.74) emerges from the elementary fact that $s^{\gamma-1}\log s$ is bounded on any interval of the form $[d, \infty)$, $d > 0$.

4. Proof of Theorem 2.1. We first show that, for I sufficiently large, there exists a positive constant $R(I)$ such that if $\Pi_+(\xi,\eta)$ is sufficiently large (satisfying $\mathscr{H}(I)$) and $\zeta \in \Sigma(\xi,\eta,I)$, then

$$\operatorname{supp}(\zeta) \subset \Pi_+(R(I)), \tag{4.1}$$

modulo a set of zero measure. From Lemma 3.3, there exists $I_3 > I_1$ such that if $I > I_3$, then

$$\sigma(I) > \frac{5}{2}aN\|\zeta_0\|_1. \tag{4.2}$$

Fix $I > I_3$ and consider $\zeta \in \Sigma(\xi,\eta,I)$ for some $\Pi_+(\xi,\eta)$ satisfying $\mathscr{H}(I)$. From (4.2) and definition of σ, we infer that

$$\frac{5}{2}aN\|\zeta\|_1 \le \Psi(\zeta) \le \frac{1}{2}\|\zeta\|_1 \sup_{x\in\operatorname{supp}(\zeta)} K_+\zeta(x), \tag{4.3}$$

thus

$$\sup_{x\in\operatorname{supp}(\zeta)} K_+\zeta(x) \ge 5aN. \tag{4.4}$$

Since $K_+\zeta \in C(\mathbb{R}^2)$, it attains its maximum relative to $\overline{\operatorname{supp}(\zeta)}$ at z, say. Therefore, by applying (4.4), we obtain

$$5aN \le K_+\zeta(z) \le N\min\{z_1,z_2\}, \tag{4.5}$$

whence $\min\{z_1,z_2\} \ge 5a$. Without loss of generality, we may assume that $\mathfrak{I}(\zeta) \ge 1$, hence, by (3.74) we obtain

$$5aN \le K_+\zeta(z) \le PI(z_1z_2)^{-\gamma}, \tag{4.6}$$

so

$$z_1z_2 \le \left(\frac{PI}{5aN}\right)^{\gamma}. \tag{4.7}$$

Now we define

$$R(I) := \max\left\{\left(\frac{PI}{5aN}\right)^{\gamma}, 25a^2\right\}. \tag{4.8}$$

Then $V := \{x \in \Pi_+ \mid x_1x_2 \le R(I), \min\{x_1,x_2\} \ge 5a\}$ is not empty and $z \in V$. Note that at least a quadrant of $B_{4a}(x)$, for every $x \in V$, is contained in $\Pi_+(R(I))$ and, in fact, contained in $\Pi_+(\xi_1,R(I))$ for some $\xi_1^2 > R(I)$. By $\Pi_+^t(\xi_1,R(I))$ we denote the translation of $\Pi_+(\xi_1,R(I))$ along $\operatorname{diag}(\Pi_+)$, $\sqrt{2}t$ units. Observe that the family of translations $\{\Pi_+^t(\xi_1,R(I))\}_{0\le t\le t_0}$, where $t_0 := (I/\|\zeta_0\|_1)^{1/2}$, is uniformly contained in $\Pi_+(\xi_2,\eta_2)$, for some ξ_2 and η_2 (in fact we can take $\xi_2 = \xi_1 + t_0$). From now on we assume that $\xi > \xi_2$ and $\eta > \eta_2$. Since a quadrant of $B_{4a}(z)$, designated by Q, is contained in $\Pi_+(R(I))$ we can apply the mean value inequality and (2.5) to deduce that

$$K_+\zeta(x) \ge K_+\zeta(z) - 4aN \ge aN, \quad x \in Q, \tag{4.9}$$

where the last inequality is obtained from (4.4). To seek a contradiction we assume that $E := \mathrm{supp}(\zeta)\setminus\Pi_+(R(I))$ has a positive measure and write $\zeta = \zeta_0+\zeta_1$, where

$$\zeta_1 := \zeta\chi_E. \tag{4.10}$$

Since $|Q| = 4\pi a^2 > |\mathrm{supp}(\zeta)| = \pi a^2$, there exists a measure preserving bijection, denoted by T, from E onto a subset of $Q\setminus\mathrm{supp}(\zeta)$, say G, see Royden [15]. Now define

$$\zeta_2 := \zeta_1\circ T^{-1}, \tag{4.11}$$

on the range of T and zero elsewhere, that is,

$$\zeta_2 = (\zeta_1\circ T^{-1})\chi_{\mathrm{im}(T)}, \tag{4.12}$$

where $\mathrm{im}(T)$ is the range of T, and let $\zeta' := \zeta_0+\zeta_2$. Clearly $\zeta'\in\mathcal{F}(\xi,\eta)$. We show that $\mathfrak{I}(\zeta') < \mathfrak{I}(\zeta)$:

$$\begin{aligned}\mathfrak{I}(\zeta') &= \int_{\Pi_+} x_1x_2\zeta_0+\int_{\Pi_+} x_1x_2\zeta_2\\ &= \int_{\Pi_+} x_1x_2\zeta_0+\int_{\Pi_+} x_1x_2\zeta_1\circ T^{-1}\\ &= \int_{\Pi_+} x_1x_2\zeta_0+\int_{E} (x_1x_2\circ T)\zeta_1\\ &< \int_{\Pi_+} x_1x_2\zeta_0+\int_{\Pi_+} x_1x_2\zeta_1\\ &= \mathfrak{I}(\zeta).\end{aligned}\tag{4.13}$$

On the other hand, we have

$$\Psi(\zeta')-\Psi(\zeta) = \int_{\Pi_+}(\zeta_2-\zeta_1)K_+\zeta+\Psi(\zeta_2-\zeta_1) > \int_{\Pi_+}(\zeta_2-\zeta_1)K_+\zeta, \tag{4.14}$$

since K_+ is strictly positive, see Emamizadeh [10]. Hence

$$\begin{aligned}\Psi(\zeta')-\Psi(\zeta) &> \int_{\Pi_+}\zeta_2K_+\zeta-\int_{\{x\in\Pi_+\mid x_1x_2>R(I)\}}\zeta_1K_+\zeta\\ &\geq aN\int_{\Pi_+}\zeta_2-\int_{\{x\in\Pi_+\mid x_1x_2>R(I)\}}\zeta_1K_+\zeta,\end{aligned}\tag{4.15}$$

by (4.9). Now we proceed to estimate $\int_{\{x\in\Pi_+\mid x_1x_2>R(I)\}}\zeta_1K_+\zeta$. For this purpose we set

$$\mathrm{supp}(\zeta) = J_1\cup J_2, \tag{4.16}$$

where

$$\begin{aligned}J_1 &:= \Big\{x\in\mathrm{supp}(\zeta)\mid x_1x_2>R(I),\ \min\{x_1,x_2\}\geq\frac{a}{2}\Big\},\\ J_2 &:= \Big\{x\in\mathrm{supp}(\zeta)\mid x_1x_2>R(I),\ \min\{x_1,x_2\}<\frac{a}{2}\Big\}.\end{aligned}\tag{4.17}$$

If $x \in J_1$, then by (3.74)

$$K_+\zeta(x) \le PI(x_1x_2)^{-\gamma} \le PIR(I)^{-\gamma}. \tag{4.18}$$

On the other hand, if $x \in J_2$ then by (2.5)

$$K_+\zeta(x) \le N\min\{x_1, x_2\} \le \frac{a}{2}. \tag{4.19}$$

Therefore, if $x \in \operatorname{supp}(\zeta_1)$

$$K_+\zeta(x) \le \max\left\{PIR(I)^{-\gamma}, \frac{aN}{2}\right\}. \tag{4.20}$$

Assume that $R(I)$ is large enough to ensure

$$aN - \max\left\{PIR(I)^{-\gamma}, \frac{aN}{2}\right\} > 0. \tag{4.21}$$

Therefore, we obtain

$$\Psi(\zeta') - \Psi(\zeta) \ge \left(aN - \max\left\{PIR(I)^{-\gamma}, \frac{aN}{2}\right\}\right)||\zeta_1||_1 > 0. \tag{4.22}$$

This implies that $\Psi(\zeta') > \Psi(\zeta)$. Finally, we define ζ'' to be the function obtained by translating ζ' along $\operatorname{diag}(\Pi_+)$ so that $\Im(\zeta'') = I$. If we denote the amount of translation by t, then it is clear that t is the bigger root of the following algebraic equation:

$$||\zeta'||_1 t^2 + 2\left(\int_{\Pi_+}(x_1+x_2)\zeta'\right)t + \int_{\Pi_+} x_1x_2\zeta' = I. \tag{4.23}$$

Note that t depends on ζ; but we are able to find a uniform bound, independent of ζ, as follows. Solving (4.23) for t yields

$$\begin{aligned} t &= \frac{-\int_{\Pi_+}(x_1+x_2)\zeta' + \left(\left(\int_{\Pi_+}(x_1+x_2)\zeta'\right)^2 - ||\zeta'||_1(\Im(\zeta') - I)\right)^{1/2}}{||\zeta'||_1} \\ &< (||\zeta'||_1(I - \Im(\zeta')))^{1/2} < \left(\frac{I}{||\zeta'||_1}\right)^{1/2}, \end{aligned} \tag{4.24}$$

as desired. Note that the choices of ξ_2 and η_2 ensure that $\zeta'' \in \mathcal{F}(\xi, \eta, I)$. Now, by Lemma 3.11 we have

$$\Psi(\zeta'') \ge \Psi(\zeta') > \Psi(\zeta). \tag{4.25}$$

This is a contradiction to the maximality of ζ. Therefore we have been able to show that if $I > I_3$, then there exists $R(I)$ given by (4.8) such that if $\Pi_+(\xi, \eta)$ is sufficiently large ($\xi \ge \xi_2$ and $\eta \ge \eta_2$) and $\zeta \in \Sigma(\xi, \eta, I)$, then, for almost every $x \in \operatorname{supp}(\zeta)$, (4.1) holds.

However, the possibility that the vortex core runs off to infinity, as $\Pi_+(\xi, \eta)$ exhausts Π_+, still exists. We now show that this situation is ruled out once I is sufficiently large. For this purpose, fix $I > I_3$ and consider $(\zeta, \phi, \lambda) \in M(\xi, \eta, I)$. We claim that if ξ and

η are large enough then λ can not be too negative. For this purpose let $\xi \geq \xi_2$ and $\eta \geq \max\{h, \eta_2\}$, ξ_2 and η_2 are as above, where

$$h := \left(N|\lambda^*|^{-1} + 1\right)R(I), \quad \lambda^* := -\frac{aN}{3R(I)}, \tag{4.26}$$

such that $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$. We show that

$$\lambda > \lambda^*. \tag{4.27}$$

To seek a contradiction suppose that $\lambda \leq \lambda^*$. Without loss of generality we may assume that $R(I) \geq 1$. Let $x \in W := \{y \in \Pi_+(\xi,\eta) \mid y_1y_2 > h\}$. Then

$$\begin{aligned} K_+\zeta(x) - \lambda x_1x_2 &> -\lambda x_1x_2 = |\lambda|x_1x_2 > |\lambda|h \\ &= |\lambda|\left(N|\lambda^*|^{-1} + 1\right)R(I) > (N + |\lambda|)R(I). \end{aligned} \tag{4.28}$$

Now consider $x \in \operatorname{supp}(\zeta)$. If $\max\{x_1,x_2\} \geq 1$, then $\min\{x_1,x_2\} \leq x_1x_2$, hence $\min\{x_1,x_2\} \leq R(I)$. If, however, $\max\{x_1,x_2\} < 1$ then $\min\{x_1,x_2\} < 1 \leq R(I)$. Therefore in either case we have $\min\{x_1,x_2\} \leq R(I)$. This, in turn, implies that

$$K_+\zeta(x) - \lambda x_1x_2 \leq N\min\{x_1,x_2\} - \lambda x_1x_2 < (N + |\lambda|)R(I), \tag{4.29}$$

whence

$$\sup_{x\in\operatorname{supp}(\zeta)} \left(K_+\zeta(x) - \lambda x_1x_2\right) \leq (N + |\lambda|)R(I). \tag{4.30}$$

Therefore $K_+\zeta(x) - \lambda x_1x_2$ takes greater values on a nonempty subset of $\Pi_+(\xi,\eta)$, namely W, than its supremum on $\operatorname{supp}(\zeta)$. This is impossible, since ζ is essentially an increasing function of $K_+\zeta(x) - \lambda x_1x_2$ on $\Pi_+(\xi,\eta)$. Hence we derive (4.27). For the rest of the proof we fix $I > I_0 := \max\{I_1, I_2, I_3\}$. Let $\xi > \xi_2$, $\eta > h$ (as above) be such that $\Pi_+(\xi,\eta)$ satisfies $\mathcal{H}(I)$. Consider $(\zeta,\phi,\lambda) \in M(\xi,\eta,I)$. Now fix $x \in \operatorname{supp}(\zeta) \setminus \operatorname{diag}(\Pi_+)$ such that $\min\{x_1,x_2\} < a/6$. Then by Lemmas 3.9 and 3.12, in conjunction with (4.27),

$$\begin{aligned} aN &\leq K_+\zeta(x) - \lambda x_1x_2 \\ &\leq \frac{4R(I)x_1x_2}{\pi|x_1^2 - x_2^2|}\|\zeta\|_1 + N\min\{x_1,x_2\} - \lambda^* x_1x_2 \\ &\leq \frac{4R(I)x_1x_2}{\pi|x_1^2 - x_2^2|}\|\zeta\|_1 + N\min\{x_1,x_2\} - \lambda^* R(I) \\ &\leq \frac{4R(I)x_1x_2}{\pi|x_1^2 - x_2^2|}\|\zeta\|_1 + \frac{aN}{6} + \frac{aN}{3}. \end{aligned} \tag{4.31}$$

Hence

$$|x_1^2 - x_2^2| < \frac{8R(I)\|\zeta_0\|_1}{a\pi N}. \tag{4.32}$$

To summarise, we have shown that if $x \in \operatorname{supp}(\zeta)$ is such that $\min\{x_1,x_2\} > a/6$, then $x \in \Pi_+(R(I)) \cap \{y \in \Pi_+ \mid \min\{y_1,y_2\} > a/6\}$; otherwise x satisfies (4.32). This clearly concludes the existence part of the theorem.

Now consider $\zeta \in \Sigma(I)$. Then there exists $\hat{\xi} > 0$ such that $\overline{\operatorname{supp}(\zeta)}$ is a compact subset of $D(\hat{\xi}) := (0,\hat{\xi}) \times (0,\hat{\xi})$ and, according to Lemma 3.1,

$$\zeta = \phi \circ (K_+\zeta - \lambda x_1 x_2), \quad \text{a.e. in } D(\hat{\xi}), \tag{4.33}$$

for some increasing function ϕ and $\lambda \in \mathbb{R}$. Note that from Lemma 3.9

$$\kappa := \operatorname{ess\,sup}\{K_+\zeta(x) - \lambda x_1 x_2 \mid x \in \operatorname{supp}(\zeta)\} \geq aN > 0. \tag{4.34}$$

Since the level sets of $K_+\zeta - \lambda x_1 x_2$, on $\operatorname{supp}(\zeta)$, have zero measure, in particular we have

$$\left| \{x \in \operatorname{supp}(\zeta) \mid K_+\zeta - \lambda x_1 x_2 = \kappa\} \right| = 0. \tag{4.35}$$

Therefore

$$K_+\zeta - \lambda x_1 x_2 > \kappa, \quad \text{a.e. in } \operatorname{supp}(\zeta). \tag{4.36}$$

Thus we may suppose that $\phi(s) = 0$ for $s \leq \kappa$. Now if we define $F(s) := \int_0^s \phi(t)\,dt$, then Lemma 3.7 yields

$$2\int_{D(\hat{\xi})} F \circ \psi - 2\lambda I = \int_{\partial D(\hat{\xi})} (F \circ \psi)(x \cdot \vec{n}), \tag{4.37}$$

where $\psi := K_+\zeta - \lambda x_1 x_2$. We claim that for $x \in \partial D(\hat{\xi})$ we have $\psi \leq \kappa$. Otherwise, by the continuity of ψ we can find $B_\epsilon(x)$ such that $B_\epsilon(x) \cap \operatorname{supp}(\zeta)$ has positive measure, since $\overline{\operatorname{supp}(\zeta)}$ is a compact subset of $D(\hat{\xi})$, and $\psi(s) > \kappa$ for $s \in B_\epsilon(x)$; but this is a contradiction to (4.33). Therefore, if $x \in \partial D(\hat{\xi})$ we have $F \circ \psi(x) = 0$. Hence from (4.37) we deduce that $\lambda > 0$, as required.

Now fix $x \in \operatorname{supp}(\zeta)$. Since $\lambda > 0$, we can employ Lemma 3.9 to obtain

$$aN \leq K_+\zeta(x) - \lambda x_1 x_2 < K_+\zeta(x) \leq N \min\{x_1, x_2\}. \tag{4.38}$$

Thus $\min\{x_1, x_2\} \geq a$. This proves the vortex core avoids $\partial\Pi_+$. The validity of (2.12) is established as in Emamizadeh [11]. □

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