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A COVERING THEOREM FOR ODD TYPICALLY-REAL FUNCTIONS

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ABSTRACT. An analytic function $f(z) = z + a_2z^2 + \ldots$ in $|z| < 1$ is typically-real if $Im\, f(z) Im\, z \geq 0$. The largest domain G in which each odd typically-real function is univalent (one-to-one) and the domain $\bigcap f(G)$ for all odd typically real functions f are obtained.

KEY WORDS AND PHRASES. *Typically-real functions, domain of univalence, covering theorems.*

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1. INTRODUCTION.

An analytic function $f(z) = z + a_2z^2 + \ldots$ in the unit disk E ($|z| < 1$) is in the class T of typically-real functions if and only if there exists a nondecreasing function γ on $[0,\pi]$ such that $\gamma(\pi) = 1$, $\gamma(0) = 0$, and

$$f(z) = \int_0^\pi \frac{z d\gamma(t)}{1 - 2z \cos t + z^2}, \tag{1.1}$$

[1]. The function γ when normalized on $(0,\pi)$ by $\gamma(t) = (\gamma(t+) + \gamma(t-))/2$ is uniquely determined by f.

The domain of univalence of the class T is known [2] to be

$$G = \{z : |z - i| < \sqrt{2}\} \cap \{z : |z + i| < \sqrt{2}\} . \tag{1.2}$$

Brannon and Kirwan [3] proved that the largest domain contained in f(G) for every function in T is $|w| < 1/4$.

In this paper we obtain the corresponding results for the class T_0 of odd typically-real functions. Recently Goodman [4] determined the largest domain that is contained in f(E) for every $f \in T$. The analog of this result for the class T_0 is an open problem.

2. The domain of univalence of T_0

THEOREM 2.1. The domain of univalence for T_0 is the domain G of (1.2).

PROOF. Since $T_0 \subset T$, each $f \in T_0$ is univalent in G. The theorem is established, therefore, if we can show that there is a function $f \in T_0$ that is not univalent in any domain D that properly contains G. Let $f(z) = z(1 + z^2)/(1 - z^2)^2 = \frac{1}{2}z/(1 - z)^2 + \frac{1}{2}z/(1 + z)^2$. This function is clearly in T_0 since T is a linear class. The function

$$\zeta = \frac{2z}{1 + z^2} \tag{2.1}$$

maps G onto $|\zeta| < 1$. By the change of variables (2.1), the function f has the form

$$f(z) = \tfrac{1}{2}z/(1 - 2z + z^2) + \tfrac{1}{2}z/(1 + 2z + z^2)$$

$$= \tfrac{1}{4}\zeta/(1 - \zeta) + \tfrac{1}{4}\zeta/(1 + \zeta) = \tfrac{1}{2}\zeta/(1 - \zeta^2) \quad .$$

Since $\zeta/(1 - \zeta^2)$ is not univalent in any domain that properly contains $|\zeta| < 1$, we conclude that f is not univalent in any domain that properly contains G.

3. A covering theorem for T_0

THEOREM 3.1. The largest domain U contained in f(G) for every $f \in T_0$ is the domain that includes the origin, is bounded in the right half-plane by $w = \rho e^{i\theta}$, where

$$\rho = \begin{cases} (\cos\theta)/2 , & 0 \le |\theta| \le \pi/4, \\ 1/(4|\sin\theta|) , & \pi/4 < |\theta| \le \pi/2, \end{cases}$$

and is symmetric relative to the imaginary axis.

PROOF. By,(1.1) we have

$$-f(-z) = \int_0^\pi \frac{z d\gamma(t)}{1 + 2z\cos t + z^2} = \int_0^\pi \frac{z d[1 - \gamma(\pi - \tau)]}{1 - 2z\cos\tau + z^2} .$$

If $f(z) = -f(-z)$, then by the uniqueness of γ we have $\gamma(t) = 1 - \gamma(\pi - t)$ for $t \in [0,\pi]$. In particular, $\gamma(\pi/2) = 1/2$. For $f \in T_0$, therefore,

$$f(z) = \int_0^{\pi/2} \frac{z d\gamma(t)}{1 - 2z\cos t + z^2} + \int_{\pi/2}^{\pi} \frac{z d\gamma(t)}{1 - 2z\cos t + z^2}$$

$$= \int_0^{\pi/2} \frac{z d\gamma(t)}{1 - 2z\cos t + z^2} + \int_{\pi/2}^{0} \frac{z d\gamma(\pi - t)}{1 - 2z\cos(\pi - t) + z^2}$$

$$= \int_0^{\pi/2} \left[\frac{z}{1 - 2z\cos t + z^2} + \frac{z}{1 + 2z\cos t + z^2}\right] d\gamma(t) .$$

By the change of variables (2.1), we obtain

$$f(z) = \int_0^{\pi/2} \frac{\zeta\, d\gamma(t)}{1 - \zeta^2\cos^2 t} , \quad \gamma(\pi/2) = 1/2, \ \gamma(0) = 0 . \tag{3.1}$$

Let $z \in \partial G$. By (2.1) we have that the corresponding ζ is on the unit circle $|\zeta| = 1$. For fixed $\zeta = e^{i\theta}$, $-\pi < \theta \le \pi$, the function $2f(z)$ is by (3.1) in the closed convex hull H of the circular arc $w(s) = e^{i\theta}/(1 - se^{2i\theta})$, $s \in [0,1]$. For each λ, $0 \le \lambda \le 1$, the point $\lambda e^{i\theta} + (1 - \lambda)i/\sin\theta$ is on the linear portion of H. Let $D(\lambda)$ denote the square of the distance from such a point to the origin. If $\theta \ne 0,\pi$, we have

$$D(\lambda) = |\lambda e^{i\theta} + \tfrac{1}{2}(1 - \lambda)i/\sin\theta|^2 = 1 + (1-\lambda^2)/(4\sin^2\theta) \quad .$$

This function of λ has a minimum at $\lambda = 1 - 2\sin^2\theta$ and $0 \leq \lambda \leq 1$ provided $|\sin\theta| \leq \sqrt{2}/2$. Thus, the distance from any point of H to the origin is not less than $[D(1 - \sin^2\theta)]^{\frac{1}{2}} = |\cos\theta|$ when $0 < |\theta| \leq \pi/4$ or $3\pi/4 \leq |\theta| < \pi$. For other $\theta \in (-\pi,\pi]$ the distance from any point of H to the origin is $\min\{1, 1/(2|\sin\theta|)\} = 1/(2|\sin\theta|)$ for $\pi/4 < |\theta| < 3\pi/4$ and 1 for $\theta = 0$ or π. Since the convex hull H contains for each $z \in \partial G$ the values of $2f(z)$ for all $f \in T_0$ and since every point of H is the value of $2f(z)$ for some $f \in T_0$, we conclude that U is the exact domain covered by all $f \in T_0$ when $z \in G$.

REFERENCES

1. Robertson, M. S., On the coefficients of a typically-real function, Bull. Amer. Math. Soc., 41 (1935), 565-572.

2. Merkes, E. P., On typically-real functions in a cut plane, Proc. Amer. Math. Soc., 10 (1959), 863-868.

3. Brannan, D. A. and Kirwan, W. E., A covering theorem for typically-real functions, Glasgow Math. J., 10 (1969), 153-155.

4. Goodman, A. W., The domain covered by a typically-real function, Proc. Amer. Math. Soc., 64 (1977), 233-237.