ON LOCALLY DIVIDED INTEGRAL DOMAINS
AND CPI-OVERRINGS

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ABSTRACT. It is proved that an integral domain R is locally divided if and

only if each CPI-extension of R (in the sense of Boisen and Sheldon) is R-flat

(equivalently, if and only if each CPI-extension of R is a localization of

R) . Thus, each CPI-extension of a locally divided domain is also locally

divided. Treed domains are characterized by the going-down behavior of their

CPI-extensions. A new class of (not necessarily treed) domains, called CPI-

closed domains, is introduced. Examples include locally divided domains,

quasilocal domains of Krull dimension 2, and quasilocal domains with the QQR-

property. The property of being CPI-closed behaves nicely with respect to the

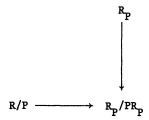D + M construction, but is not a local property.

1. INTRODUCTION.

In [3], Boisen and Sheldon recently introduced the notion of a CPI-extension

of a (commutative integral) domain. For the reader's convenience, we recall

the definition of this type of overring and summarize salient results from [3]

at the beginning of section 2. Boisen and Sheldon [3, p. 729] have noted that

a proper CPI-extension may be integral. Indeed, by combining [6, Corollary 2.6]

and [7, Proposition 2.1], it follows that each CPI-extension of a domain R is

integral if and only if R is a quasilocal going-down ring, in the sense of [5].

Since a proper integral overring cannot be flat [14, Proposition 2], one might

expect a rather different class of domains R to be characterized by the property

of having each CPI-extension of R being R-flat. The main result of section 2,

Theorem 2.4, establishes that the domains thus characterized are the locally

divided domains introduced in [6]. Prüfer domains are perhaps the most natural

examples of locally divided domains. It was shown in [6] that any locally divided

domain is a (not necessarily quasilocal) going-down ring, and that the converse

holds in the root-closed case. As byproducts, Proposition 2.3 establishes that

a CPI-extension T of a domain R is R-flat (if and) only if T is a local-

ization (in the sense of "ring of fractions" in [4, p. 57]) of R ; and Coro-

llary 2.6 establishes that each CPI-extension of a locally divided domain is

also locally divided. The final result in section 2 shows how going-down behavior

of CPI-extensions serves to characterize treed domains.

Divided domains are precisely the domains which coincide with each of their

CPI-extensions. More generally, section 3 is devoted to studying domains R

which are "CPI-closed," in the sense that each CPI-extension of a CPI-extension

of R is itself a CPI-extension of R . (A more useful characterization of

CPI-closed domains is given in Proposition 3.2.) Examples of CPI-closed domains

include the locally divided domains; the quasilocal domains of Krull dimension 2;

and quasilocal domains with the QQR-property (see Corollary 3.3, Remark 3.5 and

Proposition 3.7, respectively). Despite expectations raised by the first-cited

example, the second family of examples illustrates that a CPI-closed domain

need not be treed. The quasilocal, treed CPI-closed domains are characterized

in Proposition 3.9 and Remark 3.10(a). As shown by Example 3.6, being CPI-closed

is not a local property, since the ring of polynomials in two variables over a

field is not CPI-closed. Examples of non-CPI-closed domains with arbitrary

Krull dimension exceeding 1 then result from the D + M-construction (see

Proposition 3.11). In Remarks 3.8 and 3.10(b), (c), we raise some open questions

relating divided domains, domains with the QQR-property, certain types of CPI-

closed domains, and the Δ-domains of Gilmer and Huckaba [10].

2. LOCALLY DIVIDED DOMAINS.

As defined by Boisen and Sheldon [3], a CPI-extension of the domain R is

an overring of R of the form $R + PR_P$, for some prime ideal P of R . The

terminology "CPI" stands for "complete pre-image," and is well chosen inasmuch

as $R + PR_P$ is easily shown to be (canonically isomorphic to) the pullback, in

the category of commutative rings with unit, of the diagram

$$
\begin{array}{ccc}
 & & R_P \\
 & & \downarrow \\
R/P & \longrightarrow & R_P/PR_P
\end{array}
$$

where the vertical map is the canonical surjection and the horizontal map is the

inclusion. Reasoning as in [11, Remark 3.9], we find that applying the contra-

variant functor Spec to the above diagram produces a pushout diagram in the

(dual) category of affine schemes. In particular, $Spec(R + PR_p)$ may be viewed set-theoretically as the quotient space of the disjoint union of $Spec(R_p)$ and $Spec(R/P)$ in which the prime PR_p of R_p is identified with the zero prime P/P of R/P. We next summarize some more precise order-theoretic information gleaned from [3, Section 2]. The method used in the proof of [6, Lemma 2.4] to study the case in which $R + PR_p$ is integral over R also extends to yield the same facts.

LEMMA 2.1. (Boisen and Sheldon). Let $T = R + PR_p$ be the CPI-extension of the domain R corresponding to the prime P of R. Then the contraction map $Spec(T) \to Spec(R)$ sets up an isomorphism of partially ordered sets (with respect to inclusion) between $Spec(T)$ and the set of primes of R which are comparable to P. If Q is a prime of R which is contained in P, then Q corresponds to $QR_p \in Spec(T)$. If Q is a prime of R which contains P, then Q corresponds to $Q + PR_p \in Spec(T)$.

The next result collects some facts which will be used below.

LEMMA 2.2. Let P and Q be comparable primes of the domain R. Let T be the CPI-extension of R with respect to P, i.e., $T = R + PR_p$. Then:

(a) If $Q \subset P$, then $T_{QR_p} = R_Q$.

(b) If $P \subset Q$, then $T_{R \backslash Q} = R_Q + PR_p = T_Q + PR_p$.

PROOF. (a): As noted in Lemma 2.1, $QR_p \cap R = Q$ if $Q \subset P$. Hence, $R_Q \subset T_{QR_p}$. For the reverse inclusion, observe $T \subset R_p$, so that $T_{QR_p} \subset (R_p)_{QR_p} = R_Q$, the final step coming from [8, Corollary 4.3].

(b): Let $P \subset Q$. As the inclusions $T_{R \backslash Q} \subset T_Q + PR_p$ and $R_Q + PR_p \subset T_{R \backslash Q}$ are obvious, it remains only to prove that $T_Q + PR_p \subset R_Q + PR_p$. To this end, consider a typical element x of the left hand side. Now, $x = t(r + pz^{-1})^{-1}$ for some $t \in T$, $r \in R \backslash Q$, $p \in P$ and $z \in R \backslash P$. As $T = R + PR_p$ and PR_p

is the set of nonunits of R_P , it follows that, in order to prove $x \in R_Q + PR_P$,

we may assume $t \in R$. A short calculation reveals that $x = tr^{-1} + y$, when

$y = -ptr^{-1}(rz + p)^{-1}$. Since $tr^{-1} \in R_Q$ and $y \in PR_P$, the proof is complete.

Before presenting the main result of this section (Theorem 2.4), we isolate

a fragment which is of some independent interest.

PROPOSITION 2.3. Let $T = R + PR_P$ be the CPI-extension of the domain R

corresponding to the prime P of R . Then T is a flat R-module if and only

if T is a localization of R .

PROOF. The "if" half is immediate, as any localization is flat [4, Theorem 1,

p. 68]. Conversely, let T be R-flat. As any domain (in particular, T) is the

intersection of its localizations at maximal ideals and the maximal ideals of

T are classified in Lemma 2.1, [14, Theorem 2] yields $T = \cap R_M$, where M ranges

over the set of maximal ideals of R which contain P . If $S = R \backslash (\cup M)$, it is

clear that $R_S \subset T$, and so it suffices to prove that $T \subset R_S$.

To that end, let $t \in T$, and consider the conductor $I = \{x \in R: xt \in R\}$.

Note that, for each maximal ideal M of R which contains P , we have $I \not\subset M$

since $T \subset R_M$. If $t \notin R_S$, then $I \cap S = \phi$ and, by Zorn's lemma, I is

contained in a prime ideal N which is also disjoint from S . If P + N were

contained in a maximal ideal Q of R , then Q would be an M of the above

type (since $P \subset Q$) , whence $I \not\subset Q$ by the above argument, although $I \subset N \subset Q$.

Consequently, $P + N = R$, and so $p + n = 1$, for some $p \in P$ and $n \in N$.

Since $N \subset R \backslash S$, there exists a maximal ideal W of R such that $P \subset W$ and

$n \in W$. Then

$$1 = p + n \in P + W \subset W + W = W$$

the desired contradiction, to complete the proof.

Recall from [6] that a domain R is said to be <u>divided</u> in case $P = PR_P$

for each prime ideal P of R . (This notion was introduced in [1] by Akiba who termed such objects "AV-domains.") Any divided domain is quasilocal (cf. [1, Theorem 1]). A domain R is called <u>locally divided</u> if R_M is divided for each maximal ideal M of R (equivalently, by [6, Remark 2.7(b)], for each prime M of R). As noted in [6, Remark 2.7(b)], any localization of a locally divided domain is itself locally divided; and any locally divided domain is a going-down ring, in the sense of [5, p. 448]. (While the converse of the preceding assertion is false [6, Example 2.9], any root-closed going-down ring must be a locally divided domain [6, Corollary 2.8].) Thus, by either [1, Theorem 1] or [5, Theorem 2.2], any locally divided domain R is <u>treed</u>, in the sense that no maximal ideal of R contains incomparable prime ideals.

THEOREM 2.4. For any domain R , the following six conditions are equivalent:

(1) R is locally divided;

(2) For each prime P of R , the CPI-extension $R + PR_P$ is R-flat;

(3) For each prime P of R , the CPI-extension $R + PR_P$ is a localization of R ;

(4) For any comparable primes $P \subset Q$ of R , the corresponding CPI-extensions satisfy $R + PR_P \subset R + QR_Q$;

(5) For any comparable primes $P \subset Q$ of R , the containment $PR_P \subset PR_Q$ holds;

(6) For any comparable primes $P \subset Q$ of R , the containment $PR_P \subset R_Q$ holds.

PROOF. (5) $\Leftrightarrow$ (1): Establishing (1), i.e., that R_Q is a divided domain for each maximal (equivalently, for each prime) ideal Q of R , amounts to showing, for each prime P of R which is contained in Q , that PR_Q coincides with $(PR_Q)(R_Q)_{PR_Q}$. By the result cited in the proof of Lemma 2.2 (a), the latter prime is just $(PR_Q)R_P$ which, since $R_Q \subset R_P$, simplifies to PR_P .

As $P \subset Q$ also forces $PR_Q \subset PR_P$, it is now clear that (5) $\Leftrightarrow$ (1).

(1) $\Rightarrow$ (2): Let $T = R + PR_P$ for some prime P of R . To show that T is R-flat, it is enough, by [4, Proposition 15, p. 91], to establish that $T_{R \setminus M}$ is R_M-flat for each maximal ideal M of R . If $P \subset M$, then Lemma 2.2(b) gives $T_{R \setminus M} = R_M + PR_P$, while the hypothesis (1) [with the aid of the implication (1) $\Rightarrow$ (5) established above] leads to $PR_P = PR_M$, so that $T_{R \setminus M} = R_M + PR_M = R_M$ which is, indeed, R_M-flat. Finally, in case $P \not\subset M$, the hypothesis (1) is not needed:

$$T_{R \setminus M} = R_M + PR_P R_M = R_M + R_M R_P = R_M R_P = (R_M)_{R \setminus P} \; ,$$

the second equality arising since $PR_M = R_M$. Thus, if $P \not\subset M$, then $T_{R \setminus M}$ is a localization of (and, hence, is flat over) R_M .

(2) $\Leftrightarrow$ (3): Apply Proposition 2.3.

(2) $\Rightarrow$ (4): Assume, by (2), that the primes $P \subset Q$ of R induce R-flat CPI-extensions $S = R + PR_P$ and $T = R + QR_Q$. As in the proof of Proposition 2.3, $S = \cap R_M$ and $T = \cap R_N$ where M (resp., N) ranges over the set of maximal ideals of R which contain P (resp., Q) . Since $P \subset Q$, each ideal (index) of the form N is also of type M , whence $S \subset T$, as desired.

Finally, observe that the implications (4) $\Rightarrow$ (6) and (6) $\Rightarrow$ (5) are straightforward, and the proof is complete.

The remainder of this section analyzes further the nonreversible implications

$$\text{locally divided domain} \Rightarrow \text{going-down ring} \Rightarrow \text{treed domain.}$$

(The simplest known example of a treed domain which is not a going-down ring is due to W.J. Lewis and appears in [12, Example 6.4].) As motivation for the next result, note that the examples in [5, Corollary 4.4(ii)] combine with

[6, Lemma 2.2(b)] to show that, for each positive integer n , there exists

a divided domain which has Krull dimension n and possesses an overring which

is not treed.

PROPOSITION 2.5. Let T be an R-flat overring of a domain R . If R

is locally divided (resp., a going-down ring; resp., treed), then T has the

corresponding property.

PROOF. It suffices to prove that T_N has the appropriate property for

each maximal ideal N of T . Let $M = N \cap R$. As R_M inherits the appro-

priate property from R and R-flatness of T forces $R_M = T_N$ [14, Theorem 2],

the proof is complete.

COROLLARY 2.6. Let $T = R + PR_P$ be the CPI-extension of a domain R

arising from a prime P of R . If R is locally divided (resp., a going-

down ring; resp., treed), then T has the corresponding property.

PROOF. If R is locally divided, then the implication (1) $\Rightarrow$ (2) in Theorem

2.4 assures that T is R-flat, and so Proposition 2.5 applies, showing that

T is also locally divided.

The assertions about going-down rings and treed domains are not corollaries

of Proposition 2.5, but are instead simple consequences of Lemma 2.1. Details

may be safely omitted.

It should be noted that it is possible to give a proof of the assertion

in Corollary 2.6 concerning locally divided domains without recourse to the

notion of flatness. Indeed, one may show that T inherits from R the pro-

perty of satisfying condition (5) in Theorem 2.4. There are three cases to

consider, determined by the nature of the containments between the prime defining

T and the primes figuring in (5). Each case is treated with the aid of appro-

priate parts of Lemma 2.2. Details are left to the reader.

In view of the example of Lewis mentioned above, it is somewhat interesting that treed domains may be characterized by appropriate going-down phenomena: as Proposition 2.7 indicates, the CPI-overrings hold the key. Another consequence of Proposition 2.7 is that, if the "flat" property in condition (2) of Theorem 2.4 is weakened to the "going-down" property, then the class of domains thereby chatacterized is enlarged from the locally divided domains to the treed domains. Finally, we note that $(2) \Rightarrow (1)$ in Proposition 2.7 may be viewed as a generalization of [1, Theorem 1].

PROPOSITION 2.7. For any domain R, the following are equivalent:

(1) R is treed;

(2) $R \subset T$ has the going-down property for each CPI-extension T of R.

PROOF. $(1) \Rightarrow (2)$. Suppose that $T = R + PR_P$ for some prime P of R. Verifying that $R \subset T$ has the going-down property amounts to showing that whenever $P_2 \subset P_1$ are primes of R and Q_1 is a prime of T such that $Q_1 \cap R = P_1$, then there exists a prime Q_2 of T such that both $Q_2 \subset Q_1$ and $Q_2 \cap R = P_2$. Given Q_1, we infer from Lemma 2.1 that P_1 is comparable to P. To produce a satisfactory Q_2 _via_ the recipe in Lemma 2.1, it is enough to show that P_2 is also comparable to P. As $P_2 \subset P_1$, we may suppose that $P_1 \not\subset P$, in which case $P \subset P_1$. Then, if R is treed, P_2 and P must be comparable, for they both lie within (any maximal ideal containing) P_1.

$(2) \Rightarrow (1)$: Let P and Q be prime ideals of R which are each contained in a given maximal ideal M of R. Let $T = R + PR_P$, the CPI-extension of R with respect to P. Recall that $Q_1 = M + PR_P$ is a prime ideal of T such that $Q_1 \cap R = M$. As $Q \subset M$, the requirement that $R \subset T$ have the going-down property produces a prime Q_2 of T such that $Q_2 \cap R = Q$ (and $Q_2 \subset Q_1$). Since Q is in the image of the contraction map $\text{Spec}(T) \to \text{Spec}(R)$, Lemma 2.1

yields the comparability of Q and P , to complete the proof.

3. CPI-CLOSED DOMAINS.

The definition of this section's main object of study is motivated by the observation that it is possible to iterate the process of taking CPI-extensions. More precisely, we shall say that a domain R is CPI-closed if each CPI-extension of each CPI-extension of R is itself a CPI-extension of R . Trivally, any divided domain is CPI-closed. Indeed, so is any locally divided domain, but the proof of this and the furnishing of more examples must await the next two results.

LEMMA 3.1. Let P and Q be comparable primes of a domain R , and let $T = R + PR_P$, the CPI-extension of R with respect to P . Let W be the prime of T such that $W \cap R = Q$. (By Lemma 2.1, $W = QR_P$ if $Q \subset P$, and $W = Q + PR_P$ if $P \subset Q$.) Then the CPI-extension of T with respect to W is $R + PR_P + QR_Q$.

PROOF. One need merely simplify $T + WT_W$ with the help of the appropriate part of Lemma 2.2. Details are left to the reader.

As an immediate consequence, we have:

PROPOSITION 3.2. A domain R is CPI-closed if and only if, whenever P and Q are comparable primes of R , the sum of the CPI-extension of R with respect to P and the CPI-extension of R with respect to Q is itself a CPI-extension of R .

COROLLARY 3.3. Any locally divided domain is CPI-closed.

PROOF. Observe that condition (4) of Theorem 2.4 implies the criterion given in Proposition 3.2.

Similar reasoning also establishes the next result.

COROLLARY 3.4. If a domain R is such that the set of CPI-extensions of R is totally ordered with respect to inclusion, then R is CPI-closed.

As usual, dim(R) will denote the Krull dimension of R .

REMARK 3.5. Each of the following two conditions is sufficient for a domain R to be CPI-closed:

(a) $\dim(R) \leq 1$;

(b) R is quasilocal and $\dim(R) = 2$.

Accordingly, a CPI-closed domain need not be treed; a treed CPI-closed domain need not be a going-down ring; and a CPI-closed going-down ring need not be locally divided.

PROOF. The fact that (a) and (b) are each sufficient follows, as the criterion in Proposition 3.2 is easily seen to be satisfied in these cases. (Moreover, (a) is a special case of Corollary 3.3.) As for the final assertion; consider the (quasilocal) nontreed domain R in [13, Example 2.28]; the (quasilocal) example of Lewis [12, Example 6.4] of a treed domain which is not a going-down ring; and the (quasilocal) going-down ring which is not a divided domain, constructed by Boisen and Sheldon [2, Example 1.6] and studied further in [6, Example 2.9]. By (b), each of these three rings is CPI-closed, completing the proof.

We pause to present an example of a domain which is not CPI-closed. By combining Remark 3.5(b) and the observations to be made in Example 3.6, we shall see that a domain R need not be CPI-closed even when R_P is CPI-closed for each prime P of R .

EXAMPLE 3.6. Let X and Y be algebraically independent indeterminates over a field K . Then the polynomial ring R = K[X, Y] is not CPI-closed.

PROOF. Deny. Then, if we consider the comparable primes P = XR and Q = XR + YR , Proposition 3.2 supplies a prime W so that

$$R + PR_P + QR_Q = R + WR_W .$$

As $WR_W \cap (R + PR_P)$ is a prime of $R + PR_P$ whose intersection with R is W , Lemma 2.1 shows that W and P are comparable. Similarly, W and Q are

comparable. Thus, since Q is maximal and has height 2 , either W = P or
W = Q .

Assume that W = P . Then

$$Y(1 + Y + X)^{-1} = f + Xgh^{-1}$$

for some elements f, g and h of R such that $h \notin W$. Crossmultiply and
substitute X = 0 . If F_0 denotes the result of substituting X = 0 into a
polynomial F , the result may be written as

$$h_0 Y = f_0 h_0 (1 + Y) .$$

Since $h \notin W$, we have $h_0 \neq 0$, and so cancellation yields $Y = f_0(1 + Y)$, a
contradiction since R is a unique factorization domain.

The resulting case, W = Q , is handled similarly. Write

$$XY^{-1} = f + we^{-1}$$

for some $f \in R$, $w \in W$ and $e \in R\backslash W$. Multiply by Ye and then set Y = 0 .
If e_0 results from e by setting Y = 0 , we thus obtain $e_0 X = 0$, whence
$e_0 = 0$ and $e \in YR \subset W$, the desired contradiction. This completes the proof.

Before presenting another family of CPI-closed domains, we recall from [9]
that a domain R is said to have the QQR-property in case each overring of R
is an intersection of localizations of R . It is known that a domain R is a
Prüfer domain if and only if R is integrally closed and has the QQR-property
[9, Corollary 1.7]. There exist (quasilocal) domains with the QQR-property
which are not Prüfer domains [9, Example 4.3]; by a result of Davis (cf. also
[9, Theorem 1.4]), any such domain R satisfies $\dim(R) = \infty$.

PROPOSITION 3.7. Let R be a quasilocal domain with the QQR-property. Then
R is CPI-closed. Indeed, if P and Q are comparable primes of R such that

$PR_P \neq P$ and $QR_Q \neq Q$, then $R + PR_P = R + QR_Q$.

PROOF. The proof consists of applying some results of Gilmer and Heinzer [9]. Since valuation domains, being divided, are CPI-closed, we may suppose that R is not a valuation domain. Then R', the integral closure of R, is a Prüfer domain [9, Corollary 1.7]; and R' is contained in each proper overring of R [9, Theorem 1.10]. In view of Proposition 3.2, it suffices to establish the final assertion, i.e., that $R + PR_P = R + QR_Q$ under the stated conditions. Observe that $A = R + PR_P$ is a valuation domain, for it is quasilocal (by Lemma 2.1) and it is an overring of the Prüfer domain R'. As $B = R + PR_P + QR_Q$ is a CPI-extension of A (by Lemma 3.1) and A is divided, it follows that $A = B$ and, in particular, $R + QR_Q \subset A$. To establish the reverse inclusion, reverse the roles of P and Q in the preceding argument. This completes the proof.

REMARK 3.8. We have seen that the class of valuation domains may be properly extended in (at least) two ways: to the class of divided domains, and to the class of quasilocal domains having the QQR-property. It is easily seen that the former of these "extended classes" is not contained in the latter. Indeed, Gilmer and Huckaba [10, Example 3] construct a quasilocal domain D which does not have the QQR-property, although [6, Lemma 2.2(b)] reveals that D is divided. As for the reverse (non)containment, we do not know of a quasilocal domain with the QQR-property which is not divided (but we suspect that such exists). In any event, it should be noted that the class of quasilocal CPI-closed domains properly contains the "union" of the two "extended classes," as the result of [9] quoted just prior to the statement of Proposition 3.7 shows that the final example in Remark 3.5, which has already been seen to be a quasilocal nondivided CPI-closed domain, also fails to have the QQR-property.

While it was natural in section 2 to consider embedding the class of locally divided domains in the class of treed domains, Example 3.6 reveals that, in

passing from the class of locally divided domains to the larger class of CPI-closed

domains, some nontreed domains (e.g., K[X, Y]) have been introduced. The next

result interposes a class of quasilocal treed domains properly between the divided

domains and the CPI-closed domains, and incidentally generalizes Corollary 3.4.

PROPOSITION 3.9. Let the domain R be such that each sum of two CPI-exten-

sions of R is itself a CPI-extension of R. Then R is CPI-closed, quasi-

local and treed.

PROOF. Proposition 3.2's criterion clearly holds, and so R is CPI-closed.

It remains only to show that any two primes of R, say M and N, are comparable.

Suppose, instead, that $M \not\subset N$ and $N \not\subset M$. By hypothesis, $R + MR_M + NR_N$ is a

ring, in fact of the form $R + PR_P$ for some prime P of R. The incomparability

assumptions give $MR_N = R_N$ and $NR_M = R_M$, respectively, so that $(MR_M)(NR_N) = R_M R_N$.

Thus, $R + PR_P = R_S$, where $S = \{ab \in R : a \in R\backslash M, b \in R\backslash N\}$. Since PR_P is a

prime of R_S which intersects R in P, we have $P \cap S = \emptyset$, and so $P \subset M \cap N$.

Since P and M are therefore comparable, Lemma 2.1 implies that some prime of

R_S intersects R in M, whence $M \cap S = \emptyset$. However, $M \cap S \supset M \cap (R\backslash N) \neq \emptyset$,

the desired contradiction.

REMARK 3.10. (a) It is clear from Proposition 3.2 that the converse of

Proposition 3.9 is also valid.

(b) The second of the three examples in Remark 3.5 shows that the "treed"

conclusion in Proposition 3.9 cannot be strengthened to "going-down ring," and

hence certainly cannot be strengthened to "divided." Note that this example

satisfies the hypothesis of Proposition 3.9 since it satisfies the ostensibly

more stringent assumption of Corollary 3.4. We say "ostensibly," for we do not

know (but doubt) whether, conversely, "the sum of any two CPI-extensions of the

domain R is itself a CPI-extension of R" implies "the set of CPI-extensions

of R is totally ordered by inclusion." Of course, the implication does hold

if dim(R) $\leq$ 3 .

(c) Despite (b), one <u>can</u> strengthen the hypothesis in Corollary 3.4 in order

to get a "going-down" conclusion. Indeed, Papick [12, Lemma 2.41] has shown,

with the aid of [10, Theorem 4], that if the set of <u>all</u> the overrings of a domain

R is totally ordered by inclusion, then R is a quasilocal i-domain (in the

terminology of [13]) and, hence, R is a going-down ring. The crucial point is

that such an R is a Δ-domain. (Following Gilmer and Huckaba [10], we say that

a domain R is a Δ-domain in case each sum of two overrings of R is a ring.)

It should be noted that a quasilocal i-domain need not have its overrings forming

a totally ordered set, since [10, Example 4] presents a quasilocal i-domain which

is not a Δ-domain; and [10, Propositions 10 and 11] exhibits a Δ-domain whose set

of overrings fails to be totally ordered.

Apart from the semantic similarity between the criterion in Proposition 3.2

and the definition of Δ-domain, one should note that any quasilocal QQR-domain

is both CPI-closed and a Δ-domain (by Proposition 3.7 and [10, Theorem 5]).

We do not know of a Δ-domain which is not CPI-closed. However, a CPI-closed

domain need not be a Δ-domain; indeed, a divided domain need not be a Δ-domain.

For a quasilocal example of Krull dimension 1, use either the ring D_6 in

[10, Proposition 10] or the ring D in [10, Example 4]; an integrally closed

example may be fashioned from [5, Theorem 4.2], by recalling that any integrally

closed Δ-domain must be a Prüfer domain [10, Theorem 4(2)]. What makes all these

examples work is the phenomenon that the property of being a Δ-domain is not pre-

served by the D + M construction. (Neither is the QQR-property: see [10, Example

3].) Our final result will establish quite different behavior for CPI-closed

domains. Besides this point of contrast, Proposition 3.11 also produces exam-

ples, with <u>arbitrary</u> Krull dimension exceeding 1 , of the behavior described

in Remark 3.5 and Example 3.6.

PROPOSITION 3.11. Let $K + M$ be a valuation domain with nonzero maximal ideal M, such that K is a field. Let D be a (proper) subring of K. Then $D + M$ is CPI-closed if and only if D is CPI-closed.

PROOF. As the primes of $R = D + M$ and the localizations of R at those primes have been catalogued [8, Theorem A, p. 560], one readily verifies the following. If P is a prime of D, the CPI-extension of R with respect to $P + M$ is $E + M$, where E is the CPI-extension of D with respect to P; moreover, R is the CPI-extension of R with respect to each prime which is not of the form $P + M$. Since the "+" in $K + M$ is a direct sum, the conclusion now follows easily from Proposition 3.2.

Finally, observe that each of the following five types of rings is detectable locally: locally divided domain; going-down ring; treed domain; domain having the QQR-property (by [9, Theorem 1.9]); and Δ-domain (by [10, Theorem 3]). However, as shown by Remark 3.5(b) and Example 3.6, being a CPI-closed domain is not a local property. We close by asking whether one can find an interesting new class of domains for which CPI-closedness would be detectable locally. One criterion for "interesting" might be that, as is the case with all the above classes, the new class be required to include all Prüfer domains.

NOTE ADDED IN PROOF (AUGUST 20, 1979). By reasoning as in the proofs of Proposition 2.3 and Theorem 2.4, one sees that the following three conditions are equivalent for an integral domain R. (1) R is locally divided and each nonzero prime ideal of R lies in a unique maximal ideal of R; (2) For each nonzero $P \in \mathrm{Spec}(R)$, there exists a prime ideal Q of R such that $R + PR_P = R_Q$; (3) For each nonzero $P \in \mathrm{Spec}(R)$, there exists a (uniquely determined) maximal ideal M of R such that $R + PR_P = R_M$.

REFERENCES

1. AKIBA, T. A note on AV-domains, _Kyoto Univ. Education Ser. B_, 31 (1967) 1-3.

2. BOISEN, M.B. Jr. and SHELDON, P.B. Pre-Prüfer rings, _Pac. J. Math._ 58 (1975) 331-344.

3. BOISEN, M.B., Jr. and SHELDON, P.B. CPI extensions: overrings of integral domains with special prime spectrums, _Can. J. Math._ 29 (1977) 722-737.

4. BOURBAKI, N. _Commutative Algebra_, Addison-Wesley, Reading, Mass., 1972.

5. DOBBS, D.E. On going down for simple overrings, II, _Comm, in Algebra_ 1 (1974) 439-458.

6. DOBBS, D.E. Divided rings and going-down, _Pac. J. Math._ 67 (1976) 353-363.

7. DOBBS, D.E. Coherence, ascent of going-down, and pseudo-valuation domains, _Houston J. Math._ 4 (1978) 551-567.

8. GILMER, R.W. _Multiplicative Ideal Theory_, Queen's papers on pure and appl. math., Queen's University, Kingston, Ont. 12 (1968).

9. GILMER, R.W. and HEINZER, W.J. Intersections of quotient rings of an integral domain, _J. Math. Kyoto Univ._ 7 (1967) 133-150.

10. GILMER, R.W. and HUCKABA, J.A. Δ-rings, _J. Algebra_ 28 (1974(414-432.

11 GREENBERG, B. Global dimension of Cartesian squares, _J. Algebra_ 32 (1974) 31-43.

12. PAPICK, I.J. Topologically defined classes of going-down domains, _dissertation_, Rutgers University, New Brunswick, N.J., 1975.

13. PAPICK, I.J. Topologically defined classes of going-down domains, _Trans. Amer. Math. Soc._ 219 (1976) 1-37.

14. Richman, R. Generalized quotient rings, _Proc. Amer. Math. Soc._ 16 (1965) 794-799.