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AN ORDERED SET OF NÖRLUND MEANS

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ABSTRACT. Nörlund methods of summability are studied as mappings from ℓ_1 into ℓ_1. Those Nörlund methods that map ℓ_1 into ℓ_1 are characterized. Inclusion results are given and a class of Nörlund methods is shown to form an ordered abelian semigroup.

KEY WORDS AND PHRASES. *Inclusion Theorem, ℓ-ℓ method, ordered abelian semigroup, Nörlund method.*

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1. INTRODUCTION.

Let p be a complex sequence $p_0 \neq 0$, and let $P_n = \sum_{k=0}^{n} p_k$, n=0,1,2,..., denote the n+1-st partial sum of p. Suppose the sequence P_n is eventually non-zero. Let K be the least positive integer so that $P_n \neq 0$ for all $n \geq K$. Define the Nörlund method of summability N_p by

$$N_p[n,k] = \begin{cases} p_{n-k}/P_0, & \text{if } 0 \leqq n < K,\ k \leqq n, \\ p_{n-k}/P_n, & \text{if } n \geqq K,\ k \leqq n \text{ and} \\ 0 & \text{, otherwise.} \end{cases}$$

Let N denote the collection of all such Nörlund methods N_p. If $P_n \nrightarrow 0$ for all $n \geqq 0$, that is $K = 0$, then

$$N_p[n,k] = \begin{cases} p_{n-k}/P_n, & \text{if } k \leqq n \text{ and} \\ 0 & \text{, otherwise.} \end{cases}$$

Throughout we let $\hat{P}_n = P_0$ if $0 \leqq n < K$ and $\hat{P}_n = P_n$ if $n \geqq K$. The N_p transform of a sequence x is then given by $N_p x$, where

$$(N_p x)_n = (1/\hat{P}_n) \sum_{k=0}^{n} p_{n-k} x_k$$

for all $n \geqq 0$.

A matrix summability method is called an ℓ-ℓ method if and only if it maps the space $\ell_1 \equiv \ell$ into itself. In [4], Knopp and Lorentz proved that the matrix method A is ℓ-ℓ if and only if there exists some $M > 0$ such that

$$\sup_k \{ \sum_{n=0}^{\infty} |a_{nk}| \} < M.$$

In the special case of a Nörlund method N_p we have:

THEOREM 1. The Nörlund method N_p is ℓ-ℓ if and only if

(i) $p \in \ell$, and

(ii) $\hat{P}_n \nrightarrow 0$ as $n \to \infty$.

PROOF. First suppose that (i) and (ii) hold. Since $p \in \ell$ it implies $\lim_n \hat{P}_n$ exists and by (ii) is non-zero. So there exist strictly positive numbers

H and δ such that $\sum_n |p_n| < H$ and $|\hat{P}_n| > \delta$ for all $n \geqq 0$. Thus for each fixed k,

$$\sum_{n=k}^{\infty} |N_p[n,k]| = \sum_{n=k}^{\infty} |p_{n-k}/\hat{P}_n|$$

$$< (1/\delta) \sum_{n=k}^{\infty} |p_{n-k}|$$

$$< H/\delta.$$

Thus by the Knopp-Lorentz Theorem N_p is ℓ-ℓ.

Now suppose that N_p is ℓ-ℓ. Then

$$\sum_{n=k}^{\infty} |p_{n-k}/\hat{P}_n| = O(1).$$

In particular if n=k it implies $|1/\hat{P}_n| = O(1)$ and hence $\hat{P}_n \not\to 0$ as $n \to \infty$.

Now suppose that $p \notin \ell$. We assert that the series $\sum_n (|p_n / \hat{P}_n|)$ diverges.
For sufficiently large n,

$$|\hat{P}_n| = |P_n| = |\sum_{k=0}^{n} p_k| \leqq \sum_{k=0}^{n} |p_k| \equiv S_n.$$

Then for sufficiently large N

$$\sum_{n=N+1}^{N+m} (|p_n / \hat{P}_n|) \geqq (1/S_{N+m}) \sum_{n=N+1}^{N+m} |p_n|$$

$$= 1 - S_N/S_{N+m}.$$

But $S_{N+m} \to \infty$ as $m \to \infty$. So choose m large enough such that $S_{N+m} > 2S_N$.

Then $\sum_{n=N+1}^{N+m} (|p_n|/|\hat{P}_n|) > \frac{1}{2}$. The theorem now follows.

COROLLARY 1 [1, Theorem 4]. Let N_p be a Nörlund method with $p_n \geqq 0$, $p_0 > 0$. Then N_p is ℓ-ℓ if and only if $p \in \ell$.

2. We make the following definitions.

DEFINITION. Let N_ℓ denote the collection of all $N_p \in N$ that are ℓ-ℓ methods.

DEFINITION. Let $\ell(N_p)$ consist of all sequences x such that $N_p x$ is in ℓ.

DEFINITION. Given two Nörlund methods N_p and N_q, N_q is ℓ-stronger than N_p if and only if $\ell(N_p) \subseteq \ell(N_q)$. The method N_q is strictly ℓ-stronger than N_p provided $\ell(N_p) \subsetneq \ell(N_q)$, and N_p and N_q are ℓ-equivalent provided $\ell(N_p) = \ell(N_q)$.

DEFINITION. Given N_p and N_q define formally:

(i) $p(z) = \sum_{n=0}^{\infty} p_n z^n$, $P(z) = \sum_{n=0}^{\infty} \hat{P}_n z^n$,

(ii) $q(z) = \sum_{n=0}^{\infty} q_n z^n$, $Q(z) = \sum_{n=0}^{\infty} \hat{Q}_n z^n$, and

(iii) $a(z) = p(z)/q(z) = \sum_{n=0}^{\infty} a_n z^n$, $b(z) = q(z)/p(z) = \sum_{n=0}^{\infty} b_n z^n$.

The next propositions follow by an argument similar to the one used in Theorem 18 of [3] .

PROPOSITION 1. If $N_p \in N_\ell$, then the series $\sum_n p_n z^n$ and $\sum_n \hat{P}_n z^n$ converge for $|z|<1$.

PROPOSITION 2. If N_p, $N_q \in N_\ell$, then the series $a(z) = \sum_n a_n z^n$ and $b(z) = \sum_n b_n z^n$ have positive radii of convergence and, moreover

(i) $p_n = a_n q_0 + \dots + a_0 q_n$,

(ii) $\hat{P}_n = a_n \hat{Q}_0 + \dots + a_0 \hat{Q}_n$,

(iii) $q_n = b_n p_0 + \dots + b_0 p_n$, and

(iv) $\hat{Q}_n = b_n \hat{P}_0 + \dots + b_0 \hat{P}_n$.

PROPOSITION 3. Suppose $N_p \in N_\ell$ and the sequence $S = \{S_n\}$ is in $\ell(N_p)$. Then the series $S(z) = \sum_n S_n z^n$ has positive radius of convergence.

PROOF. Let $h(z) = 1/p(z)$. By Proposition 1, $p(z)$ defines an analytic function for $|z|<1$. Since $p_0 \neq 0$, by the continuity of $p(z)$ there exists some $\alpha \in (0,1)$ such that $p(z) \neq 0$ for all $z \in (-\alpha,\alpha)$. Therefore $h(z) = \sum_n h_n z^n$ has positive radius of convergence. Now for $n \geq 0$,

$$\sum_{k=0}^{n} [h_{n-k}\hat{P}_k(N_pS)_k] = S_n.$$

Since $S \in \ell(N_p)$, it follows that the series $\sum_{k=0}^{\infty} P_k(N_pS)_k z^k$ converges for $|z|<1$.

Therefore $\sum_n S_n z^n$ has positive radius of convergence since

$$\sum_{n=0}^{\infty} S_n z^n = \{h(z)\}\{\sum_{k=0}^{\infty} \hat{P}_k(N_pS)_k z^k\}.$$

3. The symmetric product $g = p*q$ of the sequences p and q is defined by $g_n = p_0q_n + \dots + p_nq_0$ for all $n \geq 0$. Given Nörlund methods N_p and N_q in N we say $N_g = N_{p*q}$ is the symmetric product of N_p and N_q provided $N_g \in N$.

In order to prove an inclusion result for two Nörlund methods in N_ℓ we need the following lemma.

Lemma 1. Let the complex sequences p and q be given and define $r = p*q$. Suppose N_p, $N_r \in N$. Then in order that $\ell(N_p) \subseteq \ell(N_r)$ it is necessary and sufficient that there is some $M > 0$, independent of k, such that

$$|\hat{P}_k| \sum_{n=k}^{\infty} |q_{n-k}/\hat{R}_n| < M.$$

Proof. Let $x = \{x_n\}$ be any sequence. Then

$$(N_rx)_n = (1/\hat{R}_n) \sum_{k=0}^{n} [q_{n-k}\hat{P}_k(N_px)_k].$$

Let $e_{nk} = q_{n-k}\hat{P}_k/\hat{R}_n$ if $k \leq n$, and 0 if $k > n$. Then by the Knopp-Lorentz Theorem $\ell(N_p) \subseteq \ell(N_r)$ if and only if there exists some $M > 0$ such that

$$\sup_k \{ \sum_{n=k}^{\infty} |e_{nk}| \} < M.$$

That is,

$$\sup_k \{ |\hat{P}_k| \sum_{n=k}^{\infty} |q_{n-k}/\hat{R}_n| < M.$$

The lemma can now be used to get the desired inclusion result.

Theorem 2. Suppose N_p, $N_q \in \mathcal{N}_\ell$. The $\ell(N_p) \subseteq \ell(N_q)$ if and only if $b = \{b_n\} \in \ell$.

Proof. In the previous lemma replace the sequence q with the sequence b which implies $r = p*b = q$. Then $\ell(N_p) \subseteq \ell(N_r) = \ell(N_q)$ if and only if there exists some $M > 0$, independent of k, such that

$$|\hat{P}_k| \sum_{n=k}^{\infty}| b_{n-k}/\hat{Q}_n| < M.$$

But since N_p, $N_q \in \mathcal{N}_\ell$, $|\hat{P}_k|$ and $|\hat{Q}_k|$ respectively are bounded by two strictly positive constants. Thus $|\hat{P}_k| \sum_{n=k}^{\infty} |b_{n-k}/\hat{Q}_n| < M$, independent of k, is equivalent to $b \in \ell$.

Corollary 2. Suppose N_p, $N_q \in \mathcal{N}_\ell$. Then

(i) $\ell(N_p) = \ell(N_q)$ if and only if both $a = \{a_n\} \in \ell$ and $b = \{b_n\} \in \ell$

and

(ii) $\ell(N_p) \subsetneq \ell(N_q)$ if and only if $a = \{a_n\} \notin \ell$ and $b = \{b_n\} \in \ell$.

Corollary 3. Suppose $N_p \in \mathcal{N}_\ell$ and $h(z) = 1/p(z)$. Then $\ell(N_p) = \ell$ if and only if $h \in \ell$.

Proof. Let I be the identity matrix so that $\ell(I) = \ell$. Then

$I(z) = \sum_{n=0}^{\infty} i_n z^n = 1$; that is $i_0 = 1$ and $i_n = 0$ for all $n \geqq 1$. Therefore

$a(z) = p(z)/I(z) = p(z)$ and $b(z) = I(z)/p(z) = h(z)$. The corollary now follows.

Example. The binary matrix $B = (b_{nk})$ is given by $b_{nk} = 1$, if $n=k=0$, $1/2$, if $k=n-1$ or $k=n$, $n > 1$, and 0 otherwise. Thus B is the ℓ-ℓ Nörlund method defined by $p_0=1=p_1$, $p_n=0$ for $n > 2$. It now follows by Corollary 3 that $\ell \subsetneqq \ell(B)$.

The next result addresses the question of when two Nörlund methods N_p, $N_q \in N_\ell$ are comparable. We prove that changing only the first term in the generating sequence N_p can result in a method $N_q \in N_\ell$ satisfying $\ell(N_p) \cap \ell(N_q) = \ell$. That is, the methods are not comparable.

Theorem 3. Suppose $N_p \in N_\ell$. Let $q = (p_0', p_1, \ldots)$ with $p_0' \neq p_0$. If the sequence q satisfies $\liminf_{n\to\infty} |Q_n| \neq 0$ then N_q is ℓ-ℓ and moreover

$$\ell(N_p) \cap \ell(N_q) = \ell.$$

Proof. Since $q \in \ell$, $N_q \in N_\ell$, $|\hat{P}_n|$ and $|\hat{Q}_n|$ respectively lie between two strictly positive constants. Now for any sequence x

$$(N_q x)_n = (p_0' - p_0)x_n/\hat{Q}_n + [(N_p x)_n][\hat{P}_n/\hat{Q}_n],$$

and hence

$$|(N_q x)_n| + |\hat{P}_n/\hat{Q}_n||(N_p x)_n| \geqq |p_0' - p_0||x_n|/|\hat{Q}_n|.$$

Therefore, if $x \in \ell(N_p)$, $x \notin \ell$, then $x \notin \ell(N_p)$.
Similarly if $x \in \ell(N_q)$, $x \notin \ell$, then $x \notin \ell(N_p)$.

Corollary 4. Suppose p is a positive number sequence in ℓ. Let $q = (p_0', p_1, \ldots)$, where $p_0' > 0$ and $p_0' \neq p_0$. Then N_p and N_q are ℓ-ℓ with $\ell(N_p) \cap \ell(N_q) = \ell$.

Theorem 4. Suppose p is a sequence in ℓ, with P_n eventually non-zero. Let $q = (p_1, p_2, \ldots)$. If Q_n is eventually non-zero, then N_p, $N_q \in N_\ell$ with $\ell(N_p) \cap \ell(N_q) = \ell$.

The next theorem is a special case of Theorem 2.

Theorem 5. If N_p and N_q are Nörlund methods with

(i) $p_{n+1}/p_n \geqq p_n/p_{n-1}$, $n > 0$, $p_0 = 1$, $p_n > 0$,

(ii) $q_n \geqq 0$, $q_0 = 1$, and

(iii) $p, q \in \ell$,

then $\ell(N_p) \subseteq \ell(N_q)$.

Proof. By Theorem 22 of [3]

$\{p(z)\}^{-1} = 1 - c_1 z - c_2 z^2 - \ldots$, where $c_n > 0$ for all $n \geq 0$ and $\sum_n c_n \leqq 1$.

Then for small $|z|$, $q(z)/p(z) = \sum_{n=0}^{\infty} (q_n\gamma_0 + \ldots + q_0\gamma_n)z^n$, where $\gamma_n = -c_n$ for $n > 0$ and $\gamma_0 = 1$. So that if $q(z)/p(z) = \sum_{n=0}^{\infty} b_n z^n$, we have

$$\sum_{n=0}^{\infty} |b_n| \leqq \sum_{n=0}^{\infty} (|q_0||\gamma_n| + \ldots + |q_n||\gamma_0|)$$

$$= \{\sum_{n=0}^{\infty} |q_n|\}\{\sum_{n=0}^{\infty} |c_n|\}$$

$$< \infty.$$

Therefore by Theorem 2, $\ell(N_p) \subseteq \ell(N_q)$.

Corollary 6. If N_p and N_q are ℓ-ℓ Nörlund methods such that

(i) $p_{n+1}/p_n \geqq p_n/p_{n-1}$, $n > 0$, $p_0=1$, $p_n > 0$,

and

(ii) $q_{n+1}/q_n \geqq q_n/q_{n-1}$, $n > 0$, $q_0=1$, $q_n > 0$,

then $\ell(N_p) = \ell(N_q)$.

For example if $p_n = \rho^n$ and $q_n = \tau^n$ where $0 < \rho, \tau < 1$, then $\ell(N_p) = \ell(N_q)$. Moreover by Corollary 3, $\ell(N_p) = \ell = \ell(N_q)$. If $\gamma_n = 1/n^k$, $k > 1$ and $n > 0$, then $\ell(N_p) = \ell(N_\gamma)$.

4. We now show that $\mathcal{N}_\ell$ forms an ordered abelian semigroup. The order relation is set inclusion between the absolute summability fields, and the binary operation is the symmetric product of the generating sequences. We need the following lemmas.

Lemma 2. Suppose $N_q \in \mathcal{N}_\ell$ and p is a sequence for which $\lim_{n\to\infty} \hat{P}_n = \hat{P} \neq 0$. Let $r = p*q$. Then $\lim_{n\to\infty} (\hat{R}_n/\hat{P}\hat{Q}_n) = 1$.

Proof. We assert that N_q is a regular method. By the Silverman-Toeplitz Theorem, see for example [6], it suffices to show that

(i) $q_{n-\nu}/\hat{Q}_n \to 0$ as $n \to \infty$ for each fixed $\nu \geqq 0$, and

(ii) $\sum_{\nu=0}^{n} |q_\nu| = O(|\hat{Q}_n|)$.

But since $N_q \in \mathcal{N}_\ell$, (i) and (ii) follow. Now for all n sufficiently large

$$\hat{R}_n/\hat{Q}_n = (1/\hat{Q}_n) \sum_{k=0}^{n} r_k = (1/\hat{Q}_n) \sum_{k=0}^{n} q_{n-k}\hat{P}_k,$$

which is the N_q transform of the convergent sequence $\{P_n\}$. Thus $\lim_{n\to\infty} (\hat{R}_n/\hat{P}\hat{Q}_n) = 1$.

Lemma 3. If N_p, $N_q \in \mathcal{N}_\ell$ and $r = p*q$, then $N_r \in \mathcal{N}_\ell$.

Proof. By Lemma 1, we have $\lim_{n\to\infty} \hat{R}_n = \hat{P}\hat{Q} \neq 0$. In order to have $N_r \in \mathcal{N}_\ell$, we need to show N_r is an ℓ-ℓ method. By Theorem 1, it suffices to show that $r \in \ell$. But

$$\sum_{n=0}^{\infty} |r_n| \leqq \{ \sum_{n=0}^{\infty} |p_n| \} \{ \sum_{n=0}^{\infty} |q_n| \}.$$

$$< \infty.$$

Lemma 4. If N_p, $N_q \in \mathcal{N}_\ell$, $r = p*q$, then N_r is ℓ-ℓ with $\ell(N_p) \cup \ell(N_q) \subseteq \ell(N_r)$.

Proof. By Lemma 2, $N_r \in \mathcal{N}_\ell$. By Theorem 2 it follows that $\ell(N_p) \cup \ell(N_q) \subseteq \ell(N_r)$.

Lemma 5. Suppose N_p, N_q, $N_s \in \mathcal{N}_\ell$, $\mu = p*s$, and $\nu = q*s$.

(i) If $\ell(N_p) \subseteq \ell(N_q)$, then $\ell(N_\mu) \subseteq \ell(N_\nu)$.

(ii) If $\ell(N_p) \subsetneq \ell(N_q)$, then $\ell(N_\mu) \subsetneq \ell(N_\nu)$.

Proof of (i). Let $b(z) = q(z)/p(z)$ and $c(z) = \nu(z)/\mu(z)$. By Theorem 2, $b \in \ell$. We need to show that $c \in \ell$. Now for $|z|<1$,

$$\nu(z) = \sum_{n=0}^{\infty} \nu_n z^n = \{\sum_{n=0}^{\infty} q_n z^n\}\{\sum_{n=0}^{\infty} s_n z^n\}$$

$$= q(z)s(z).$$

Similarly $\mu(z) = p(z)s(z)$. Therefore

$$\sum_{n=0}^{\infty} c_n z^n = c(z) = \nu(z)/\mu(z) = b(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Thus $\ell(N_\mu) \subseteq \ell(N_\nu)$.

The proof of (ii) follows by Theorem 2 and Corollary 2.

A semigroup with order relation $<$ is an ordered semigroup provided (i) $a<b$ and $b<c$ implies $a<c$, and (ii) $a<b$ implies $ac<bc$ for all c. We now have the following theorem.

Theorem 6. With "strictly ℓ-weaker than " as the order relation and $*$ as the binary operation, $\mathcal{N}_\ell$ is an ordered abelian semigroup.

Proposition 4. Let $N_p \in \mathcal{N}_\ell$. Define $p_{-1} = 0$ and $q_n = p_{n-1} + p_n$ for $n \geq 0$. If Q_n is eventually non-zero, then $\ell(N_p) \subsetneq \ell(N_q)$.

Proof. First note that $N_q \in \mathcal{N}_\ell$. Moreover, it follows that $q(z) = (1+z)p(z)$ for $|z|<1$. Then by Corollary 2, $\ell(N_p) \subsetneq \ell(N_q)$.

Proposition 5. There exists infinite chains of Nörlund methods from N_ℓ.

Proof. Let $p^{(1)}(z) = \sum_{n=0}^{\infty} p_n^{(1)} z^n$, where $\{p_n^{(1)}\} \varepsilon \ell$, $p_n^{(1)} \geqq 0$ for $n \geqq 0$, and $p_0^{(1)} > 0$. Then $N_{p^{(1)}} \varepsilon N_\ell$. Define

$$p^{(n)}(z) = (1+z)^{n-1} \sum_{k=0}^{\infty} p_k^{(n-1)} z^k, \text{ for } n \geqq 2.$$

Then $N_{p^{(n)}} \varepsilon N_\ell$ for $n \geqq 1$. Moreover by Proposition 4 we have

$$\ell(N_{p^{(1)}}) \subsetneqq \ell(N_{p^{(2)}}) \subsetneqq \cdots \subsetneqq \ell(N_{p^{(n)}}) \subsetneqq \cdots .$$

The next theorem is a different version of Lemma 3.

Theorem 7. If $N_q \ \varepsilon\ N_\ell$, $\lim_{n\to\infty} P_n = P \neq 0$, and $r = p*q$, then $\ell(N_p) \subseteq \ell(N_r)$.

Proof. By Lemma 1 it suffices to show there exists some $M > 0$, independent of k, such that

$$|\hat{P}_k| \sum_{n=k}^{\infty} |q_{n-k}/\hat{R}_n| < M.$$

Since $N_q \ \varepsilon\ N_\ell$, there exists some $H > 0$ such that

$$|\hat{Q}_{n-k}/\hat{Q}_n| < H$$

for all $n \geqq k$ and for all $k \geqq 0$. Then by Lemma 1, we have

$$|\hat{P}\hat{Q}_{n-k}/\hat{R}_n| = |\hat{P}\hat{Q}_n/\hat{R}_n|\,|\hat{Q}_{n-k}/\hat{Q}_n| < \infty$$

for all $n \geqq k$. Thus the result follows.

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