Internat. J. Math. & Math. Sci.
Vol. 4 No. 3 (1981) 513-528

LNC POINTS FOR m-CONVEX SETS

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(Received April 8, 1980 and in revised form September 4, 1980)

ABSTRACT. Let S be closed, m-convex subset of R^d, S locally a full d-dimensional, with Q the corresponding set of lnc points of S. If q is an essential lnc point of order k, then for some neighborhood U of q, $Q \cap U$ is expressible as a union of k or fewer $(d - 2)$-dimensional manifolds, each containing q. For S compact, if to every $q \in Q$ there corresponds a $k > 0$ such that q is an essential lnc point of order k, then Q may be written as a finite union of $(d - 2)$-manifolds.

For q any lnc point of S and N a convex neighborhood of q, $N \cap \text{bdry } S \not\subseteq Q$. That is, Q is nowhere dense in bdry S. Moreover, if $\text{conv}(Q \cap N) \subseteq S$, then $Q \cap N$ is not homeomorphic to a $(d - 1)$-dimensional manifold.

KEY WORDS AND PHRASES. *Points of nonconvexity, m-Convex Sets.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *Primary 52.A20, 52.A40.*

1. INTRODUCTION.

Let S be a subset of R^d. The set S is said to be m-convex, $m \geq 2$, if and only if for every m distinct points in S, at least one of the $\binom{m}{2}$ line segments determined by these points lies in S. If the m-convex set S is not j-convex for $j < m$, then S is exactly m-convex. A point x in S is said to be a point of local convexity of S if and only if there is some neighborhood N of x such that if $y,z \in S \cap N$, then $[y,z] \subseteq S$. If S fails to be locally convex at some point q in S , then q is called a point of local nonconvexity (lnc point) of S .

Few studies have been made concerning points of local nonconvexity for m-convex sets. Valentine [3] has proved that for S a compact 3-convex subset of R^d with Q the corresponding set of lnc points of S , if int ker $S \neq \phi$ and $Q \subseteq$ int conv S , then Q consists of a finite number of disjoint closed (d - 2)-dimensional manifolds. The purpose of this paper is to obtain an analogue of Valentine's result for m-convex sets.

The following familiar terminology will be used: For points x,y in S , we say x sees y via S if and only if the corresponding segment [x,y] lies in S . Points $x_1,\ldots,x_n$ in S are visually independent via S if and only if for $1 \le i < j \le n$, x_i does not see x_j via S . Throughout the paper, aff S , conv S , ker S , int S , rel int S , bdry S , and cl S will be used to denote the affine hull, convex hull, kernel, interior, relative interior, boundary, and closure, respectively, of the set S .

Also, for points x and y , R(x,y) will denote the ray emanating from x through y , and for point x and set T , cone (x,T) will represent $\cup\{R(x,t) : t \in T\}$.

Finally, S will be a closed subset of R^d which is locally a full d-dimensional - i.e., for s in S and N any neighborhood of s , $\dim(S \cap N) = d$. And Q will denote the set of lnc points of S .

2. ESSENTIAL LNC POINTS OF ORDER K .

We begin with the following definitions for the closed set S and its corresponding collection of lnc points Q . The first definition is an adaptation of Definition 1 in [1] .

DEFINITION 1. Let $q \in Q$. We say that q is essential if and only if there is some neighborhood N' of q such that for every convex neighborhood N of q with $N \subseteq N'$, $(S \cap N) \sim Q$ is connected.

DEFINITION 2. We say that $q \in Q$ has order k if and only if there is some neighborhood N' of q such that the following are true.

1) $\mathrm{Conv}(Q \cap N') \subseteq S$.

2) For every convex neighborhood N of q with $N \subseteq N'$, $(S \cap N) \sim \mathrm{conv}(Q \cap N)$ contains at least one k-tuple of points which are visually independent via S and no $(k + 1)$-tuple of points visually independent via S .

3) For every convex neighborhood N of q with $N \subseteq N'$, $\dim \mathrm{conv}(Q \cap N) = \dim \mathrm{conv}(Q \cap N')$. If this dimension is d , then $q \in \mathrm{int}\, \mathrm{conv}(C \cup (Q \cap N))$ for each component C of $(S \cap N) \sim \mathrm{conv}(Q \cap N)$ If this dimension is $d - 1$, then $q \in \mathrm{rel\ int}\, (S \cap \mathrm{aff}(Q \cap N))$.

The following lemmas will be useful.

LEMMA 1. Let S be a closed m-convex set in R^d , with Q the corresponding set of lnc points of S . Then $Q \subseteq \mathrm{cl}(S \sim Q)$.

PROOF. Suppose on the contrary that for some point q in Q and some neighborhood N of q , $N \cap (S \sim Q) = \phi$. Then $S \cap N \subseteq Q$. Select x_1, x_1' in $S \cap N$ which are visually independent via S , and let $M, M' \subseteq N$ be neighborhoods of x_1 and x_1' respectively so that no point of M sees any point of M' via S . Since $x_1' \in Q$, choose x_2, x_2' in $M' \cap S$ which are visually independent via S . By an obvious induction, we obtain m visually independent points $x_1, x_2, \ldots, x_m$, contradicting the m-convexity of S . Our assumption is false and $Q \subseteq \mathrm{cl}(S \sim Q)$.

LEMMA 2. Let N be a convex neighborhood for which $\mathrm{conv}(Q \cap N) \subseteq S$, let $x \in S \cap N$, and let Q_x denote the subset of $\mathrm{conv}(Q \cap N)$ which x sees via S . Then $\mathrm{conv}(Q_x \cup \{x\}) \subseteq S$.

PROOF. Let $y \in \mathrm{conv}(Q_x \cup \{x\})$ to prove that $y \in S$. Then by Carathéodory's theorem, $y \in \mathrm{conv}\{z_1, \ldots, z_{k+1}\}$ for an appropriate $k + 1$ member subset of $Q_x \cup \{x\}$, $k \le d$. If $y \in \mathrm{cl\ conv}(Q \cap N) \subseteq S$, the argument is finished, so assume that $y \notin \mathrm{cl\ conv}(Q \cap N)$. Hence one of the z_i points above must be x , and we may assume that $y \in \mathrm{conv}\{x, z_1, \ldots, z_k\}$, where $z_i \in Q_x$ for $1 \le i \le k$. Further, we assume that k is minimal. Then

$P \equiv \text{conv}\{x, z_1, \ldots, z_k\}$ is a k-simplex having y in its relative interior.

We use an inductive argument to finish the proof. Clearly the result is true for $k = 1$. For $k \geq 2$, assume that the result is true for all natural numbers less than k, to prove for k. Thus we may assume that every proper face of P lies in S.

Since $y \notin \text{cl conv}(Q \cap N)$, there is a hyperplane H strictly separating y from $\text{cl conv}(Q \cap N)$, and clearly $\{x, y\}$ and $\{z_1, \ldots, z_k\}$ lie in opposite open halfspaces determined by H.

Let H' be a hyperplane parallel to H and containing y, and let L be a line in H' with $y \in L$. Then $L \cap P$ is an interval $[a, b]$ where a and b lie in facets of P. Hence by our induction hypothesis, $[x, a] \cup [x, b] \subseteq S$. Clearly $Q \cap N$ and $\{x\}$ lie on opposite sides of H', so there can be no lnc point of S in $\text{conv}\{x, a, b\}$. Therefore, by a lemma of Valentine [4, Corollary 1], $\text{conv}\{x, a, b\} \subseteq S$. Thus $y \in S$ and the lemma is proved.

The following theorem is an analogue of Valentine's result for 3-convex sets.

THEOREM 1. Let S be a closed m-convex set in R^d, S locally a full d-dimensional, with Q the corresponding set of lnc points for S. If q is an essential lnc point of order k, then for some neighborhood U of q, $U \cap Q$ is expressible as a union of k or fewer $(d - 2)$-dimensional manifolds, each containing q.

PROOF. Let N' be a convex neighborhood of q satisfying Definitions 1 and 2. The proof will require three cases, each determined by the dimension of $\text{conv}(Q \cap N')$.

CASE 1. Assume that for every neighborhood M of q with $M \subseteq N'$, $\dim \text{conv}(Q \cap M) = d$. We proceed by induction on the order of q. If the order of q is 2, then $S \cap N'$ is 3-convex, and $S' = \text{cl}(S \cap N')$ is compact and 3-convex. Letting Q' denote the set of lnc points of S', clearly $Q' = \text{cl}(Q \cap N')$. It is easy to show that every lnc point of a 3-convex set lies in the kernel of that set, so $Q' \subseteq \ker S'$ and hence $\text{int} \ker S' \neq \phi$. Also, since q satisfies Definition 2, $q \in \text{int conv } S'$. Thus by [3, Lemma 4 and 5], there is a neighborhood U of q such that $Q \cap U$ is a $(d-2)$-dimensional manifold.

Inductively, assume that the result is true for order $q < k$ to prove for order $q = k$. Since a closed m-convex set is locally starshaped [2, Lemma 2], without loss of generality assume that $S \cap N'$ is starshaped relative to q. Let V be a neighborhood in $\text{int conv}(Q \cap N')$ and select a point $p \in N'$ so that $q \in \text{int conv}(\{p\} \cup V) \equiv W$. Since $q \in \text{int conv}(C \cup (Q \cap W))$ for every component C of $(S \cap W) \sim \text{conv}(Q \cap W)$, we may select $x \in (S \cap W) \sim \text{conv}(Q \cap W)$ so that $R(x,q)$ intersects $\text{int conv}(Q \cap W)$. Finally, select a convex neighborhood N of q, $N \subseteq W$, so that for all r in $N \cap \text{bdry conv}(Q \cap W)$, $R(x,r)$ intersects $\text{int conv}(Q \cap W)$, $[R(x,r) \sim [x,r)] \cap N \subseteq \text{conv}(Q \cap W)$, and $[x,r) \cap \text{conv}(Q \cap W) = \phi$.

Let T denote the subset of $N \cap \text{conv}(Q \cap W)$ seen by x. By the proof of Lemma 2, $\text{conv}(T \cup \{x\}) \subseteq S$. Let K denote the closure of the set $\text{conv}(T \cup \{x\}) \cup \text{conv}(Q \cap W)$, with Q_k the corresponding set of lnc points of K We assert that $Q \cap T = Q_k \cap N$: By our construction, for r in $Q \cap T$, clearly $r \in Q_k$, so $r \in Q_k \cap N$. To obtain the reverse inclusion, for r in $Q_k \cap N$, certainly $r \in \text{conv}(Q \cap W) \cap \text{conv}(T \cup \{x\})$, so r is a point of $N \cap \text{conv}(Q \cap W)$ which x sees via S, and $r \in T$. Now if r were not in Q, then r would not be an lnc point of S, so for some neighborhood A of r, $S \cap A$ would be convex and hence disjoint from Q. Without loss of generality, assume that $A \subseteq N$. Since $R(x,r)$ intersects $\text{int conv}(Q \cap W)$, select v in $A \cap \text{int conv}(Q \cap W) \cap R(x,r) \subseteq \text{int}(S \cap A)$ and select w in $(x,r) \cap A \subseteq S \cap A$. Then since $S \cap A$ is convex, $r \in (v,w) \subseteq \text{int}(S \cap A)$. Let H be a hyperplane supporting $\text{conv}(Q \cap W)$ at r, with x in the open halfspace H_1 determined by H. Using Valentine's lemma [4, Corollary 1], it is not hard to show that x sees $S \cap A \cap H_1$ via S, and since $r \in \text{int}(S \cap A)$, x sees some neighborhood A' of r via S, $A' \subseteq A$. But since $A' \subset N$, this implies that $r \in \text{int conv}(A' \cup \{x\}) \subseteq \text{int conv}(T \cup \{x\}) \subseteq \text{int } K$, contradicting the fact that

$r \in Q_k$. We conclude that $Q_k \cap N \subseteq Q \cap T$, the sets are equal, and our assertion is proved.

To complete Case 1, unfortunately it is necessary to examine two subcases:

CASE 1a. If $\mathrm{conv}(T \cup \{x\})$ has dimension d , then by a previous argument the set K and the point $q \in K$ satisfy the hypotheses of [3, Lemma 4]. Hence for some neighborhood U' of q , $Q_k \cap U'$ is a $(d - 2)$-dimensional manifold.

Now let C denote the component of $(S \cap W) \sim \mathrm{conv}(Q \cap W)$ which contains x , and let $S' = \mathrm{cl}(S \sim C)$. Select a convex neighborhood M of q , $M \subseteq N \subseteq W$, so that $S' \cap M$ contains no point of $\mathrm{cone}(x,T) \sim \mathrm{conv}(Q \cap W)$. Then for y in $(S' \cap M) \sim \mathrm{conv}(Q \cap W)$, we assert that $[y,x] \not\subseteq S \cap M$: If $[y,x] \subseteq S \sim \mathrm{conv}(Q \cap W)$, then $y \in C$, impossible. And if $[y,x] \cap \mathrm{conv}(Q \cap W) \neq \phi$, then y would lie in $\mathrm{cone}(x,T)$, again impossible.

Thus $S' \cap M$ has at most $k - 1$ visually independent points not in $\mathrm{conv}(Q \cap W)$. If q is an lnc point of S' , then q is an essential lnc point of S' of order at most $k - 1$. Letting Q' denote the set of lnc points of S' , Q' contains all lnc points of $S \cap M$ which do not lie in $Q \cap T = Q_k \cap N$. By an inductive argument, for an appropriate neighborhood U of q , $Q' \cap U$ is expressible as a union of $k - 1$ or fewer $(d - 2)$-manifolds which contain q . For simplicity of notation assume that $U \subseteq U' \cap N$. Then $Q \cap U = (Q' \cap U) \cup (Q \cap T \cap U) = (Q' \cap U) \cup (Q_k \cap U)$ is a union of k or fewer $(d - 2)$-manifolds, the desired result.

If q is not an lnc point of S' , select the neighborhood U of q so that $S' \cap U$ is convex, $U \subseteq U' \cap N$. Then $Q \cap U = Q_k \cap U$ is a $(d - 2)$-manifold. This finishes Case 1a.

CASE 1b. Suppose that Case 1a does not occur. Hence $\mathrm{conv}(T \cup \{x\})$ has dimension $\leq d - 1$. By a previous argument for some neighborhood N of q , $Q_k \cap N = Q \cap T$. Also, since $\dim \mathrm{conv}(Q \cap W) = d$ and $\dim \mathrm{conv}(T \cup \{x\}) \leq d - 1$, it is clear that $Q_k \cap N$ is exactly the set of points of intersection of $\mathrm{conv}(Q \cap W)$ with $(\mathrm{conv}(T \cup \{x\})) \cap N$, so $T = Q_k \cap N \subseteq Q$.

Recall that N is a neighborhood of q satisfying the definition of essential, so $(S \cap N) \sim Q$ is locally convex and connected and hence polygonally connected. Select points v,w in $K \cap N$, $v < q < w$, with $v \in (x,q)$ and $w \in \text{int conv}(Q \cap W)$. Let λ be a polygonal path in $(S \cap N) \sim Q$ from v to w. Then $\lambda \cup [x,v]$ is a path in $S \sim Q$ from x to w. Now by our definition of W, bdry conv$(Q \cap W)$ separates N into two disjoint connected sets. Let $v = t_1,\ldots,t_n = w$ denote the consecutive vertices of λ, and assume that they are labeled so that t_j is the first point of λ in conv$(Q \cap W)$. Clearly $j > 1$. Then $[x,t_1] \cup [t_1,t_2] \subseteq S \sim Q$. Furthermore, by our choice of N, we assert that there can be no lnc point r in int conv$\{x,t_1,t_2\}$: Otherwise, clearly r would lie in $N \cap$ bdry conv$(Q \cap W)$, so $[R(x,r) \sim [x,r)] \cap N \subseteq$ conv$(Q \cap W)$. Since $R(x,r) \sim [x,r)$ intersects (t_1,t_2), then $(t_1,t_2) \cap$ conv$(Q \cap W) \neq \phi$, contradicting our choice of t_j. Then by a generalization of Valentine's lemma [4, Corollary 1], $[x,t_2] \subseteq S$. For $j > 2$, the above argument may be used to show that $[x,t_2] \subseteq S \sim Q$. An easy induction gives $[x,t_{j-1}] \subseteq S \sim Q$ and $[x,t_j] \subseteq S$. Thus $t_j \in T$. However, this is impossible since $t_j \notin Q$ and we know that $T \subseteq Q$. We conclude that Case 1b cannot occur, dim conv$(T \cup \{x\}) = d$, and the previous argument in Case 1a guarantees our result.

CASE 2. Assume that N' may be selected so that for M' any convex neighborhood of q and $M' \subseteq N'$, dim conv$(Q \cap M') = d - 1$. Let M be such a neighborhood of q, and let $H = \text{aff}(Q \cap M)$. By Definition 2, we have $q \in \text{rel int}(S \cap H)$, so without loss of generality we may assume that $M \cap H \subseteq S$. Also assume that $M \cap S$ is starshaped relative to q.

Select k visually independent points $x_1,\ldots,x_k$ in $S \cap M$. Since S is locally a full d-dimensional, clearly these points may be selected in $(S \cap M) \sim H$. For each i, consider the set T_i in $M \cap H$ seen by x_i. By arguments used in the proof of Lemma 2, it is easy to show that conv$(\{x_i\} \cup T_i) \subseteq S$. Also, using the definition of essential, one may show that T_i is a $(d - 1)$-dimensional set.

For simplicity of notation, assume that q is the origin in R^d and that H

is orthogonal to the vector $e_1 = (1,0,\dots,0)$. Let H_1, H_2 denote distinct open halfspaces determined by H , labeled so that e_1 is in H_1 . Finally, define S_i to be the closure of the set

$$\operatorname{conv}(\{x_i\} \cup T_i) \cup ((M \cap H) \times [q,z])$$

where $z = -e_1$ if $x_i \in H_1$ and $z = e_1$ if $x_i \in H_2$.

For each i , it is easy to show that the set Q_i of lnc points of S_i lies in Q . Furthermore, every point of $Q \cap M$ is an lnc point for some S_i set. Now S_i is 3-convex, $q \in (\operatorname{int}\operatorname{conv} S_i) \cap Q_i$, and it is easy to see that $\operatorname{int}\ker S_i \neq \phi$ for each i . Hence by Valentine's theorem there is a neighborhood U_i of q so that $U_i \cap Q_i$ is a $(d-2)$-dimensional manifold. Thus for an appropriate neighborhood U of q , $U \cap Q$ is a union of k $(d-2)$-manifolds, each containing q .

CASE 3. In case $\operatorname{conv}(Q \cap M)$ has dimension $\leq d-2$ for some neighborhood M of q , we assert that $\operatorname{conv}(Q \cap M) = Q \cap M$ and hence $Q \cap M$ is a convex set of dimension $d-2$ by a result in [1].

Without loss of generality, assume that M is a convex neighborhood of q satisfying Definition 1. Let S' denote the closure of the set $S \cap M$, $Q' = \operatorname{cl}(Q \cap M)$ the corresponding set of lnc points of S' . Since M satisfies Definition 1, $S' \sim Q'$ is connected. By a previous lemma, $Q' \subseteq \operatorname{cl}(S' \sim Q')$, so $S' \sim Q' \subseteq S' \subseteq \operatorname{cl}(S' \sim Q')$, and S' is connected. We have S' closed, connected, and $S' \sim Q'$ connected, so $S' = \operatorname{cl}(\operatorname{int} S')$ by [1, Lemma 1]. Also, by the argument in [1, Lemma 4], the set $S' \sim \operatorname{aff} Q'$ is connected.

Now let r be a point in $\operatorname{conv}(Q \cap M)$ to show that $r \in Q$. Let A denote the subset of $S' \sim \operatorname{aff} Q'$ which r sees via S . By repeating arguments in [1, Lemma 5], it is easy to show that A is open and closed in $S' \sim \operatorname{aff} Q'$ and that $A \neq \phi$. Hence $A = S' \sim \operatorname{aff} Q'$, and r sees $S' \sim \operatorname{aff} Q'$ via S .

Finally, select x,y in $S' \sim \operatorname{aff} Q'$ with $[x,y] \nsubseteq S$ and $y \notin \operatorname{aff}(Q' \cup \{x\})$ (Clearly this is possible since $S' = \operatorname{cl}(\operatorname{int} S')$.) By Valentine's lemma [4], there must be some lnc point in $\operatorname{conv}\{x,y,r\} \sim [x,y]$, but by our choice of x and y , there can be no lnc point p in $\operatorname{conv}\{x,y,r\} \sim ([x,y] \cup \{r\})$: Otherwise,

$y \in \text{aff}\{p,x,r\} \subseteq \text{aff}(Q' \cup \{x\})$, impossible. Hence r must belong to Q and $\text{conv}(Q \cap M) \subseteq Q \cap M$. The reverse inclusion is obvious, $\text{conv}(Q \cap M) = Q \cap M$, and the assertion is proved.

The set S' is a closed connected set whose corresponding set of lnc points is convex and satisfies Definition 1 in [1]. Hence by the corollary to Theorem 2 in [1], Q' has dimension $d - 2$. This completes Case 3 and finishes the proof of the theorem.

COROLLARY 1. Let S be a compact m-convex set in R^d , S locally a full d-dimenisonal, with Q the corresponding set of lnc points of S . Assume that for every point q in Q , there is some $k > 0$ such that q is an essential lnc point of order k . Then Q is a finite union of $(d - 2)$-dimensional manifolds.

PROOF. Since Q is compact, the result is an immediate consequence of Theorem 1.

The following examples reveal that Theorem 1 fails in case q does not satisfy both Definition 1 and Definition 2, part 3.

EXAMPLE 1. It is easy to find examples which show that q must be essential in Theorem 1. For $d \geq 3$, simply consider two d-dimensional convex sets which meet in a single point q .

EXAMPLE 2. To see that Definition 2, part 3 is required when $\dim \text{conv}(Q \cap N) = d$, let $d = 2$ and identify R^2 with the complex plane. Let S_1 be the infinite sided polygon having consecutive vertices $\exp 0$, $\exp \frac{\pi i}{2}, \ldots, \exp \frac{(2^n - 1)\pi i}{2^n}$, $n \geq 0$. Similarly, let S_2 be the infinite sided polygon with vertices $\exp 0$, $\exp \frac{\pi i}{4}, \exp \frac{5\pi i}{8}, \ldots, \exp \frac{(2^{n+1} - 3)\pi i}{2^{n+1}}$, $n \geq 1$. (See Figure 1.) The set $S = \text{cl}(\text{conv } S_1 \cup \text{conv } S_2)$ is 3-convex, and its lnc points are essential. However, for every neighborhood N of $q = \exp \pi i$ and every component C of $(S \cap N) \sim \text{conv}(Q \cap N)$, $q \notin \text{int conv}(C \cup (Q \cap N))$. Clearly $Q \cap N$ is not expressible as a finite union of $(d - 2)$-manifolds. The example may be generalized to higher dimensions.

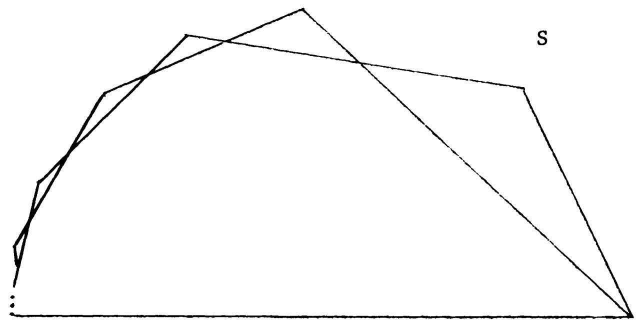

Figure 1.

EXAMPLE 3. To see that Definition 2, part 3 must be satisfied when $\dim \operatorname{conv}(Q \cap N) = d - 1$, let $d = 3$ and identify the $x - y$ plane H with the complex plane. In this plane let P be the infinite sided polygon having vertices $v_n = \exp \frac{(2^n - 1)\pi i}{2^n}$, $n \geq 0$. At each vertex v_n, $n \geq 1$, strictly separate v_n from the remaining vertices with a line L_n so that L_n cuts each edge of P adjacent to v_n and so that no two L_n lines intersect in conv P. (See Figure 2.) Each line L_n determines a closed triangular subset T_n of conv P.

Let R be the rectangle in the $x - y$ plane whose vertices are $(1,0)$, $(-1,0)$, $(-1,-1)$, $(1,-1)$, and define

$$A_0 = \operatorname{conv} P \sim \cup\{T_n : n \geq 1\},$$

$$A_1 = \cup\{T_n : n \equiv 0 \mod 3 \text{ or } n \equiv 1 \mod 3\} \cup A_0 \cup R,$$

$$A_2 = \cup\{T_n : n \equiv 0 \mod 3 \text{ or } n \equiv 2 \mod 3\} \cup A_0 .$$

Finally, let $S_1 = \operatorname{cl} A_1 \times [\theta, e_3]$ and $S_2 = \operatorname{cl} A_2 \times [\theta, -e_3]$, where $e_3 = (0,0,1)$ and $\theta = (0,0,0)$. Clearly both S_1 and S_2 are convex and closed. Label the halfspaces determined by H so that $S_1 \subseteq \operatorname{cl} H_1$ and $S_2 \subseteq \operatorname{cl} H_2$.

Let B denote a 3-dimensional parallelepiped in $\operatorname{cl} H_2$, with $B \cap H = B \cap (S_1 \cup S_2) = R$. The set B may be constructed so that the point $q = (-1,0,0)$ is interior to $\operatorname{conv}(S_1 \cup S_2 \cup B)$. Hence letting S denote the 4-convex set $S_1 \cup S_2 \cup B$, it is not hard to show that $q \in \operatorname{int} \operatorname{conv}(S \cap N)$ for every neighborhood N of q.

Note that the set Q of lnc points of S is exactly

$\cup \{L_i \cap T_i : i \neq 0 \mod 3\} \cup [q,r]$, where $r \not= (1,0,0)$. For every ne..nbc-rhood N of q , dim conv(Q $\cap$ N) = d - 1 , yet S does not tisfy part 3 of Definition 2 and Q $\cap$ N is not a finite union of (d - 2)-manifolds. Further-more, it is interesting to notice that for every neighborhood N of - and for every component C of (S $\cap$ N) ~ conv(Q $\cap$ N) , C is exactl (S $\cap$ N) ~ conv(Q $\cap$ N) , and q $\in$ int conv(C $\cup$ (Q $\cap$ N)) = int conv(S $\cap$ N) . Thus the require-ment that q belong to int conv(C $\cup$ (Q $\cap$ N)) i- -ot sufficient to guarantee our result in case dim conv(Q $\cap$ N) = d - 1

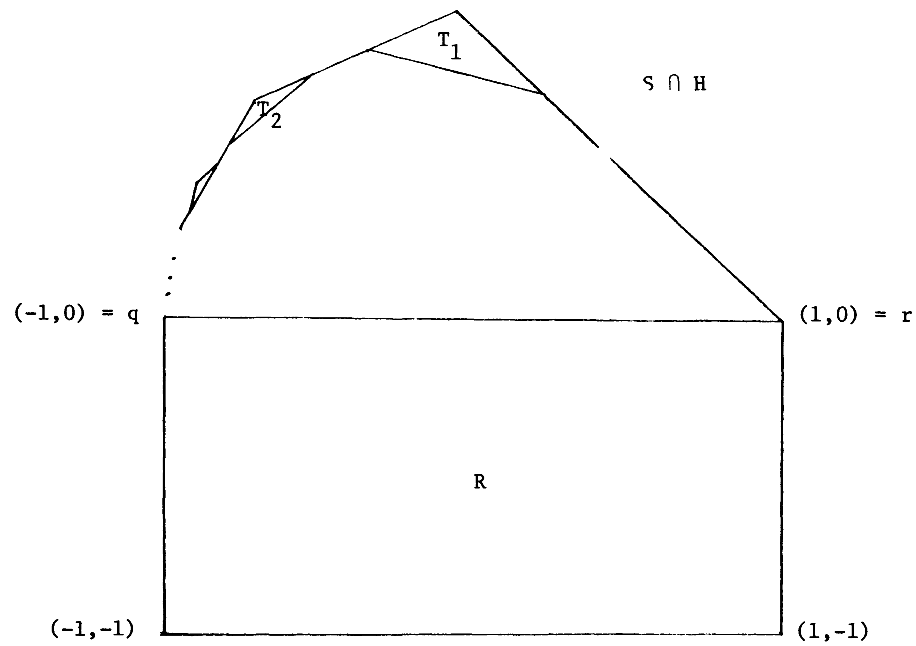

Figure 2.

The author would like to thank the referee for providing three additional examples given below. The first of these (Example 4) reveals that the conclusion of Theorem 1 may hold without Definition 2, part 1.

EXAMPLE 4. Let S be the closed set in Figure 3. (S is a cube from which a smaller cube has been removed.) The lnc point q of S satisfies Definition 1 and parts 2 and 3 of Definitions 2 for k = 3 . Definition 2, part 1 does not hold. However, Q is expressible as a union of three d - 2 = 1 dimensional manifolds.

Whether Theorem 1 is true without Definition 2, part 1 remains an open question.

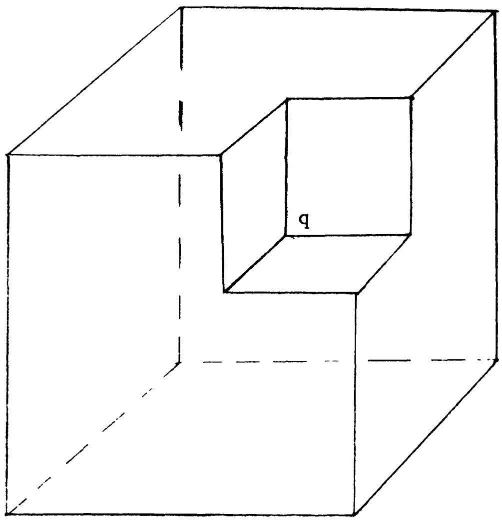

Figure 3.

Furthermore, the conclusion of Theorem 1 can hold when q is not essential, as Example 5 reveals. (Compare to Example 1 in which q is not essential and Theorem 1 fails.)

EXAMPLE 5. For $d \geq 2$, let S be a union of two d-polytopes which intersect in a common $(d - 2)$-dimensional face Q. Then the lnc points of S are not essential, yet Q is a $(d - 2)$-dimensional manifold.

It would be interesting to obtain an extension of Theorem 1 to include the situations of Examples 4 and 5.

The final example by the referee illustrates Theorem 1.

EXAMPLE 6. Let S be the union of four stacked cubes of equal size in Figure 4. The point q is an essential lnc point of order 3, and Q is expressible as a union of three 1-dimensional manifolds, each containing q.

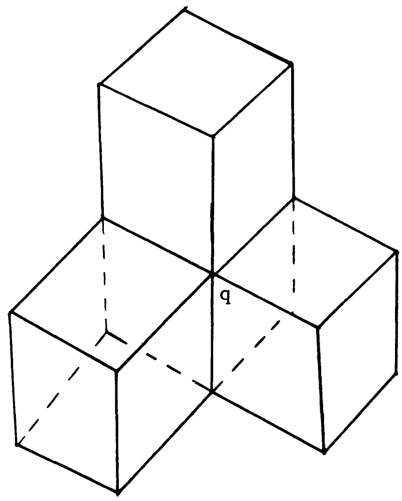

Figure 4.

3. Q IS NOWHERE DENSE IN BDRY S .

The final theorem will require the following easy lemma.

LEMMA 3. Let S be a closed m-convex set in R^d, Q the corresponding set of lnc points of S. Let N be a convex neighborhood. If $S \cap N$ is exactly k-convex, then there exist points $x_1,\dots,x_{k-1}$ in $(S \cap N) \sim Q$ which are visually independent via $S \cap N$.

PROOF. Select $y_1,\dots,y_{k-1}$ visually independent via $S \cap N$, and let $N_1,\dots,N_{k-1} \subseteq N$ be corresponding neighborhoods of $y_1,\dots,y_{k-1}$ respectively, such that no point of N_i sees any point of N_j via S, $1 \le i < j \le k-1$. By Lemma 1, each N_i contains some point x_i in $S \sim Q$, and the points $x_1,\dots,x_{k-1}$ are the required visually independent points.

THEOREM 2. Let S be a closed m-convex set in R^d, S locally a full d-dimensional, with Q the set of lnc points of S. For q in Q and N any convex neighborhood of q, $N \cap \text{bdry } S \not\subseteq Q$. That is, Q is nowhere dense in bdry S.

PROOF. Assume on the contrary that $N \cap \text{bdry } S \subseteq Q$ for some convex neighborhood N of q. We assert that for some point r in $Q \cap N$ and some neighborhood U of r, $\text{conv}(Q \cap U) \subseteq S$: Suppose on the contrary that no such r exists. Select two points x,y in $S \cap N$ whose corresponding segment $[x,y]$

is not in S . The segment $[x,y]$ intersects bdry S , and since S is closed, clearly we may select points x',y' in bdry $S \cap [x,y]$ with $[x',y'] \not\subseteq S$. For convenience of notation, assume $x = x'$ and $y = y'$. Since $[x,y] \not\subseteq S$, there exist disjoint convex neighborhoods N_1 and N_2 for x and y respectively, $N_1 \cup N_2 \subseteq N$, so that no point of N_1 sees any point of N_2 via S . Since $x,y \in N \cap \text{bdry } S \subseteq Q \cap N$, $\text{conv}(Q \cap N_1) \not\subseteq S$ and $\text{conv}(Q \cap N_2) \not\subseteq S$.

Now repeat the argument for each of N_1 and N_2 . By an obvious induction, we obtain a collection of m visually independent points of S , contradicting the fact that S is m-convex. Hence our supposition is false and for some point r in $Q \cap N$ and for some neighborhood U of r , $\text{conv}(Q \cap U) \subseteq S$, the desired result.

Therefore, without loss of generality we may assume that $\text{conv}(Q \cap N) \subseteq S$. Also assume that $S \cap N$ is exactly j-convex, $3 \le j \le m$. By Lemma 3, there exist points $x_1,\ldots,x_{j-1}$ in $(S \cap N) \sim Q$ which are visually independent via S , and clearly at most one x point, say x_1 is in $\text{conv}(Q \cap N)$. Now if every point of $\text{conv}(Q \cap N) \cap \text{bdry } S$ sees one of $x_2,\ldots,x_{j-1}$ via S , delete x_1 from our listing. Otherwise, some $z \in \text{conv}(Q \cap N) \cap \text{bdry } S$ does not see any x_i via S , $2 \le i \le j - 1$, and for some neighborhood M of z , $M \subseteq N$, no point of $S \cap M$ sees any x_i via S , $2 \le i \le j - 1$. Select $x_0 \in (S \cap M) \sim \text{conv}(Q \cap N)$. (Clearly such an x_0 exists since $z \in Q$.) Replacing x_1 by x_0 , we have $x_0,x_2,\ldots,x_{j-1}$, a collection of j visually independent points, and since $S \cap N$ is exactly j-convex, every point of $\text{conv}(Q \cap N) \cap \text{bdry } S$ sees one of these points via S . Hence in either case we have a collection of points $y_1,\ldots,y_k$ in $(S \cap N) \sim \text{conv}(Q \cap N)$ such that every point of $\text{conv}(Q \cap N) \cap \text{bdry } S$ sees one of these points via S , $j - 2 \le k \le j - 1$.

For the moment, suppose that for every neighborhood $U \subseteq N$ with $U \cap Q \neq \phi$, $\dim \text{conv}(Q \cap U) = d$. Let Q_i denote the subset of $Q \cap N$ seen by y_i , $1 \le i \le k$. By Lemma 2, $\text{conv}(\{y_i\} \cup Q_i) \subseteq S$ for each i . Since $y_i \notin \text{conv}(Q \cap N)$, certainly $\dim \text{conv}(\{y_i\} \cup Q_i) \le d - 1$, for otherwise $\text{conv}(\{y_i\} \cup Q_i)$ would capture some point of Q in its interior, impossible.

Thus $Q \cap N$ lies in a finite union of flats, each having dimension $\leq d - 1$. Moreover, since for every neighborhood $U \subseteq N$ with $U \cap Q \neq \phi$, $U \cap Q$ does not lie in a hyperplane, it follows that $U \cap \text{bdry } S \not\subseteq Q_i$. That is, Q_i is necessarily nowhere dense as a subset of bdry S. Then $Q \cap N = \cup Q_i$ is a finite union of sets, each nowhere dense in bdry S, and by standard arguments $Q \cap N$ is nowhere dense in bdry S. We have a contradiction, our supposition is false, and $\dim \text{conv}(Q \cap U) \leq d - 1$ for some neighborhood $U \subseteq N$ with $U \cap Q \neq \phi$. Since S is a full d-dimensional, $\dim \text{conv}(Q \cap U) = d - 1$ for such a neighborhood U. For convenience of notation, assume that $\dim \text{conv}(Q \cap N) = d - 1$.

We assert that since $N \cap \text{bdry } S \subseteq Q$ and $\dim \text{conv}(Q \cap N) = d - 1$, then $Q \cap N$ is convex: For x,y in $Q \cap N$ and $x < z < y$, we will show that $z \in \text{bdry } S$. Otherwise, there would be a neighborhood V of z interior to S, with $V \subseteq N$. Since $x \in \text{bdry } S$, there is a sequence $\{x_n\}$ in $R^d \sim S$ converging to x, and for each x_n and each p in V, $(x_n,p) \cap \text{bdry } S \neq \phi$. A parallel statement holds for y. This implies that $\dim \text{conv}(N \cap \text{bdry } S) = d$ and $\dim \text{conv}(Q \cap N) = d$, impossible. We have a contradiction, and z must belong to bdry S. Hence $z \in (\text{bdry } S) \cap N \subseteq Q \cap N$, and $Q \cap N$ is indeed convex.

Again let Q_i denote the subset of $Q \cap N$ seen by y_i, $1 \leq i \leq k$. Since $\text{conv}(\{y_i\} \cup Q_i) \subseteq S$ for every i, Q_i is necessarily a convex subset of $Q \cap N$, and since $\dim(Q \cap N) = d - 1$, some Q_i set, say Q_1, has dimension $d - 1$. Then the set $\text{conv}(\{y_1\} \cup Q_1)$ is a full d-dimensional. Our previous argument may be repeated to obtain a finite set of visually independent points $z_1,\ldots,z_n$ in $(S \cap \text{cone}(y_1,Q_1)) \sim \text{conv}(\{y_1\} \cup Q_1)$, each z_i seeing a subset T_i of Q_1 having dimension at most $d - 2$, with $Q_1 = \cup\{T_i : 1 \leq i \leq n\}$. Clearly this is impossible, our assumption is false, and $N \cap \text{bdry } S \not\subseteq Q$ for every neighborhood N of q. This completes the proof of Theorem 2.

Techniques identical to those employed in the proof of Theorem 2 may be used to obtain the following result.

COROLLARY 1. Let S be a closed m-convex set in R^d with Q the set of lnc points of S. Then if $\text{conv}(Q \cap N) \subseteq S$ for some neighborhood N, $Q \cap N$

cannot be homeomorphic to a (d - 1)-dimensional manifold.

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