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THE CONVOLUTION-INDUCED TOPOLOGY ON $L_\infty(G)$ AND LINEARLY DEPENDENT TRANSLATES IN $L_1(G)$

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ABSTRACT. Given a locally compact Hausdorff group G, we consider on $L_\infty(G)$ the τ_c-topology, i.e. the weak topology under all convolution operators induced by functions in $L_1(G)$. As a major result we characterize the trigonometric polynomials on a compact group as those functions in $L_1(G)$ whose left translates are contained in a finite-dimensional set. From this, we deduce that τ_c is different from the w^*-topology on $L_\infty(G)$ whenever G is infinite. As another result, we show that τ_c coincides with the norm-topology if and only if G is discrete. The properties of τ_c are then studied further and we pay attention to the τ_c-almost periodic elements of $L_\infty(G)$.

KEY WORDS AND PHRASES. *Locally compact group, convolution operator, topology induced by convolution, linearly dependent translates, almost periodic functions.*

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1. INTRODUCTION.

The reader intending to read the following paper should have some familiarity with such basic texts as Hewitt and Ross or Dunford and Schwartz.

For a locally compact Abelian group G, Argabright and Gil de Lamadrid [1] considered almost periodicity of measures with respect to several topologies. A special case of this general notion, namely almost periodicity with respect to the τ_c-topology on $L_\infty(G)$, has been used in Crombez and Govaerts [2] in order to characterize those multipliers from $L_1(G)$ to $L_\infty(G)$ which are almost periodic in the strong operator topology. Throughout this paper, unless explicitly stated otherwise, G will denote a locally compact Hausdorff group with left Haar measure. For such an arbitrary G the τ_c-topology is not weaker than the w^*-topology and not stronger than the norm topology on $L_\infty(G)$. The question as to whether there are neighborhoods in the τ_c-topology which are also neighborhoods in the w^*-topology leads us to consider the apparently completely different problem of determining those functions $f \neq 0$ in $L_1(G)$ such that all left translates of f are in a finite-dimensional subspace of $L_1(G)$ (a related problem was recently investigated in Edgar and Rosenblatt [3] for Abelian groups). We prove that such functions only exist for compact G, and then they are exactly the trigonometric polynomials. From this result we derive that the τ_c-topology is always different from the w^*-topology whenever G is infinite. However, a further investigation shows that for compact G these two topologies coincide on every norm-bounded subset of $L_\infty(G)$, and so we may conclude that for compact G $L_1(G)$ is the dual of $(L_\infty(G), \tau_c)$. Among the other results we mention that except for discrete G the τ_c-topology is always different from the norm-topology(section 3), and that for fixed g in $L_\infty(G)$ the map $s \to {}_s g$ from G to $(L_\infty(G), \tau_c)$ is continuous (section 4). In section 5 we give some further results about τ_c-almost periodic functions.

For complex-valued functions f and g on G and $a \in G$, we define the left translate ${}_a f$ and the convolution $f*g$ by means of ${}_a f(x) = f(ax)$ and $(f*g)(x) = \int_G (xy) g(y^{-1}) dy$ (we warn the reader that in some of the references, e.g. [4] and [5], different conventions are used). Each function f in $L_1(G)$ induces by convolution an operator T_f on $L_\infty(G)$; the weak topology on $L_\infty(G)$ under all convolution operators $T_f : L_\infty(G) \to (L_\infty(G), \|\ \|_\infty)$ is denoted by τ_c. By w^* and $\|\ \|_\infty$ we denote the $(L_\infty(G), L_1(G))$, i.e., weak * topology, and the essential supremum norm topology respectively, on $L_\infty(G)$. All other nonexplained notation is taken from Hewitt and Ross [6].

2. FUNCTIONS IN $L_1(G)$ WITH FINITE-DIMENSIONAL SPAN OF TRANSLATES.

From the definitions we immediately derive $w^* \leq \tau_c \leq \|\ \|_\infty$. Investigation of the possibility that some τ_c-neighborhood is also a w^*-neighborhood leads to a special class of functions in $L_1(G)$, as Proposition 1 shows. For convenience we take as a subbase at 0 for w^* the sets

$\{h\in L_\infty(G) : |\int_G f(x)h(x^{-1})dx| <\varepsilon\}$, where $f\in L_1(G)$ and $\varepsilon>0$; we write $<f,h>$ for $\int_G f(x)h(x^{-1})dx$.

PROPOSITION 1. For $0\neq f\in L_1(G)$ the following are equivalent:

(i) There exists an $\varepsilon>0$ such that the τ_c-neighborhood determined by f and ε is a w^*-neighborhood.

(ii) The set of left translates of f is part of a finite-dimensional subspace of $L_1(G)$.

(iii) There exist $a_1,\dots,a_n$ in G such that, for each a in G, scalars $c_1,\dots,c_n$ may be found such that ${}_af = \sum_{i=1}^n c_i\, {}_{a_i}f$.

(iv) Given $\varepsilon>0$, there exists $a_1,\dots,a_n$ in G and $\delta>0$ such that, for $g\in L_\infty(G)$, the inequality $|<{}_{a_i}f, g>|<\delta$ for all $i=1,\dots,n$ implies $\|f*g\|_\infty<\varepsilon$.

PROOF (i)→(ii). Suppose that the set $\{g\in L_\infty(G) : \|f*g\|_\infty<\varepsilon\}$ is a w^*-neighborhood of zero. Then we may find functions $f_i (i=1,\dots,r)$ in $L_1(G)$ and $\delta>0$ such that, whenever $g\in L_\infty(G)$ and $|\int_G f_i(x)g(x^{-1})dx|<\delta$ for all i, then $\|f*g\|_\infty<\varepsilon$. Each f_i determines a linear functional on $L_\infty(G)$; call N the intersection of their kernels. Since for any scalar c, $cg\in N$ whenever $g\in N$, there results that $|c|\ \|f*g\|_\infty<\varepsilon$ for $g\in N$ and for any scalar c; hence $f*g=0$ for g in N, or $\int_G {}_af(y)g(y^{-1})dy=0$ for any a in G and g in N. This means that, for given a in G, the linear functional determined by ${}_af$ may be written as a linear combination of the ones determined by the f_i $(i=1,\dots,r)$. So, given a in G, there exist scalars $\alpha_1,\dots,\alpha_r$ such that ${}_af = \sum_{i=1}^r \alpha_i f_i$.

(ii)→(iii). Obvious. We may choose $a_1,\dots,a_n$ in G such that the set $\{{}_{a_i}f\}_{i=1}^n$ is also a linearly independent set.

(iii)→(iv). We first remark that the assumption of (iii) implies that G is necessarily compact. Indeed, whenever (iii) is true the set $\{{}_af : a\in G\}$ of left translates of f is a norm-bounded subset of a finite-dimensional subspace of $L_1(G)$, and so this set is relatively compact with respect to the norm-topology of $L_1(G)$. However, it was shown in Crombez and Govaerts [4] that for non-compact G only f=0 has this property.

From this it also follows that there exists $B>0$ such that for all $a\in G$ $\sum_{i=1}^n |c_i|<B$

for the scalars figuring in (iii). Indeed, the function $a \to {}_a f$ from G to $L_1(G)$ is continuous, and its range is part of a finite-dimensional subspace M of $L_1(G)$; assuming, as we may, that $\{{}_{a_i} f\}_{i=1}^n$ is linearly independent, the function ${}_a f = \sum_{i=1}^n c_i \, {}_{a_i} f \to (c_1,\dots,c_n)$ from M to the n-dimensional complex space $\mathbb{C}^n$ is (well-defined and) linear, and hence continuous; so the composition of these two functions is continuous on the compact group G, from which the result follows.

Suppose then that (iii) is true, and let $\varepsilon>0$ be given. Choose $\delta>0$ such that $B\delta<\varepsilon$, with B as mentioned above. If $a_1,\dots,a_n$ are as in (iii) and $|<{}_{a_i} f,g>|<\delta$ for all $i=1,\dots,n$, then for $a \in G$ we have

$$|(f*g)(a)| = \left|\int_G \Big(\sum_{i=1}^n c_i \; {}_{a_i} f\Big)(y) g(y^{-1}) dy\right| \leq \sum_{i=1}^n |c_i| \; |<{}_{a_i} f,g>| < \varepsilon .$$

(iv)$\Rightarrow$(i). Obvious.∎

Statement (iii) in Proposition 1 leads to the following problem: determine those $f \in L_1(G)$ for which all left translates are contained in a finite-dimensional subset of $L_1(G)$. As remarked in the proof of the proposition such nonzero functions can exist only for compact G. To solve this problem, we use the theory of representations of compact groups as explained in Hewitt and Ross [6]. It is readily verified that the set of functions with the mentioned property is a linear subspace V of $L_1(G)$ containing all trigonometric polynomials. Proposition 2 shows that there are no other functions in V. (For related results in the abelian case, we refer to Schwartz [7], and to the recent paper of Laird [8] and the references mentioned there.)

PROPOSITION 2. Let $0 \neq f \in L_1(G)$ with G compact. The set $\{{}_a f : a \in G\}$ of left translates of f is contained in a finite-dimensional space iff f is a trigonometric polynomial on G.

PROOF. We first remark that f is a trigonometric polynomial iff the Fourier transform $\hat{f}$ of f is such that $\hat{f}(\sigma)=0$ except for a finite number of elements σ in the dual object Σ of G (see Hewitt and Ross [6], 28.39).

Let then $f \in L_1(G)$ be such that statement (iii) of Proposition 1 is true, i.e., ${}_a f = \sum_{i=1}^n c_i(a) \, {}_{a_i} f$ (for fixed n) and $\sum_{i=1}^n |c_i(a)| \leq B$ (this was shown in the proof of (iii)$\Rightarrow$(iv) above). Taking the Fourier transform we obtain

$[\overline{U}_a^{(\sigma)} - \sum_{i=1}^{n} c_i(a)\overline{U}_{a_i}^{(\sigma)}]\ \hat{f}(\sigma)=0$. Let D be the set of those $\sigma\in\sum$ for which $\hat{f}(\sigma)$ is different from zero. Then for each $\sigma\in D$ there is a subspace $M_\sigma\neq\{0\}$ in the representation space H_σ of $U^{(\sigma)}$ such that $\overline{U}_a^{(\sigma)} - \sum_{i=1}^{n} c_i(a)\overline{U}_{a_i}^{(\sigma)} = 0$ on M_σ, $\forall a\in G$. We choose an element $\xi^{(\sigma)}$ in M_σ with $\|\xi^{(\sigma)}\| = 1$. Since $\overline{U}^{(\sigma)}$ is irreducible, the non-zero vector $\xi^{(\sigma)}$ is a cyclic vector for $\overline{U}^{(\sigma)}$, which means that the set of all finite linear combinations of elements from $\{\overline{U}_a^{(\sigma)}\xi^{(\sigma)} : a\in G\}$ is all of H_σ; but the set $\{\overline{U}_a^{(\sigma)}\xi^{(\sigma)} : a\in G\}$ is spanned by the finitely many vectors $\overline{U}_{a_1}^{(\sigma)}\xi^{(\sigma)},\dots,\overline{U}_{a_n}^{(\sigma)}\xi^{(\sigma)}$; hence, if d_σ denotes the dimension of H_σ we always have $d_\sigma\leq n$, for each σ in D. With the choice of $\xi^{(\sigma)}$ we have $|<\overline{U}_{a_i}^{(\sigma)}\xi^{(\sigma)},\xi^{(\sigma)}>|\leq 1$ for all $\sigma\in D$ and all $i\in\{1,\dots,n\}$, where now $<,>$ denotes the inner product on H_σ. If D is infinite, we obtain an infinite family $\{(<\overline{U}_{a_1}^{(\sigma)}\xi^{(\sigma)},\xi^{(\sigma)}>,\dots,<\overline{U}_{a_n}^{(\sigma)}\xi^{(\sigma)},\xi^{(\sigma)}>\}_{\sigma\in D}$ in a compact set in the n-dimensional complex space, and so it has a cluster point; this means that, given $0<\varepsilon<\frac{1}{n}$, there exist different σ_1 and σ_2 in D such that

$$|<\overline{U}_{a_i}^{(\sigma_1)}\xi^{(\sigma_1)},\xi^{(\sigma_1)}>-<\overline{U}_{a_i}^{(\sigma_2)}\xi^{(\sigma_2)},\xi^{(\sigma_2)}>|\leq\frac{\varepsilon}{B} \quad \text{for all } i.$$

For each a in G we then have

$$|<\overline{U}_a^{(\sigma_1)}\xi^{(\sigma_1)},\xi^{(\sigma_1)}>-<\overline{U}_a^{(\sigma_2)}\xi^{(\sigma_2)},\xi^{(\sigma_2)}>| =$$

$$= |\sum_{i=1}^{n} c_i(a)(<\overline{U}_{a_i}^{(\sigma_1)}\xi^{(\sigma_1)},\xi^{(\sigma_1)}>-<\overline{U}_{a_i}^{(\sigma_2)}\xi^{(\sigma_2)},\xi^{(\sigma_2)}>)|\leq\varepsilon$$

Assuming that the Haar measure of the compact group G is normalised, it follows that

$$|\int_G (<\overline{U}_a^{(\sigma_1)}\xi^{(\sigma_1)},\xi^{(\sigma_1)}>-<\overline{U}_a^{(\sigma_2)}\xi^{(\sigma_2)},\xi^{(\sigma_2)}>)\ U_a^{(\sigma_1)}\xi^{(\sigma_1)},\xi^{(\sigma_1)}>da|\leq\varepsilon,$$

while the first member has the value $\frac{1}{d_{\sigma_1}}$. Since $d_{\sigma_1}\leq n$ (fixed), we arrive at a contradiction by our choice of ε. ∎

3. CONNECTION OF τ_c WITH OTHER TOPOLOGIES ON $L_\infty(G)$.

From Proposition 1 we immediately conclude that for non-compact G the w^*-topology is always strictly weaker than the τ_c-topology. But taking Proposition 2 into account, we infer that also for infinite compact G these two topologies are different. Indeed, it suffices to remark that there always exists a function f in $L_1(G)$ which is not a trigonometric polynomial (e.g., choose in $\sum$ a countable infinite set $\{\sigma_n\}_{n=1}^{\infty}$ of different elements; let χ_{σ_n} be the corresponding character, and put $f(x) = \sum_{n=1}^{\infty} \frac{\chi_{\sigma_n}(x)}{n^2 d^2_{\sigma_n}}$ for x in G; then $f \in L_1(G)$, and $\hat{f}(\sigma_n) = \frac{1}{n^2 d^2_{\sigma_n}} I_{H_{\sigma_n}}$, where $I_{H_{\sigma_n}}$ is the identity operator on H_{σ_n}).

Although τ_c and w^* are different for infinite compact G, they induce the same topology on every norm-bounded subset of $L_\infty(G)$, as the following proposition shows.

PROPOSITION 3. If G is compact, and B is a norm-bounded subset of $L_\infty(G)$, then τ_c and w^* coincide on B.

PROOF. It is sufficient to prove that for any τ_c-neighborhood V of 0 there exists a w^*-neighborhood W of 0 such that $W \cap B \subset V$. Suppose that $\|h\|_\infty \leq M$, $\forall h \in B$, and let $V=\{h \in L_\infty(G): \|f_i * h\|_\infty < \varepsilon$ for $i=1,\ldots,n\}$ with given $f_i \in L_1(G)$ and $\varepsilon>0$. From compactness of G and continuity of $a \to {}_a f$ from G to $(L_1(G), \| \|_1)$ it follows that each f_i is almost periodic in $(L_1(G), \| \|_1)$; this means that there exists elements $a_1,\ldots,a_m$ in G such that, for each a in G and each $i \in \{1,\ldots n\}$ a point a_j may be found $(1 \leq j \leq m)$ such that $\| {}_a f_i - {}_{a_j} f_i \| < \frac{\varepsilon}{2M}$. With this choice of a_j and for $g \in L_\infty(G)$ we have

$|(f_i * g)(a) - (f_i * g)(a_j)| \leq \| {}_a f_i - {}_{a_j} f_i \|_1 \| g \|_\infty$, or for g in B, $|(f_i * g)(a)| \leq |\langle {}_{a_j} f_i, g\rangle| + \frac{\varepsilon}{2}$.

Put $W=\{h \in L_\infty(G): |\langle {}_{a_j} f_i, h\rangle| < \frac{\varepsilon}{2}$ for all $i,j\}$. Then W is a w^*-neighborhood of 0, and for h in $W \cap B$ we obtain $\|f_i * h\|_\infty < \varepsilon$. ∎

COROLLARY 1. For compact G, any w^*-convergent sequence is τ_c-convergent. Indeed, the set consisting of the elements in the sequence together with its limit is w^*-compact, and hence also norm bounded. ∎

COROLLARY 2. For compact G, $L_1(G)$ is the dual of $(L_\infty(G), \tau_c)$.

PROOF. For a compact group G there is a connection between the τ_c-topology and the so-called bounded weak*-topology bw^* (see Holmes [9], p. 150; this topology is called the bounded X-topology in Dunford and Schwartz [10], p. 427); indeed, we have $\tau_c \leq bw^*$. The result then follows from the fact that $L_1(G)$ is the dual of $(L_\infty(G), bw^*)$. ∎

The following proposition characterizes those groups for which τ_c and $\| \|_\infty$

coincide.

PROPOSITION 4. τ_c coincides with $\|\ \|_\infty$ iff G is discrete.

PROOF. For discrete G we have that the τ_c-topology and the $\|\ \|_\infty$-topology are equal. For if e is the identity of G, then δ_e is a convolution identity for $L_1(G)$, and the convolution operator T_{δ_e} induced by δ_e is the identity map on $L_\infty(G)$.

Let then G be non-discrete. Given $\varepsilon>0$ and $f_1,\ldots,f_n$ in L1(G), choose a compact subset K in G such that $\int_{G/K} |f_i(x)|dx<\frac{\varepsilon}{2}$, for each $i=1,\ldots,n$. Let $\eta>0$ be such that $\int_A |f_i(x)|dx<\frac{\varepsilon}{2}$ for all $i=1,\ldots,n$ and all measurable $A\subset G$ with $\mu(A)<\eta$, where μ denotes left Haar measure. Further, let U be a compact symmetric neighborhood of the identity e of G with $0<\mu(U)<\eta$. Let g be the function defined by $g(x)=1$ for $x\in U$, and $g(x) = 0$ on $G\setminus U$. For $1 \leqslant i \leqslant n$ and $x\in G$ we obtain

$$|(f_i*g)(x)|\leqslant \int_{G\setminus K} |f_i(y)|dy+|\int_K f_i(y)g(y^{-1}x)dy|.$$

Both terms on the right-hand side are dominated by $\frac{\varepsilon}{2}$, since $g(y^{-1}x)$ is zero except when $y\in xU^{-1}$, and since $\mu(xU^{-1})<\eta$. Hence $\|f_i*g\|_\infty<\varepsilon$, although $\|g\|_\infty=1$. This shows that no τ_c-neighborhood of 0 lies wholly in any $\|\ \|_\infty$-ball of radius less than 1. Hence τ_c is coarser than $\|\ \|_\infty$.■

4. FURTHER PROPERTIES OF THE τ_c-TOPOLOGY.

The proofs of Propositions 5 and 8 that follow were kindly suggested to us by Robert B. Burckel. Both results also appear in Crombez and Govaerts[2].

PROPOSITION 5. Any norm-closed ball in $L_\infty(G)$ is τ_c-complete.

PROOF. Let $\{g_\alpha\}$ be a τ_c-Cauchy net in a ball in $L_\infty(G)$. Let g be a w^*-cluster point of this net, such that a subnet $\{g_\beta\}$ w^*-converges to g. Then $\{(f*g_\beta)(x)\}$ converges to $(f*g)(x)$ for all x in G and all f in $L_1(G)$. Given $\varepsilon>0$ and $f\in L_1(G)$, there exists α_ε such that $\|f*g_\alpha-f*g_{\alpha'}\|_\infty\leqslant\varepsilon$ for all α, $\alpha'\geqslant\alpha_\varepsilon$. Since all these functions are continuous and $\|\ \|_\infty$ here is genuine supremum, we derive $|(f*g_\alpha)(x)-(f*g_{\alpha'})(x)|\leqslant\varepsilon$ for all $\alpha,\alpha'\geqslant\alpha_\varepsilon$, for all $x\in G$. In this last inequality we take $\alpha'=\beta$ and let β recede to infinity; then this leads to

$|(f*g_\alpha)(x)-(f*g)(x)| \leq \varepsilon$ for all $\alpha \geq \alpha_\varepsilon$ and all $x \in G$, i.e.,

$||f*g_\alpha - f*g||_\infty \leq \varepsilon$ for all $\alpha \geq \alpha_\varepsilon$.■

In particular, we derive from Proposition 5 that a set in $L_\infty(G)$ is τ_c-relatively compact iff it is τ_c-totally bounded. We also have that the closed absolutely convex hull of a τ_c-compact set is again τ_c-compact. Denoting by cl_{τ_c} the closure in the τ_c-topology, we have

PROPOSITION 6. If $g \in L_\infty(G)$, then $g \in cl_{\tau_c}(L_1*g)$.

PROOF. Given $\varepsilon > 0$ and n functions k_i in $L_1(G)$ determining a τ_c-neighborhood V of g in $L_\infty(G)$, and denoting by $\{e_\lambda\}_{\lambda\in\Lambda}$ an approximate identity in $L_1(G)$, we see that $||k_i*(e_\lambda*g)-k_i*g||_\infty$ may be made arbitrarily small. Hence V contains elements of the form $e_\lambda*g$.■

COROLLARY 3. Let S be a τ_c-closed L_1-submodule of $L_\infty(G)$. Then $S=cl_{\tau_c}(L_1*S)$.

COROLLARY 4. Let S be a τ_c-closed L_1-submodule of $L_\infty(G)$. Then S is left translation invariant.

PROOF. Given g in S and $a \in G$ we show that any τ_c-neighborhood of ${}_ag$ contains a function in L_1*S, from which the result will follow. Denote by Δ the modular function of G. Let V be the τ_c-neighborhood of ${}_ag$ determined by $f_1,\ldots,f_n$ in $L_1(G)$ and $\varepsilon > 0$. There always exist $k \in L_1(G)$ and $h \in S$ such that

$||(f_i)_a*g-(f_i)_a*(k*h)||_\infty < \frac{\varepsilon}{\Delta(a)}$. Then $||f_i*_ag-f_i*_a(k*h)||_\infty < \varepsilon$. Hence V contains the function ${}_ak*h \in L_1*S$.■

Since $w^* < \tau_c$, Proposition 6 and its corollaries are stronger than the corresponding resutls in Crombez and Govaerts [5].

Given $g \in L_\infty(G)$, the map $s \to {}_sg$ from G to $(L_\infty(G), || \; ||_\infty)$ is continuous iff g is locally a.e. equal to a function in $C_{ru}(G)$. (Here, as in [11], $C_{ru}(G)$ is the set of all right uniformly continuous, bounded, complex-valued functions on G). However, using the τ_c-topology on L (G) we obtain continuity for any $g \in L_\infty(G)$.

PROPOSITION 7. Leg g be a function in $L_\infty(G)$. Then the maps $s \to g_s$ and $s \to {}_s g$ from G to $(L_\infty(G), \tau_c)$ are continuous.

PROOF. That the map $s \to g_s$ is continuous is trivial, since for any f in $L_1(G)$ $f*g \in C_{ru}(G)$ and $f*g_s = (f*g)_s$. To prove that $s \to {}_s g$ is continuous, consider the composition of the maps $G \to L_1(G) \times \mathbb{C} \to L_\infty(G)$ given by $s \to (f_s, \Delta(s)) \to \Delta(s) f_s * g = f * {}_s g$. Each map is continuous, and so the result follows.∎

5. SOME MORE RESULTS ON τ_c-ALMOST PERIODIC FUNCTIONS.

In this final section we always suppose G to be Abelian. The notion of τ_c-almost periodic (τ_c-AP) function in $L_\infty(G)$ was introduced in [2] in order to characterize those multipliers which are strongly almost periodic.

PROPOSITION 8. A function g in $L_\infty(G)$ is τ_c-AP iff $f*g$ is $\|\ \|_\infty$-almost periodic for each f in $L_1(G)$.

PROOF. We first notice that $f*g_a = (f*g)_a$ for any a in G; so if we set $0_g = \{g_a : a \in G\}$, then $f*0_g = 0_{f*g}$.

If 0_g is relatively τ_c-compact, then its continuous image $f*0_g = 0_{f*g}$ in $(C_{ru}(G), \|\ \|_\infty)$ is relatively compact, so $f*g$ is norm almost periodic. Conversely, by definition of τ_c the map

$$g \to (f*g)_{f \in L_1(G)} \in \prod_{f \in L_1(G)} f*L_\infty(G)$$

is a homeomorphism from τ_c into the product of the norm topologies on the right. Evidently the image of 0_g lies in the subspace $\prod_{f \in L_1(G)} 0_{f*g}$. If each $f*g$ is norm almost periodic, then this last product is relatively compact, and so 0_g is relatively τ_c-compact.∎

Denoting by AP the $\|\ \|_\infty$-almost periodic functions in $L_\infty(G)$, we obtained in [2] that τ_c-AP = AP for G discrete, and τ_c-AP = $L_\infty(G)$ for G compact (both results are of course clear now by Proposition 4 and Proposition 3, respectively). We always have that AP $\subseteq \tau_c$-AP. From Proposition 8 we derive: $L_1(G)*\tau_c\text{-AP} \subsetneq \text{AP}$. Since $L_1(G)*\text{AP} = \text{AP}$ (see Crombez and Govaerts [4]), we also get $L_1(G)*\tau_c\text{-AP} = \text{AP}$. Hence we obtain from Proposition 8 that τ_c-AP is the largest linear subspace S of $L_\infty(G)$

such that $L_1(G)*S = AP$. The set τ_c-AP is an L_1-submodule of $L_\infty(G)$ which is obviously τ_c-closed. From Corollary 3 we may conclude that $\tau_c\text{-AP}=cl_{\tau_c}\,AP$. In particular, for compact G we have that $L_\infty(G) = cl_{\tau_c}\,C(G)$, where C(G) denotes the set of continuous functions on G.

PROPOSITION 9. G is compact iff $\tau_c\text{-AP}=L_\infty(G)$.

PROOF. Suppose that $\tau_c\text{-AP} = L_\infty(G)$. Then $AP = L_1(G)*\tau_c\text{-AP} = L_1(G)*L_\infty(G) = C_{ru}(G)$, the last equality coming from Hewitt and Ross [6], 32.45(b). Pick $0 \neq f \in C_{ru}(G)$ with compact support K. If G is not compact there exist infinitely many disjoint translates a_jK of K. Clearly the subset $\{{}_{a_j^{-1}}f\}_{j=1}^{\infty}$ of the left orbit of f is not totally bounded.∎

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