STABILITY IMPLICATIONS ON THE ASYMPOTOTIC BEHAVIOR OF NONLINEAR SYSTEMS

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ABSTRACT. In this paper we generalize Bownds' Theorems (1) to the systems
\[ \frac{dY(t)}{dt} = A(t)Y(t) \]
and
\[ \frac{dX(t)}{dt} = A(t)X(t) + F(t,X(t)). \]
Moreover, we also show that there always exists a solution \( X(t) \) of
\[ \frac{dX(t)}{dt} = A(t)X + B(t) \]
for which \( \lim_{t \to \infty} \sup ||X(t)|| > 0 \) if there exists a solution \( Y(t) \) for which
\( \lim_{t \to \infty} \sup ||Y(t)|| > 0 \).

KEY WORDS AND PHRASES. stable, norm, linear systems, null solution, Schauder-Tychonoff Theorem, uniformly converges, equicontinuous.

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1. INTRODUCTION.

In this paper we shall study the stability behavior of the following systems
\[ \frac{dY(t)}{dt} = A(t)Y(t), \quad 0 < t < \infty \quad (1.1) \]
and
\[ \frac{dX(t)}{dt} = A(t)X(t) + F(t,X(t)), \quad 0 < t < \infty \quad (1.2) \]
where \( A(t) \) is a continuous matrix on \( \mathbb{R}^n \) for all \( 0 < t < \infty \), \( F(t,X(t)) \) is a real valued continuous \( n \)-vector defined on \( [0,\infty) \times \mathbb{R}^n \) and \( X(t) \) and \( Y(t) \) are \( n \)-vectors.

Consider special equations of (1.1) and (1.2)
\[ y'' + a(t)y = 0, \quad 0 \leq t < \infty \quad (1.3) \]
and
\[ x'' + a(t)x = g(t,x,x'), \quad 0 \leq t < \infty \quad (1.4) \]
where $a(t) \in C[0, \infty)$ and $g(t,x,x')$ is continuous on $[0, \infty) \times \mathbb{R} \times \mathbb{R}$. From some theorems of stability theory, Bownds [1] showed that (1.3) has a solution $y(t)$ with property
\[
\lim_{t \to \infty} \sup (|y(t)| + |y'(t)|) > 0
\]  
(1.5)

He also established that (1.4) has the property (1.5) provided that the zero solution of (1.3) is stable and there exists a function $\gamma(t) \in L[0, \infty)$ such that
\[
|g(t,x,x')| \leq \gamma(t) (|x| + |x'|)
\]
for $(t,x,x') \in [0, \infty) \times \mathbb{R} \times \mathbb{R}$.

Thus in the following section we shall extend the above results to systems (1.1) and (1.2). In section 3 we shall consider a nonhomogeneous system
\[
\frac{dX(t)}{dt} = A(t)X(t) + B(t), \quad 0 \leq t < \infty
\]  
(1.6)
where $B(t)$ is a continuous vector for $0 \leq t < \infty$. We shall prove that there always exists a solution $X(t)$ of (1.6) for which $\lim_{t \to \infty} \|X(t)\| > 0(= \infty)$, if there exists a solution $Y(t)$ of (1.1) for which $\lim_{t \to \infty} \|Y(t)\| > 0(= \infty)$. Here $\| \cdot \|$ is an appropriate vector (or matrix) norm.

2. ASYMPTOTIC BEHAVIOR FOR (1.1) AND (1.2).

Before stating main theorems, let us recall a theorem from Coppel [2, p. 60].

THEOREM 2.1. (Hartman [2, p. 60]). Suppose that, for every solution $Y(t)$ of (1.1), the limit
\[
\lim_{t \to \infty} \|Y(t)\|
\]  
(2.1)
exists and is finite. If there exists a nontrivial solution $Y(t)$ of (1.1) for which the limit (2.1) is zero, then
\[
\int_{t_0}^{t} t A(s) ds \to -\infty \quad \text{as } t \to \infty.
\]

From the above theorem we will obtain the following corollary which is a generalization of Theorem 1 in [1].

COROLLARY 2.1. Suppose that
\[
\int_{t_0}^{\infty} t A(s) ds < \infty.
\]
Then there exists a nontrivial solution $Y(t)$ of (1.1) for which
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\[ \lim_{t \to \infty} \|Y(t)\| > 0. \]

**Proof.** Suppose, to the contrary, that all solutions \( Y(t) \) of (1.1) satisfy
\[ \lim_{t \to \infty} \|Y(t)\| = 0. \]
From Theorem 2.1 we obtain
\[ \int_{t_0}^{t} t A(s)ds \to -\infty \quad \text{as} \quad t \to \infty. \]
This leads to a contradiction. The corollary then follows.

Throughout this paper we shall denote \( \Phi(t) \), the fundamental matrix of (1.1) with initial condition \( \Phi(0) = I \) (identity matrix).

Now we shall prove the following theorem via the Schauder-Tychonoff Theorem [2, p. 9].

**Theorem 2.2.** Suppose that the null solution of (1.1) is stable and that there exists a solution \( Y(t) \) of (1.1) for which
\[ \lim_{t \to \infty} \|Y(t)\| > 0. \] (2.2)
Suppose also that there exists \( \gamma(t) \in L^1[0, \infty) \) such that for some positive constant \( \ell \)
\[ \|F(t,x)\| \leq \gamma(t) \|x\|^\ell. \] (2.3)
Then there exists a nontrivial solution \( X(t) \) of (1.2) for which
\[ \lim_{t \to \infty} \|X(t)\| > 0. \]

**Proof.** Since the null solution of (1.1) is stable, there exists a positive constant \( k \) such that
\[ \|\Phi(t) \Phi^{-1}(s)\| \leq k \] (2.4)
for all \( 0 \leq t \leq s \) and there exists a nontrivial solution \( Y(t) \) of (1.1) for which (2.2) holds and
\[ \|Y(t)\| \leq 1 - \varepsilon \] (2.5)
for \( t \geq t_0 \) and for given small positive constant \( \varepsilon \) (<1).

Since \( \gamma(t) \in L^1[0, \infty) \), there exists \( T_0 (> t_0) \) such that
\[ k \int_{t}^{\infty} \gamma(s)ds \leq \varepsilon \quad \text{for all} \quad t \geq T_0. \] (2.6)
Via the Schauder-Tychonoff Theorem we shall establish the existence of a solution
of the integral equation
\[ X(t) = Y(t) - \phi(t) \int_{t}^{\infty} \phi^{-1}(s) f(s, X(s)) \, ds, \quad t \geq T_0. \tag{2.7} \]

Consider the set
\[ F = \{ U; U(t) = X(t) \text{ is continuous on } J_0 = [T_0, \infty) \text{ and} \]
\[ \| U(t) \| \leq 1 \text{ for } t \geq T_0 \} \]
and define the operator \( T \) by
\[ TU(t) = Y(t) - \int_{t}^{\infty} \phi(t) \phi^{-1}(s) f(s, U(s)) \, ds. \tag{2.8} \]

First, we shall show that \( TF \subset F \). Taking the norm to both sides of (2.8) and using (2.3), (2.4), (2.5), and (2.6), we obtain for \( U \in F \)
\[ \| TU(t) \| \leq \| Y(t) \| + \int_{t}^{\infty} \| \phi(t) \phi^{-1}(s) f(s, U(s)) \| \, ds \]
\[ \leq 1 - \varepsilon + k \int_{t}^{\infty} \| f(s, U(s)) \| \, ds \]
\[ \leq 1 - \varepsilon + k \int_{t}^{\infty} \gamma(s) \| U(s) \| \, ds \]
\[ \leq 1 - \varepsilon + k \int_{t}^{\infty} \gamma(s) \, ds \]
\[ \leq 1 - \varepsilon + \varepsilon = 1. \]

It is clear that \( TU(t) \) is continuous on \( J_0 \). This proves \( TF \subset F \).

Second, we shall show that \( T \) is continuous. Suppose that the sequence \( \{ U_n \} \)
in \( F \) converges uniformly to \( U \) in \( F \) on every compact subinterval of \( J_0 \). We claim that \( TU_n \) converges uniformly to \( TU \) on every compact subinterval of \( J_0 \). Let \( \varepsilon_1 \) be a small positive number satisfying \( \varepsilon_1 < 1 \). Since \( \gamma(t) \in L_1[t_0, \infty) \), there exists \( T_1 > T_0 \) so that for \( t \geq T_1 \)
\[ k \int_{t}^{\infty} \gamma(s) \, ds < \frac{\varepsilon_1}{4}. \tag{2.9} \]

By the uniform convergence, there is an \( N = N(\varepsilon_1, T_1) \) such that if \( n \geq N \), then
\[ \| f(s, U_n(s)) - f(s, U(s)) \| < \frac{\varepsilon_1}{2kT_1}, \quad T_0 \leq s \leq T_1. \tag{2.10} \]
Then using (2.8), (2.9), (2.10), (2.3), (2.4), and the fact that \( \| U_n(t) \| \leq 1 \) and \( \| U(t) \| \leq 1 \) for \( T_0 \leq t < \infty \), we obtain the following inequalities
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\[ \left| |TU_n(t) - TU(t)|\right| = \left| \int_T^\infty \phi(t)\phi^{-1}(s)f(s,U_n(s))ds - \int_T^\infty \phi(t)\phi^{-1}(s)f(s,U(s))ds \right| \]

\[ \leq \int_T^1 \left| |\phi(t)\phi^{-1}(s)| \right| \left| |f(s,U_n(s)) - f(s,U(s))| \right|ds \]

\[ + \int_T^\infty \left| |\phi(t)\phi^{-1}(s)| \right| \left| |f(s,U_n(s))| \right|ds + \int_T^\infty \left| |\phi(t)\phi^{-1}(s)| \right| \left| |f(s,U(s))| \right|ds \]

\[ \leq k \int_T^1 \left| |f(s,U_n(s)) - f(s,U(s))| \right|ds + 2k \int_T^\infty \gamma(s)ds \]

\[ < \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1 \quad \text{for} \quad n \geq N. \]

This shows that \(TU_n\) converges uniformly to \(TU\) on every compact subinterval of \(J\).

Hence \(T\) is continuous.

Third, we claim that the functions in the image set \(T_F\) are equicontinuous and bounded at every point of \(J\). Since \(T_F \subset F\), it is clear that the functions in \(T_F\) are uniformly bonded. Now we show that they are equicontinuous at each point of \(J\). For each \(U \in F\), the function \(z(t) = TU(t)\) is a solution of the linear system

\[ \frac{dV}{dt} = A(t)V + f(t,U(t)) \]

Since \(\left| |z(t)| \right| = \left| |TU(t)| \right| \leq 1\) and \(\left| |f(t,U(t))| \right|\) is uniformly bounded for \(U \in F\) on any finite \(t\) interval, we see that \(\frac{dz}{dt}\) is uniformly bounded on any finite interval. Therefore, the set of all such \(z\) is equicontinuous at each point of \(J\) (see [2, p.6]).

All of the hypotheses of the Schauder-Tychonoff Theorem are satisfied. Thus there exists a \(U \in F\) such that \(U(t) = TU(t)\); that is, there exists a solution \(X(t)\) of

\[ X(t) = Y(t) - \phi(t) \int_T^\infty \phi^{-1}(s)f(s,x(s))ds \]

Thus, from the hypotheses and the above equality, we obtain

\[ \lim_{t \to \infty} \sup \left| |X(t) - Y(t)| \right| = 0 \quad (2.11) \]

Since \(\lim_{t \to \infty} \left| |Y(t)| \right| > 0\), (2.11) implies that \(\lim_{t \to \infty} \left| |X(t)| \right| > 0\). This proves the theorem.

It is clear that (1.4) can be written as the form (1.2) with
where \( X = \text{colum}(x, x') \). Thus we can apply Theorem 2.2 to (1.4) to obtain the following corollary which is a generalization of Theorem 2 in [1].

**COROLLARY 2.2.** Suppose that the null solution of (1.3) is stable and that there exists \( \gamma(t) \in L^1_{[t_0, \infty)} \) such that for some positive constant \( \ell \)
\[
||g(t,x,x')|| \leq \gamma(t) (|x| + |x'|)^2.
\]
Then there exists a nontrivial solution \( x(t) \) of (1.4) for which
\[
\limsup_{t \to \infty} (|x| + |x'|) > 0.
\]

**PROOF.** Since \( r A(t) = 0 \) for Corollary 2.1, we know that there exists a solution \( Y(t) \) of (1.1) for which
\[
\limsup_{t \to \infty} ||Y(t)|| > 0.
\]
If we take \(|X(t)| = |x| + |x'|\), then the corollary follows from Theorem 2.2.

3. **ASYMPTOTIC BEHAVIOR FOR (1.6).**

In this section we shall show that if there exists a solution \( Y(t) \) of (1.1) for which
\[
\limsup_{t \to \infty} ||Y(t)|| > 0 \quad (\ast \infty),
\]
then there exists a solution \( X(t) \) of (1.6) for which
\[
\limsup_{t \to \infty} ||X(t)|| > 0 \quad (\ast \infty).
\]

**THEOREM 3.1.** Suppose that there exists a solution \( Y(t) \) of (1.1) for which
\[
\limsup_{t \to \infty} ||Y(t)|| > 0. \tag{3.1}
\]
Then there exists a solution \( X(t) \) of (1.6) for which
\[
\limsup_{t \to \infty} ||X(t)|| > 0.
\]

**PROOF.** From the variation of constants formula we know that any solution \( X(t) \) of (1.6) can be written as the form below
\[
X(t) = \Phi(t)c + \Phi(t) \int_0^t \Phi^{-1}(s) B(s)ds \tag{3.3}
\]
Hence we shall choose \( c \) so that \( Y(t) = \Phi(t)c \) satisfies (3.1).

First, let us suppose
\[
\limsup_{t \to \infty} ||\Phi(t)\int_0^t \Phi^{-1}(s) B(s)ds|| > 0. \tag{3.4}
\]
Let $X_1(t) = X(t) - Y(t)$. It is clear that $X_1(t)$ is a solution of (1.6). Thus from (3.3) and (3.4) we obtain

$$\lim_{t \to \infty} \sup_{0 < \tau \leq t} \|X_1(t)\| = \lim_{t \to \infty} \sup_{0 < \tau \leq t} \|X(t) - Y(t)\|$$

$$= \lim_{t \to \infty} \sup_{0 < \tau \leq t} \|\phi(t) \int_0^t \phi^{-1}(s) B(s) ds\| > 0 .$$

Thus there exists a solution $X_1(t)$ of (1.6) for which (3.2) holds.

Second, suppose that

$$\lim_{t \to \infty} \|\phi(t) \int_0^t \phi^{-1}(s) B(s) ds\| = 0 . \tag{3.5}$$

Taking the norm to both sides of (3.3) and using (3.1) and (3.5) we obtain

$$\lim_{t \to \infty} \sup_{0 < \tau \leq t} \|X(t)\| \geq \lim_{t \to \infty} \sup_{0 < \tau \leq t} \|Y(t)\| - \|\phi(t) \int_0^t \phi^{-1}(s) B(s) ds\|$$

$$\geq \lim_{t \to \infty} \sup_{0 < \tau \leq t} \|Y(t)\| - \lim_{t \to \infty} \sup_{0 < \tau \leq t} \|\phi(t) \int_0^t \phi^{-1}(s) B(s) ds\|$$

$$\geq \lim_{t \to \infty} \sup_{0 < \tau \leq t} \|Y(t)\| > 0 .$$

This shows that $X(t)$ satisfies (3.2). The theorem then follows.

Using the same argument as Theorem 3.1 we also can obtain the following theorem.

**Theorem 3.2.** Suppose that there exists a solution $Y(t)$ of (1.1) for which

$$\lim_{t \to \infty} \sup_{0 < \tau \leq t} \|Y(t)\| = \infty .$$

Then there exists a solution $X(t)$ of (1.6) for which

$$\lim_{t \to \infty} \sup_{0 < \tau \leq t} \|X(t)\| = \infty .$$

**Proof.** Since the proof is almost the same as Theorem 3.1, we shall omit the detail.

**Remarks.** It is interesting to note that Hatvani and Pintér [3] have studied this type of problem for equation (1.4).

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REFERENCES


Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spacecraft dynamics.

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