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THE COMPACTUM AND FINITE DIMENSIONALITY IN BANACH ALGEBRAS

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ABSTRACT. Given a Banach algebra A, the compactum of A is defined to be the set of elements $x \in A$ such that the operator $a \to xax$ is compact. General properties of the compactum and its relation to the socle of A are discussed. Characterizations of finite dimensionality of a semi-simple Banach algebra are given in terms of the compactum and the socle of A.

KEY WORDS AND PHRASES. *Compact operators, socle, minimal idempotent, spectrum.*

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1. INTRODUCTION.

Familiarity with Banach algebras is assumed. An elementary survey of this area is Rickart [5], which is accessible to anyone with a basic knowledge of measure theory.

Let A be a Banach algebra. For $x \in A$ let T_x be the operator on A defined by $T_x(a) = xax$. Define the compactum of A to be the set $C(A) = \{x \in A: T_x$ is a compact operator$\}$. In [1], J.C. Alexander investigated the properties of a Banach algebra A which satisfies $A = C(A)$. He called this type of algebra a compact Banach algebra. The concept was also considered by Erdos, Giotopoulos and Lambrou [2]. It has its origin in a result by Vala which states that if X is a Banach space and T and T' are non-zero elements of B(X), then the operator $S \to TST'$ is compact on B(X) if and only if both T and T' are compact on X [3].

The purpose of this paper is to look at some general properties of C(A) and to give a characterization of finite dimensionality for a semi-simple Banach algebra A, using C(A). The latter result generalizes a theorem of A.W. Tullo [4].

2. GENERAL PROPERTIES.

If A is a Banach algebra with minimal left and right ideals, and if the sume of the minimal left ideals coincides with the sum of the minimal right ideals, then the resulting ideal is called the socle of A. We let S(A) denote the socle of A. If X is a Banach space then B(X) will stand for the algebra of bounded operators on X and K(X) will denote the subalgebra of B(X) consisting of compact operators. If A is an algebra and $x \in A$, then $\sigma(X)$ will denote the spectrum of x in A.

Our terminology is consistent with that of [5], and all algebras considered are over the field of complex numbers C. We recall again that a Banach algebra A is compact if A = C(A).

PROPOSITION 1. Let A be a Banach algebra. Then,

a) C(A) is a closed set.

b) If $x \in C(A)$ then the ideals xA and Ax are both contained in C(A).

c) If B is a closed subalgebra of A such that $B \subset C(A)$, then B is a compact Banach algebra.

PROOF. Let (X_n) be a sequence in C(A) which converges to x. Then for each $a \in A$ we have $||(T_x - T_{x_n})(a)|| = ||xax - x_n a x_n|| \leq ||xax - x_n ax|| + ||x_n ax - x_n a x_n|| \leq ||a|| (||x|| + ||x_n||) ||x - x_n||$. Hence $|T_x - T_{x_n}| \leq (||x|| + ||x_n||) ||x - x_n||$ where $|\cdot|$ denotes the operator norm. It follows that T_{x_n} converges to T_x, and since T_{x_n} is compact for all n, we get T_x is compact. Therefore, $x \in C(A)$. This proves (a). If $x \in C(A)$ and $y \in A$, then T_{xy} is the composition of the maps $a \to ya$, T_x and $a \to ay$, and since T_x is compact it follows that T_{xy} is compact, i.e., $xy \in C(A)$. Similarly $yx \in C(A)$, which proves (b). Part (c) follows from the definition of compact Banach algebra and the fact that if $x \in B \subset C(A)$, then the restriction of T_x to B is still compact.

PROPOSITION 2. (a) If A is finite dimensional then C(A) = A. (b) If X is a Banach space then $C(B(X)) = K(X)$.

PROOF. Part (a) is obvious, while part (b) follows by Vala's theorem which states that $T, T' \in K(X)$ if and only if the map $S \to TST'$ is compact on $B(X)$ [3].

The next two lemmas appear in [1]. We state them here without proofs.

LEMMA 1. If A is a compact Banach algebra which is not a radical algebra, then A contains an idempotent e such that eAe is finite dimensional [1, 4.3].

LEMMA 2. Let A be a semi-simple Banach algebra and $x \in A$. Then $S(A)$ exists and $x \in S(A)$ if and only if T_x has finite rank (i.e., xAx is finite dimensional).

PROPOSITION 3. Let A be a semi-simple Banach algebra. Then $C(A)$ is nonzero if and only if $S(A)$ is nonzero, and in this case $S(A) \subset C(A)$.

PROOF. Suppose that $C(A)$ is nonzero. Choose $x \in C(A)$. Then the right ideal xA is contained in $C(A)$, by Proposition 1(b). Let J be the closure of xA. Then $J \subset C(A)$ since $C(A)$ is closed. Therefore, by Proposition 1(c), J is a compact Banach algebra, and hence, by Lemma 1, it contains an idempotent e such that eJe is finite dimensional. But since $eA \subset J$, we have $eAe = e(eA)e \subset eJe$, and hence eAe is finite dimensional. It follows, by Lemma 2, that $e \in S(A)$.

If $S(A)$ is nonzero then, applying Lemma 2, we get $S(A) \subset C(A)$.

PROPOSITION 4. If A is a Banach algebra and $x \in C(A)$, then 0 is the only accumulation point for $\sigma(x)$. Moreover, if $\lambda \in \sigma(x)$, $\lambda \neq 0$ then there exists $y \in A$ commuting with x such that $T_x(y) = \lambda^2 y$.

PROOF. First note that if A has no identity and A_1 is the Banach algebra obtained by adjoining an identity to A in the usual manner, then the operator T_x on A_1 is still compact. Moreover, $\sigma(x)$ as an element of A_1 or as an element of A is the same. Therefore, we may assume that A has an identity.

Let C be a maximal commutative subalgebra containing x and let T be the restriction of T_x to C. If $\lambda \notin \sigma(T)$, then there exists $S \in B(C)$ such that $(\lambda - T)S = I$, i.e., $(\lambda - T)Sy = y$ for all $y \in C$. This is equivalent to $(\lambda - x^2)Sy = y$. If we choose $y \in C$ invertible, then $(Sy)y^{-1}$ is an inverse for $\lambda - x^2$. That is $\lambda \notin \sigma(x^2)$. This says that $\sigma(x^2) \subset \sigma(T)$. Since T is compact, the conclusion follows from the general theory for compact operators and the spectral mapping theorem.

3. CHARACTERIZATION OF FINITE DIMENSIONALITY.

For the remainder of this paper we will give a characterization of finite dimensionality for semi-simple Banach algebras. Our result generalizes a theorem of A. W. Tullo which states that a semi-simple Banach algebra A which satisfies $A=S(A)$ is finite dimensional [4]. Before stating and proving our main theorem we need two lemmas.

LEMMA 3. Let A be a semi-simple Banach algebra such that $xC(A) = (0)$ implies $x = 0$. Suppose that $J \neq (0)$ is a closed right ideal in A. Then J contains an idempotent e such that $e \in S(A)$.

PROOF. Choose $x \in J$ and $y \in C(A)$ such that $xy \neq 0$. Then, by Proposition 1(b), $xyA \subset C(A)$. We also have $xyA \subset J$. Since A is semi-simple, it follows that xyA is not a radical algebra. Hence, by Lemma 1, xyA contains an idempotent e such that exyAe is finite dimensional. But $e \in xyA$ implies that $eA \subset xyA$, and hence $eAe = e\,(eA)\,e \subset exyAe$. Therefore, eAe is finite dimensional, and it follows from Lemma 2 that $e \in S(A)$.

The next lemma appears in [6], and we include it here for the sake of completeness. We recall that an idempotent e in an algebra A is minimal if eAe is a division algebra. This is equivalent to saying that eA (Ae) is a minimal right (left) ideal [5]. By an idempotent we always mean a non-zero one.

LEMMA 4. Let A be a semi-simple normed algebra. If e and f are minimal idempotents in A then eAf is at most 1-dimensional.

PROOF. Suppose that $eAf \neq 0$ and choose x such that $exf \neq 0$. Then $exfA = eA$ by minimality of eA. Moreover, by the Gelfand-Mazur theorem, $fAf = f\mathbb{C}$ where C is the field of complex numbers. It follows that $eAf = exfAf = exf\mathbb{C}$. That is eAf is 1-dimensional.

THEOREM. Let A be a semi-simple Banach algebra such that $xC(A) = (0)$ implies $x = 0$, then the following statements are equivalent,

a) A is finite dimensional.

b) (A. W. Tullo) $A = S(A)$.

c) $S(A) = C(A)$.

d) $S(A)$ is closed.

PROOF. If (a) holds then T_x has finite rank for every $x \in A$. Therefore, $A = S(A)$ by Lemma 2. If (b) holds then (c) follows from Proposition 3. The fact that $C(A)$ is closed shows that (c) implies (d). It remains to show that (d) implies (a).

Suppose that $S(A)$ is closed. We first show that A cannot contain an infinite set of pairwise orthogonal idempotents. Suppose that to the contrary such a set $\{e_n\}_{n=1}^{\infty}$ exists. Let $J_i = e_i A$. Then J_i is a non-zero closed right ideal for each i. It follows by Lemma 3, that each J_i contains an idempotent f_i such that $f_i \varepsilon S(A)$. Let $g_i = f_i e_i$. Then $g_i \varepsilon S(A)$, and since $f_i = e_i f_i$ it follows that g_i is an idempotent and $\{g_i\}_{i=1}^{\infty}$ is a pairwise orthogonal family. Note that $g_i \neq 0$, otherwise $f_i = f_i^2 = f_i e_i f_i = g_i f_i = 0$. Now let $y_n = \sum_{i=1}^{n} \frac{g_i}{2^i \|g_i\|}$. Then $y_n \varepsilon S(A)$. Since $S(A)$ is closed it follows that $y = \lim_{n\to\infty} y_n = \sum_{i=1}^{\infty} \frac{g_i}{2^i \|g_i\|} \varepsilon S(A)$. But we have $2^i \|g_i\| y \, g_i \, y = g_i$. Therefore, the set $\{g_i\}$ is contained in yAy, and since it is an infinite set and linearly independent by the orthogonality of its elements we have yAy is infinite dimensional. This contradicts the fact that $y \varepsilon S(A)$, by Lemma 2. Hence, A contains at most a finite set of pairwise orthogonal idempotents.

Now let $\{e_1, \ldots, e_n\}$ be a set of pairwise orthogonal idempotents of maximal possible cardinality. Then each e_i is minimal. Otherwise if e_1, say, is not minimal, then e_1A is not a minimal ideal, hence it is not a minimal closed ideal [5; 2.1.10]. Thus e_1A properly contains a closed right ideal I which, by Lemma 3, contains an idempotent f, necessarily different from e_1 and $f = e_1 f$. Then $\{fe_1, e_1 - fe_1, e_2, \ldots, e_n\}$ is a pairwise orthogonal family of idempotents which contradicts the maximality of n.

Now let $f = e_1 + \ldots + e_n$. We claim that f is an identity for A. If $(1 - f) A \neq 0$, then since it is a closed ideal, Lemma 3 implies that it contains an idempotent $g = (1 - f) g$. Let $h = g(1 - f)$. Then h is an idempotent which is orthogonal to e_i for $i = 1, \ldots, n$. Then by maximality of n, we get $h = 0$. Hence $g = g^2 = g(1 - f) g = hg = 0$ which is a contradiction. Hence $(1 - f) A = 0$. Noting that the conclusion of Lemma 3 holds for closed left ideals as well, a similar argument as the above gives $A(1 - f) = 0$. Therefore f is an identity of A. It follows that $A = fAf = (\sum_{i=1}^{n} e_i) A (\sum_{i=1}^{n} e_i) = \sum_{i,j=1}^{n} e_i Ae_j$. But, by Lemma 4, $e_i Ae_j$ is at most 1-dimensional for $i,j = 1, \ldots, n$. Hence, A is finite dimensional. This shows that (d) implies (a) and concludes the proof of the theorem.

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