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UNIVALENCE OF NORMALIZED SOLUTIONS OF $W''(z) + p(z)W(z) = 0$

R.K. BROWN
Department of Mathematical Sciences
Kent State University
Kent, Ohio 44242 U.S.A.

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ABSTRACT. Denote solutions of $W''(z) + p(z)W(z) = 0$ by $W_\alpha(z) = z^\alpha[1 + \sum_{n=1}^{\infty} a_n z^n]$ and $W_\beta(z) = z^\beta[1 + \sum_{n=1}^{\infty} b_n z^n]$, where $0 < \Re\{\beta\} \leq 1/2 \leq \Re\{\alpha\}$ and $z^2p(z)$ is holomorphic in $|z| < 1$. We determine sufficient conditions on $p(z)$ so that $[W_\alpha(z)]^{1/\alpha}$ and $[W_\beta(z)]^{1/\beta}$ are univalent in $|z| < 1$.

KEY WORDS AND PHRASES. Univalent, spirallike, starlike.

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1. INTRODUCTION.

Consider the differential equation

$$W''(z) + p(z)W(z) = 0, \text{ where} \tag{1.1}$$

$$z^2p(z) = p_0 + p_1z + \cdots + p_nz^n + \cdots, \ p_0 \neq 0, \tag{1.2}$$

is holomorphic for $|z| < 1$.

The indicial equation associated with the regular singular point of the equation (1.1) at the origin is

$$\lambda^2 - \lambda + p_0 = 0 \tag{1.3}$$

and has roots which we designate by α and β, where $\alpha + \beta = 1$ and $\Re\{\alpha\} \geq 1/2 \geq \Re\{\beta\}$. We will also use the notation

$$\alpha = \alpha_1 + i\alpha_2,\ \beta = \beta_1 + i\beta_2. \tag{1.4}$$

Corresponding to the root α there is always a unique solution of (1.1) of the form

$$W_\alpha(z) = z^\alpha[1 + \sum_{n=1}^{\infty} a_n z^n] \tag{1.5}$$

valid for $|z| < 1$.

We restrict our attention in this paper to those β for which $\beta_1 = \Re\{\beta\} > 0$. We then obtain a unique solution of (1.1) of the form

$$W_\beta(z) = z^\beta[1 + \sum_{n=1}^{\infty} b_n z^n] \tag{1.6}$$

valid for $|z| < 1$.

We define two normalizations $F_\alpha(z)$ and $F_\beta(z)$ of the solutions of (1.5) and (1.6) as follows:

$$F_\alpha(z) = [W_\alpha(z)]^{1/\alpha} = z + \cdots ,$$

$$F_\beta(z) = [W_\beta(z)]^{1/\beta} = z + \cdots \tag{1.7}$$

where we choose that branch of each function for which the derivative at the origin is 1.

Next we consider the "comparison" equation

$$W''(z) + p_C^*(z)W_C(z) = 0, \text{ with} \tag{1.8}$$

$$z^2p_C^*(z) \equiv C(z^2p^*(z) - p_0^*) + p_0^*,\ C > 0, \tag{1.9}$$

and where $z^2p^*(z) = p_0^* + p_1^*z + \cdots + p_n^*z^n + \cdots$

is non-constant and holomorphic for $|z| < 1$ with p_i^*, $i = 0, 1, 2, \cdots$ real and $p_0^* \leq 1/4$. With these restrictions on $z^2p^*(z)$ the solutions of (1.8) are real on the real axis (see [1]). We will designate the exponents associated with the regular singular point of (1.8) at the origin by α^* and β^*, where $\alpha^* + \beta^* = 1$ and $\alpha^* \geq 1/2 \geq \beta^*$. As in the case of equation (1.1) we obtain for any α^* a unique solution of (1.8) of the form

$$W_{\alpha^*,C}(z) = z^{\alpha^*}[1 + \sum_{n=1}^{\infty} a_n^*(C)z^n] \tag{1.10}$$

valid for $|z| < 1$, and for any $\beta^* > 0$ a unique solution of the form

$$W_{\beta^*,C} = z^{\beta^*}[1 + \sum_{n=1}^{\infty} b_n^*(C)z^n] \tag{1.11}$$

valid for $|z| < 1$.

In [1] Robertson determined fairly general sufficient conditions on $p(z)$ relative to $p^*(z)$ under which $F_\alpha(z)$ is univalent in $|z| < 1$. In [2] Brown extended these results to $F_\beta(z)$ but only for real β satisfying $0 < \beta \le 1/2$. In the Main Theorem of this paper we present sharp sufficient conditions on $p(z)$ relative to $p_C^*(z)$ under which the function $F_\beta(z)$ is univalent and spirallike in $|z| < 1$, where β may be complex valued. We then compare these results to those of Robertson for $F_\alpha(z)$.

2. PRELIMINARIES.

S will denote the class of functions $f(z)$ holomorphic and univalent in the unit disk $D \equiv \{z: |z| < 1\}$ and normalized so that $f(0) = 0$, $f'(0) = 1$.

We shall say that $f(z) \in F_{\varphi,\alpha}$ if and only if for some real number φ, $|\varphi| < \pi/2$, and some α, $0 \le \alpha < 1$,

$$\Re\{\frac{e^{i\varphi} zf'(z)}{f(z)}\} > \alpha$$

for all $z \in D$. $F_\varphi \equiv F_{\varphi,0}$ is the class of functions called spirallike in D, [3], [4]. Functions in the subclass $S^*(\alpha) \equiv F_{0,\alpha}$ are called starlike of order α in D. $S^*(0)$ is the class of functions starlike in D. It follows that $S^*(\alpha) \subset F_\varphi \subset S$; (see [2]).

We will need the following result.

THEOREM 2.1. Let $z^2p_C^*(z)$, $W_{\alpha^*,C}(z)$, and $W_{\beta^*,C}(z)$ be defined by (1.9), (1.10), and (1.11) respectively. If for all $|z| < 1$

$$\Re\{z^2p^*(z)\} \le |z|^2p^*(|z|)$$

then for fixed C $\dfrac{|z|W'_{\alpha^*,C}(|z|)}{W_{\alpha^*,C}(|z|)}$ is monotonic decreasing for all $|z| < \min(1, R_{\alpha^*}(C))$, and $\frac{|z|W'_{\beta^*,C}(|z|)}{W_{\beta^*,C}(|z|)}$ is monotonic decreasing for all $|z| < \min(1, R_{\beta^*}(C))$ where $R_{\alpha^*}(C)$ and $R_{\beta^*}(C)$ are the smallest positive zeros of the functions $W'_{\alpha^*,C}(r)$ and $W'_{\beta^*,C}(r)$ respectively.

In the case of α^* this result is given on page 262 of [1]. For β^* the result follows from (3.16) and Theorem 3.18 of [2] after noting that if $z^2p^*(z)$ is nonconstant the equality $\Re\{z^2p^*(z)\} = |z|^2p^*(|z|)$ cannot hold for all $0 \le r \le r_1$ on any ray $\theta = \text{constant} \ne 0$.

The condition that $z^2p^*(z)$ be nonconstant is necessary to ensure strict monotonicity in the results above since if $z^2p^*(z)$ is constant so are

$$\frac{|z|W'_{\alpha^*,C}(|z|)}{W_{\alpha^*,C}(|z|)} \text{ and } \frac{|z|W'_{\beta^*,C}(|z|)}{W_{\beta^*,C}(|z|)}.$$

3. LEMMAS.

In this section we prove the lemmas used to obtain Theorem A and The Main Theorem in section 4.

Since all of the results of this section are stated for $W_\beta(z)$ and $W_{\beta^*,C}(z)$ we adopt the following notational convention:

$$\begin{aligned} W &\equiv W(z) \equiv W_\beta(z) = z^\beta[1 + \sum_{n=1}^{\infty} b_n z^n], \\ W_C &\equiv W_C(z) \equiv W_{\beta^*,C}(z) = z^{\beta^*}[1 + \sum_{n=1}^{\infty} b_n^*(C) z^n]. \end{aligned} \tag{3.1}$$

It is important to note that all of the results of this section remain valid if W and W_C are replaced by either $W_\alpha(z)$ and $W_{\alpha^*,C}(z)$ or by $W_\alpha(z)$ and $W_{\beta^*,C}(z)$ and, moreover, the proofs are obtained by making corresponding changes in the proofs given here.

In our lemmas we will investigate the rate of change of $\Re\{\frac{zW'}{W}\}$ and $\Im\{\frac{zW'}{W}\}$ on rays issuing from the origin. For this reason we designate z by $re^{i\theta}$, fix θ, vary r, and use (1.1) to obtain

$$r\frac{d}{dr}\left[\frac{zW'}{W}\right] = -z^2p(z) + \frac{zW'}{W} - \left[\frac{zW'}{W}\right]^2, \tag{3.2}$$

where W' designates differentiation with respect to z.

Taking real and imaginary parts of (3.2) we obtain

$$r\frac{d}{dr}\mathcal{R}\{\frac{zW'}{W}\} = -\mathcal{R}\{z^2p(z)\} + \mathcal{R}\{\frac{zW'}{W}\} - \mathcal{R}^2\{\frac{zW'}{W}\} + \mathcal{I}^2\{\frac{zW'}{W}\} \tag{3.3}$$

and

$$r\frac{d}{dr}\mathcal{I}\{\frac{zW'}{W}\} = -\mathcal{I}\{z^2p(z)\} + \mathcal{I}\{\frac{zW'}{W}\} - 2\mathcal{R}\{\frac{zW'}{W}\}\,\mathcal{I}\{\frac{zW'}{W}\}. \tag{3.4}$$

Also from (1.8) and the fact that W_C is real for real z, we obtain for $z \geq 0$

$$r\frac{d}{dr}\left(\frac{rW_C'(r)}{W_C(r)}\right) = -r^2p_C^*(r) + \frac{rW_C'(r)}{W_C(r)} - \left(\frac{rW_C'(r)}{W_C(r)}\right)^2. \tag{3.5}$$

Our goal is to determine conditions on $z^2p(z)$ relative to $z^2p_C^*(z)$ which will ensure that on every ray θ = constant

$$\mathcal{R}\{\frac{zW'}{W}\} - \frac{rW_C'(r)}{W_C(r)} \geq 0 \text{ for all } 0 \leq r < 1. \tag{3.6}$$

Then it will follow from (1.7) that for θ = constant

$$\mathcal{R}\{\frac{\beta zF_\beta'(z)}{F_\beta(z)}\} - \frac{rW_C'(r)}{W_C(r)} \geq 0 \text{ for all } 0 \leq r < 1. \tag{3.7}$$

Since $\mathcal{R}\{\beta\} > 0$ (3.7) implies $F_\beta(z)$ is univalent and spirallike in $|z| < R(C)$, where R(C) is the smallest positive zero of $W_C'(r)$ or 1 whichever is the smaller. We will show that C can be adjusted so that R(C) = 1 and, therefore, $F_\beta(z)$ is univalent in D.

In [1] the inequality (3.6) was obtained when $W = W_\alpha(z)$ and $W_C(r) = W_{\alpha^*,C}(r)$ by a method that relied upon the inequality

$$r\frac{d}{dr}\mathcal{R}\{\frac{zW'}{W}\} \geq -\mathcal{R}\{z^2p(z)\} + \mathcal{R}\{\frac{zW'}{W}\} - \mathcal{R}^2\{\frac{zW'}{W}\}$$

obtained from (3.3) by neglecting the term $\mathcal{I}^2\{\frac{zW'}{W}\}$. Unfortunately this inequality is not sharp enough to yield (3.6) when $W = W_\beta(z)$ and $W_C(r) = W_{\beta^*,C}(r)$ by the method of [1]. In this paper we retain the term $\mathcal{I}^2\{\frac{zW'}{W}\}$ in (3.3) and derive estimates for its rate of growth relative to that of $\frac{rW_C'}{W_C}$. These estimates enable us to establish (3.6) for $W = W_\beta(z)$ and $W_C(r) = W_{\beta^*,C}(r)$.

We introduce the following notation where $z = re^{i\theta}$, θ is constant, and r satisfies the inequalities $0 \leq r < 1$.

$$T(r) \equiv \Re\{\frac{zW'}{W}\} - r\frac{W_C'(r)}{W_C(r)}. \tag{3.8}$$

$$S(r) \equiv -\Re\{\frac{zW'}{W}\} - r\frac{W'_C(r)}{W_C(r)}. \tag{3.9}$$

$$M(r) \equiv -\Im\{\frac{zW'}{W}\} - r\frac{W_C'(r)}{W_C(r)}. \tag{3.10}$$

$$N(r) \equiv \Im\{\frac{zW'}{W}\} - r\frac{W_C'(r)}{W_C(r)}. \tag{3.11}$$

$$\tau(r) \equiv -\Re\{z^2p(z)\} + r^2p_C^*(r). \tag{3.12}$$

$$\sigma(r) \equiv \Re\{z^2p(z)\} + r^2p_C^*(r). \tag{3.13}$$

$$\mu(r) \equiv \Im\{z^2p(z)\} + r^2p_C^*(r). \tag{3.14}$$

$$\nu(r) \equiv -\Im\{z^2p(z)\} + r^2p_C^*(r). \tag{3.15}$$

$R(C)$ is the smallest positive zero of $W_C'(r)$ (3.16)

$$R^* = \min(1, R(C)). \tag{3.17}$$

In terms of this notation our goal is to establish conditions under which $T(r) > T(0)$ on every ray θ = constant, $|z| < R^*$.

From (3.3), (3.4), and (3.5) we obtain the following relations:

$$r\frac{dT(r)}{dr} = \tau(r) + T(r)(1 - \frac{2r\,W_C'(r)}{W_C(r)}) - T^2(r) + \Im^2\{\frac{zW'}{W}\}. \tag{3.18}$$

$$r\frac{dS(r)}{dr} = \sigma(r) + S(r)(1 + \frac{2rW_C'(r)}{W_C(r)}) + S^2(r) - \Im^2\{\frac{zW'}{W}\} + 2(\frac{rW_C'(r)}{W_C(r)})^2. \tag{3.19}$$

$$r\,\frac{dM(r)}{dr} = \mu(r) + M(r) + 2\,\Re\{\frac{zW'}{W}\}\Im\{\frac{zW'}{W}\} + \left(\frac{rW_C'(r)}{W_C(r)}\right)^2 \tag{3.20}$$

$$r\,\frac{dN(r)}{dr} = \nu(r) + N(r) - 2\,\Re\{\frac{zW'}{W}\}\Im\{\frac{zW'}{W}\} + \left(\frac{rW_C'(r)}{W_C(r)}\right)^2 . \tag{3.21}$$

The proofs of most of the lemmas in this section reflect a common simple theme that is set forth formally in the following lemma.

LEMMA A. Let $G(r)$ be a real-valued differentiable function on $a \leq r \leq b$. Let $G(r) > 0$ for all $a \leq r < \rho \leq b$ and $G(\rho) = 0$. Then it follows that $G'(\rho) \leq 0$.

It should be noted that from their definitions it follows that the functions $T(r)$, $S(r)$, $M(r)$ and $N(r)$ can assume a value at most a finite number of times on any segment θ = constant, $0 \leq r \leq r_2 < 1$. Thus if, for example, $T(r) = k$ for some r in $0 \leq r \leq r_2 < 1$, then there is a smallest $r \equiv \rho$ in this interval for which $T(\rho) = k$.

LEMMA 1. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. If for fixed θ we have

a) $\tau(r) \geq \tau(0)$ for all $0 < r < 1$,

b) $T(r) > T(0)$ for all $0 \leq r \leq r_1 < R^*$,

c) $\dfrac{r_1W_C'(r_1)}{W_C(r_1)} = \beta^* - |\beta_2|$,

d) $|\beta_2| \leq 2(\beta_1 - \beta^*)$,

then $T(r) > T(0)$ for all $0 < r < R^*$.

PROOF. Assume that the conclusion is false. Then there exists an r, $r_1 < r < R^*$, for which $T(r) - T(0) = 0$. Let ρ be the smallest such r. Then since $T(r) - T(0) > 0$ for all $0 < r < \rho$ it follows from Lemma A, with $G(r) = T(r) - T(0)$, that $\frac{dT(r)}{dr}\Big|_{r=\rho} \leq 0$. We will show, however, that our hypotheses imply that $\frac{dT(r)}{dr}\Big|_{r=\rho} > 0$. Thus there can be no roots of $T(r) - T(0)$ on $r_1 < r < R^*$, and consequently $T(r) > T(0)$ for all $0 < r < R^*$.

From (3.18) and a) and b) of our hypotheses we have

$$r\,\frac{dT(r)}{dr}\Big|_{r=\rho} > \tau(0) + T(0)\left(1 - \frac{2\rho W_C'(\rho)}{W_C(\rho)}\right) - T^2(0). \tag{3.22}$$

From Theorem 2.1 it follows that $f(r) \equiv 1 - \frac{2rW_C'(r)}{W_C(r)}$ is monotonic increasing on $r_1 < r < R^*$. Thus

$$r \frac{dT(r)}{dr}\Big]_{r=\rho} > \tau(0) + T(0)f(r_1) - T^2(0)$$

$$= \tau(0) + T(0)(1 - 2\beta^*) + 2|\beta_2|(\beta_1 - \beta^*) - T^2(0),$$

which by d) of our hypotheses is

$$\geq \tau(0) + T(0)(1 - 2\beta^*) + \beta_2^2 - T^2(0)$$

$$= r \frac{dT(r)}{dr}\Big]_{r=0} = 0.$$

Thus $\frac{dT(r)}{dr}\Big]_{r=\rho} > 0$. This is the desired contradiction from which it follows that $T(r) > T(0)$ for all $0 < r < R^*$.

LEMMA 2. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_1 - \beta^* > 0$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for fixed θ we have $S(r_1) > S(0)$, and for all $0 < r_1 \leq r \leq r_2 < R^*$

a) $\sigma(r) \geq \sigma(0)$,

b) $\mathcal{J}^2\{\frac{zW'}{W}\} \leq \beta_2^2$

then $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From (3.19) and a) and b) of our hypotheses we have

$$r \frac{dS(r)}{dr}\Big]_{r=\rho} \geq \sigma(0) + S(0) + S^2(0) + \frac{2\rho W_C'(\rho)}{W_C(\rho)} S(0) + 2\left(\frac{\rho W_C'(\rho)}{W_C(\rho)}\right)^2 - \beta_2^2. \quad (3.23)$$

Now use the method of proof of Lemma 1 with $G(r) \equiv S(r) - S(0)$, ρ the smallest zero of $G(r)$ on $r_1 \leq r \leq r_2$, and

$$f(r) \equiv \frac{2rW_C'(r)}{W_C(r)} S(0) + 2\left(\frac{rW_C'(r)}{W_C(r)}\right)^2.$$

From Theorem 2.1 it follows that if $\beta_1 - \beta^* > 0$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Thus from (3.22) we have

$$r \frac{dS(r)}{dr}\Big]_{r=\rho} > \sigma(0) + S(0) + S^2(0) + f(0) - \beta_2^2 = r \frac{dS(r)}{dr}\Big]_{r=0} = 0,$$

and the lemma follows as in the proof of Lemma 1.

LEMMA 3A. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $S(r_1) > S(0)$, $N(r_1) > N(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$,

b) $\nu(r) \geq \nu(0)$,

c) $0 \leq \mathcal{I}\{\frac{zW'}{W}\} \leq \beta_2$,

then it follows that $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From Lemma 2 it follows that $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$, and from (3.9) we have

$$\mathcal{R}\{\frac{zW'}{W}\} < - S(0) - \frac{rW_C'(r)}{W_C(r)} = \beta_1 + \beta^* - \frac{rW_C'(r)}{W_C(r)} \text{ for all } r_1 \leq r \leq r_2.$$

Using this inequality along with (3.21) and c) of our hypotheses we have

$$r \frac{dN(r)}{dr} > \nu(r) + N(r) - 2(\beta_1 + \beta_2 - \frac{rW_C'(r)}{W_C(r)})(N(r) + \frac{rW_C'(r)}{W_C(r)}) + (\frac{rW_C'(r)}{W_C(r)})^2 \quad (3.24)$$

where we have used the definition (3.11) in the third term.

From (3.24) we obtain

$$r \frac{dN(r)}{dr} > \nu(r)+N(r)[1-2(\beta_1+ \beta^*)] + 2N(r) \frac{rW_C'(r)}{W_C(r)} -2(\beta_1 + \beta^*)\frac{rW_C'(r)}{W_C(r)} + 3(\frac{rW_C'(r)}{W_C(r)})^2 . \quad (3.25)$$

Now use the method of proof of Lemma 1 with $G(r) \equiv N(r) - N(0)$, ρ the smallest zero of $G(r)$ in $r_1 \leq r \leq r_2$, and

$$f(r) \equiv 2[N(0) - (\beta_1 + \beta^*)] \frac{rW_C'(r)}{W_C(r)} + 3(\frac{rW_C'(r)}{W_C(r)})^2 .$$

Then from (3.25), Theorem 2.1, and a) and b) of our hypotheses we have

$$r \frac{dN(r)}{dr}\Big]_{r=\rho} > \nu(0) + N(0)[1 - 2(\beta_1 + \beta^*)] + f(\rho). \quad (3.26)$$

From Theorem 2.1 it follows that if $\beta_2 \leq \beta_1 - \beta^*$ then $f(r)$ is monotonic increasing on $0 \leq r \leq r_2$. Thus from (3.26) we have

$$r \frac{dN(r)}{dr}\Big]_{r=\rho} > \nu(0) + N(0)[1 - 2(\beta_1 + \beta^*)] + f(0)$$

$$= \nu(0) + N(0) - 2\beta_1\beta_2 + \beta^{*2}$$

$$= r\frac{dN(r)}{dr}\Big]_{r=0} = 0.$$

The lemma now follows as in the proof of Lemma 1.

COROLLARY 3A. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $\mathcal{J}\{\frac{r_1e^{i\theta}W'(r_1e^{i\theta})}{W(r_1e^{i\theta})}\} > 0$, $S(r_1) > S(0)$, $N(r_1) > N(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$,

b) $\nu(r) \geq \nu(0)$,

c) $\mathcal{J}\{\frac{zW'}{W}\} \leq \beta_2$,

d) $\frac{rW_C'(r)}{W_C(r)} \geq \max\,(\beta^* - \beta_2, 0)$,

then it follows that $\mathcal{J}\{\frac{zW'}{W}\} > 0$ and $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. To prove that $\mathcal{J}\{\frac{zW'}{W}\} > 0$ for all $r_1 \leq r \leq r_2$ note that if ρ is the smallest zero of $\mathcal{J}\{\frac{zW'}{W}\}$ in the interval $r_1 < r < r_2$, then we can apply Lemma 3A on the interval $r_1 \leq r \leq \rho$ to obtain $N(r) > N(0)$ for all $r_1 \leq r \leq \rho$. Then from (3.11) it follows that

$$\mathcal{J}\{\frac{zW'}{W}\} > \beta_2 - \beta^* + \frac{rW_C'(r)}{W_C(r)} \tag{3.27}$$

for all $r_1 \leq r \leq \rho$. (3.27) and d) of our hypotheses give

$$\beta_2 - \beta^* + \frac{rW_C'(r)}{W_C(r)} \geq 0 \text{ on } r_1 \leq r \leq \rho. \tag{3.28}$$

Then (3.27) and (3.28) imply that

$$\mathcal{I}\{\frac{\rho e^{i\theta}W'(\rho e^{i\theta})}{W(\rho e^{i\theta})}\} > 0$$

which contradicts the assumption on ρ. Thus $\mathcal{I}\{\frac{zW'}{W}\} \geq 0$ for all $r_1 \leq r \leq r_2$. Now from Lemma 3A it follows directly that $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

LEMMA 3B. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ we have $M(r_1) > M(0)$, $S(r_1) > S(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$,

b) $\mu(r) \geq \mu(0)$,

c) $\beta_2 \leq \mathcal{I}\{\frac{zW'}{W}\} \leq 0$,

then it follows that $M(r) > M(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The method of proof is the same as that of Lemma 3A. Start with (3.20) instead of (3.21) and replace the condition $0 < \beta_2 \leq \beta_1 - \beta^*$ by $|\beta_2| \leq \beta_1 - \beta^*$. The lemma then follows by establishing the contradiction $\frac{dM}{dr}\rfloor_{r=\rho} > 0$.

COROLLARY 3B. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq \beta_1 - \beta^{*'}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ, $\mathcal{I}\{\frac{r_1e^{i\theta}W'(r_1e^{i\theta})}{W(r_1e^{i\theta})}\} < 0$, $S(r_1) > S(0)$, $M(r_1) > M(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$,

b) $\mu(r) \geq \mu(0)$,

c) $\mathcal{I}\{\frac{zW'}{W}\} \geq \beta_2$.

d) $\frac{rW_C'(r)}{W_C(r)} \geq \max\ (\beta^* - |\beta_2|, 0)$,

then it follows that $M(r) > M(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The method of proof is the same as that of Corollary 3A using $M(r)$ in place of $N(r)$ throughout.

LEMMA 4A. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \frac{\beta_1 - \beta^*}{2}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $M(r_1) > M(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\mu(r) \geq \mu(0)$,

b) $T(r) \geq T(0)$,

c) $\mathcal{I}\{\frac{zW'}{W}\} \geq \beta_2$,

d) $\frac{rW_C'(r)}{W_C(r)} \geq \max(\beta^* - \beta_2, 0)$,

then it follows that $M(r) > M(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From (3.20), (3.8), and a) and b) of our hypotheses it follows that

$$r\,\frac{dM(r)}{dr} \geq \mu(0) + M(r) + 2(T(0) + \frac{rW_C'(r)}{W_C(r)})\mathcal{I}\{\frac{zW'}{W}\} + (\frac{rW_C'(r)}{W_C(r)})^2$$

with equality for $r = 0$. Using definition (3.10) we rewrite this inequality in the form

$$r\,\frac{dM(r)}{dr} \geq \mu(0) + M(r)(1 - 2T(0)) - 2(M(r) + T(0))\,\frac{rW_C'(r)}{W_C(r)} - (\frac{rW_C'(r)}{W_C(r)})^2 \quad (3.29)$$

Now use the method of proof of Lemma 1 with $G(r) = M(r) - M(0)$, ρ the smallest zero of $G(r)$ on $r_1 \leq r \leq r_2$, and

$$f(r) \equiv -2(M(0) + T(0))\,\frac{rW_C'(r)}{W_C(r)} - (\frac{rW_C'(r)}{W_C(r)})^2 .$$

From Theorem 2.1 it follows that if $\beta_2 \leq (\beta_1 - \beta^*)/2$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Then from (3.29) we have

$$\begin{aligned} r\,\frac{dM(r)}{dr}\Big]_{r=\rho} &\geq \mu(0) + M(0)(1 - 2T(0)) - 2(M(0) + T(0))(\frac{\rho W_C'(\rho)}{W_C(\rho)}) - (\frac{\rho W_C'(\rho)}{W_C(\rho)})^2 \\ &> \mu(0) + M(0)(1 - 2T(0)) - 2(M(0) + T(0))\beta^* - \beta^{*2} \\ &= r\,\frac{dM(r)}{dr}\Big]_{r=0} = 0, \end{aligned}$$

and the lemma now follows as in the proof of Lemma 1.

LEMMA 4B. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq \frac{\beta_1 - \beta^*}{2}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $N(r_1) > N(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\nu(r) \geq \nu(0)$,

b) $T(r) \geq T(0)$,

c) $\mathcal{I}\{\frac{zW'}{W}\} \leq \beta_2$

d) $\frac{rW_C'(r)}{W_C(r)} \geq \max(\beta^* - |\beta_2|, 0)$,

then it follows that $N(r) > N(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The proof proceeds precisely as that of Lemma 4A except that (3.11) is used in place of (3.10), and the condition $|\beta_2| \leq \frac{\beta_1 - \beta^*}{2}$ replaces $\beta_2 \leq \frac{\beta_1 - \beta^*}{2}$.

LEMMA 5A. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \frac{\beta_1 - \beta^*}{2}$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $S(r_1) > S(0)$, $M(r_1) > M(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\mu(r) \geq \mu(0)$,

b) $\sigma(r) \geq \sigma(0)$,

c) $T(r) \geq T(0)$,

d) $\mathcal{I}\{\frac{zW'}{W}\} \geq \beta_2$,

e) $\frac{rW_C'}{W_C} \geq \max(\beta^* - \beta_2, 0)$,

then it follows that $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. By Lemma 4A we have

$$\mathcal{I}\{\frac{zW'}{W}\} \leq -M(0) - \frac{rW_C'(r)}{W_C(r)} = \beta_2 + \beta^* - \frac{rW_C'(r)}{W_C(r)} > 0 \tag{3.30}$$

for all $r_1 \leq r \leq r_2$. Thus from (3.19) and (3.30) it follows that

$$r\frac{dS(r)}{dr} \geq \sigma(r) + S(r)(1 + \frac{2rW_C'(r)}{W_C(r)}) + S^2(r) - (-M(0) - \frac{rW_C'(r)}{W_C(r)})^2 + 2(\frac{rW_C'(r)}{W_C(r)})^2. \tag{3.31}$$

Now use the method of proof of Lemma 1 with $G(r) \equiv S(r) - S(0)$, ρ the smallest zero of $G(r)$ on $r_1 \leq r \leq r_2$ and

$$f(r) \equiv 2(S(0) - M(0))(\frac{rW_C'(r)}{W_C(r)}) + (\frac{rW_C'(r)}{W_C(r)})^2 .$$

From Theorem 2.1 it follows that if $\beta_2 \leq (\beta_1 - \beta^*)/2$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Then from (3.31) and a) through d) of our hypotheses it follows that

$$\frac{rdS(r)}{dr}\Big]_{r=\rho} \geq \sigma(0) + S(0) + S^2(0) - M^2(0) + f(0)$$

$$= \frac{rdS(r)}{dr}\Big]_{r=0} = 0,$$

and our lemma follows as in the proof of Lemma 1.

LEMMA 5B. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $\beta_2 < 0$, $|\beta_2| \leq (\beta_1 - \beta^*)/2$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ, $N(r_1) > N(0)$, $S(r_1) > S(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\nu(r) \geq \nu(0)$,

b) $\sigma(r) \geq \sigma(0)$,

c) $T(r) \geq T(0)$,

d) $\mathcal{J}\{\frac{zW'}{W}\} \leq \beta_2$,

e) $\frac{rW_C'(r)}{W_C(r)} \geq \max\ (\beta^* - |\beta_2|, 0)$,

then it follows that $S(r) > S(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The proof proceeds precisely as that of Lemma 5A except that Lemma 4B is used in place of Lemma 4A, and the condition $|\beta_2| \leq (\beta_1 - \beta^*)/2$ replaces $\beta_2 \leq (\beta_1 - \beta^*)/2$.

LEMMA 6. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for fixed θ $T(r_1) > T(0)$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\tau(r) \geq \tau(0)$,

b) $\mathcal{J}^2\{\frac{zW'}{W}\} \geq \beta_2^2$,

then $T(r) > T(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. With $G(r)$ and ρ defined as in Lemma 1, we obtain from (3.18) and our hypotheses

$$\frac{rdT(r)}{dr}\Big]_{r=\rho} > \tau(0) + T(0)(1 - 2\beta^*) - T^2(0) + \beta_2^2$$

$$= \frac{rdT(r)}{dr}\Big]_{r=\rho} = 0,$$

and the lemma follows as in the proof of Lemma 1.

LEMMA 7A. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1. Let $0 < \beta_2 \leq \beta_1 - \beta^*$, and let r_1 satisfy the inequalities $0 < r_1 < R^*$. If for a fixed θ $S(r_1) > S(0)$, $N(r_1) > N(0)$, $T(r_1) > T(0)$, $\mathfrak{I}\{\frac{r_1e^{i\theta}W'(r_1e^{i\theta})}{W(r_1e^{i\theta})}\} > 0$; and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$,

b) $\tau(r) \geq \tau(0)$,

c) $\nu(r) \geq \nu(0)$,

d) $\mathfrak{I}\{\frac{zW'}{W}\} \leq \beta_2$

e) $\frac{rW_C'(r)}{W_C(r)} \geq \max\,(\beta^* - \beta_2,\, 0)$,

then $T(r) > T(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. From (3.11) and e) of our hypotheses it follows that $(N(0) + \frac{rW_C'(r)}{W_C(r)}) \geq 0$ for all $r_1 \leq r \leq r_2$. Then from (3.18) and Corollary 3A we have

$$\frac{rdT}{dr} \geq \tau(r) + T(r)(1 - 2r\,\frac{W_C'(r)}{W_C(r)}) - T^2(r) + (N(0) + \frac{rW_C'(r)}{W_C(r)})^2 \tag{3.32}$$

for all $r_1 \leq r \leq r_2$.

Now use the method of proof of Lemma 1 with $G(r)$ and ρ defined as in Lemma 1 and with

$$f(r) \equiv -2(T(0) - N(0))\,\frac{rW_C'(r)}{W_C(r)} + (\frac{rW_C'(r)}{W_C(r)})^2 .$$

From Theorem 2.1 it follows that if $\beta_2 \leq \beta_1 - \beta^*$ then $f(r)$ is monotonic increasing on $0 < r \leq r_2$. Thus, from (3.32) and b) of our hypotheses we have

$$\frac{rdT(r)}{dr}\Big]_{r=\rho} > \tau(0) + T(0) - T^2(0) + N^2(0) + f(0)$$

$$= \tau(0) + T(0)(1 - 2\beta^*) - T^2(0) + (N(0) + \beta^*)^2$$

$$= \frac{rdT(r)}{dr}\Big]_{r=\rho} = 0,$$

and our lemma follows as in the proof of Lemma 1.

LEMMA 7B. Let $z^2p^*(z)$ satisfy the condition of Theorem 2.1. Let r_1 satisfy the inequalities $0 < r_1 < R^*$ and let $\beta_2 < 0$, $|\beta_2| \leq \beta_1 - \beta^*$. If for a fixed θ $S(r_1) > S(0)$, $M(r_1) > M(0)$, $T(r_1) > T(0)$, $\mathcal{I}\{\frac{r_1e^{i\theta}W'_\beta(r_1e^{i\theta})}{W_\beta(r_1e^{i\theta})}\} < 0$, and if for all $0 < r_1 \leq r \leq r_2 < R^*$ we have

a) $\sigma(r) \geq \sigma(0)$,

b) $\tau(r) \geq \tau(0)$,

c) $\mu(r) \geq \mu(\theta)$,

d) $\mathcal{I}\{\frac{zW'}{W}\} \geq \beta_2$,

e) $\frac{rW'_C(r)}{W_C(r)} \geq \max\ (\beta^* - |\beta_2|,\ )$,

then $T(r) > T(0)$ for all $r_1 \leq r \leq r_2$.

PROOF. The proof proceeds precisely as that of Lemma 7A except that Corollary 3B is used in place of Corollary 3A.

LEMMA 8. If for all θ, $0 \leq \theta \leq 2\pi$, and for all $0 < r < R^*$ we have a) $\sigma(r) \geq \sigma(0)$, b) $\tau(r) \geq \tau(0)$, c) $\mu(r) \geq \mu(0)$, d) $\nu(r) \geq \nu(0)$ then it follows that

e) $\frac{dS}{dr}\Big]_{r=0} \geq 0$, f) $\frac{dT}{dr}\Big]_{r=0} \geq 0$, g) $\frac{dM}{dr}\Big]_{r=0} \geq 0$ and h) $\frac{dN}{dr}\Big]_{r=0} \geq 0$

where a $\Rightarrow$ e, b $\Rightarrow$ f, c $\Rightarrow$ g, and d $\Rightarrow$ h. Moreover, strict inequality holds in e) through h) if $\beta_1 > \beta^* > 0$.

PROOF. We prove that b $\Rightarrow$ f. All four implications can be proved using the same techniques.

For $0 \leq r < R^*$ we have

$$\frac{zW'}{W} = \beta + c_1 z + \cdots + c_n z^n + \cdots \text{ and} \tag{3.33}$$

$$\frac{zW_C'(z)}{W_C(z)} = \beta^* + c_1^* z + \cdots + c_n^* z^n + \cdots \tag{3.34}$$

where $c_1 = \frac{-p_1}{2\beta}$ and $c_1^* = \frac{-Cp_1^*}{2\beta^*}$ with p_1 and p_1^* defined by (1.2) and (1.9) respectively.

Now $\tau(r) \geq \tau(0)$ if and only if $-\mathcal{R}\{z^2 p(z) - p_0\} + r^2 p_C^*(r) - p_0^* \geq 0$. Therefore, if $\tau(r) \geq \tau(0)$ for all θ, $0 \leq \theta \leq 2\pi$, and all $0 \leq r < R^*$, it follows that $(-\mathcal{R}\{p_1 z\} + Crp_1^*) \geq 0$ for sufficiently small r and for all θ.

Thus we must have

$$-\mathcal{R}\{p_1 e^{i\theta}\} + Cp_1^* \geq 0 \text{ for all } \theta. \tag{3.35}$$

Now from (3.33) and (3.34) we have

$$\left.\frac{dT}{dr}\right]_{r=0} = \mathcal{R}\{c_1 e^{i\theta}\} - c_1^*$$

$$= \mathcal{R}\{\frac{-p_1 e^{i\theta}}{2\beta}\} + \frac{Cp_1^*}{2\beta^*}.$$

Then if we write β in the form $|\beta| e^{i\varphi}$, we have

$$\left.\frac{dT}{dr}\right]_{r=0} = \mathcal{R}\{\frac{-p_1 e^{i(\theta - \varphi)}}{2|\beta|}\} + \frac{Cp_1^*}{2\beta^*}. \tag{3.36}$$

Thus $\left.\frac{dT}{dr}\right]_{r=0} \geq 0$ if and only if $-\mathcal{R}\{p_1 e^{i(\theta-\varphi)}\} + \frac{Cp_1^*}{(\frac{\beta^*}{|\beta|})} \geq 0$ (3.37)

for all θ.

However, since $\beta_1 \geq \beta^* > 0$, we have $0 < \beta^*/|\beta| \leq 1$ and it follows from (3.35) and (3.37) that $\left.\frac{dT}{dr}\right]_{r=0} \geq 0$ with strict inequality if $\beta_1 > \beta^*$.

4. THEOREM A AND THE MAIN THEOREM.

We have designated the first result of this section as Theorem A since it is our analog of Theorem A of [1] when $\gamma = 0$.

THEOREM A. Let $W \equiv W(z) \equiv W_\beta(z) = z^\beta[1 + \sum_{n=1}^{\infty} b_n z^n]$ be the unique solution of $W''(z) + p(z)W(z) = 0$ where

$$z^2p(z) = p_0 + p_1 z + \cdots + p_n z^n + \cdots$$

is holomorphic in $|z| < 1$, and $\beta = \beta_1 + i\beta_2$ with $0 < \beta_1 \leq 1/2$.

Let $W_C \equiv W_C(z) \equiv W_{\beta^*,C}(z) = z^{\beta^*}[1 + \sum_{n=1}^{\infty} b_n^*(C)z^n]$ be the unique solution of $W''(z) + z^2p_C^*(z)W(z) = 0$ where

$$z^2p_C^*(z) = C[z^2p^*(z) - p_0^*] + p_0^* \text{ with}$$

$$z^2p^*(z) = p_0^* + p_1^*z + \cdots + p_n^*z^n + \cdots,\ p_0^* \leq 1/4,$$

holomorphic in $|z| < 1$ and real on the real axis; and where $C > 0$ and $0 < \beta^* \leq 1/2$.

Let $R(C)$ be the smallest positive root of $W_C'(r)$, $0 < r < 1$, if such exists.

For $|z| < 1$ let $z^2p(z)$ and $z^2p^*(z)$ satisfy the inequalities

$$\text{(i)}\ \Re\{z^2p^*(z)\} \leq |z|^2p^*(|z|),$$

$$\text{(ii)}\ |z^2p(z) - p_0| \leq |z|^2p_C^*(|z|) - p_0^*.$$

Then for $|\beta_2| \leq (\beta_1 - \beta^*)/2$ it follows that

$$\Re\{\frac{zW'}{W}\} \geq \frac{rW_C'(r)}{W_C(r)} > 0$$

for all $|z| = r < R^* \equiv \min(R(C), 1)$.

We first note that (i) of our hypotheses ensures that $z^2p^*(z)$ satisfies the conditions of Theorem 2.1. In addition (ii) implies inequalities a) through c) of Lemma 8. For example, inequality a) of Lemma 8 is valid if and only if

$$\Re\{z^2p(z)\} + |z|^2p_C^*(|z|) \geq \Re\{p_0\} + p_0^*,$$

and this is true if and only if

$$\Re\{z^2p(z) - p_0\} \geq -\{|z|^2p_C^*(|z|) - p_0^*\}$$

which in turn follows from (ii).

We will obtain the result of Theorem A by proving that $T(r) > T(0)$ on any ray θ = constant, $0 < r < R^*$. To establish this fact we will need Lemmas 1 through 7 whose hypotheses require us to know whether $\mathcal{I}\{\frac{zW'}{W}\} \geq \beta_2$ or $\mathcal{I}\{\frac{zW'}{W}\} \leq \beta_2$. Therefore, we introduce on each ray θ = constant, $0 < r < R^*$, the points ρ_i where ρ_i and ρ_{i+1}, $\rho_i < \rho_{i+1}$, are consecutive values of r at which $\mathcal{I}\{\frac{zW'}{W}\} - \beta_2$ changes sign. We then show that $T(r) > T(0)$ on every interval $\rho_i \leq r \leq \rho_{i+1}$.

The proof requires consideration of the following four cases.

Case 1: $0 < \beta_2 < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{I}\{\frac{zW'}{W}\}]_{r=0} > 0$,

Case 2: $0 < \beta_2 < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{I}\{\frac{zW'}{W}\}]_{r=0} < 0$,

Case 3: $\beta_2 < 0$, $|\beta_2| < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{I}\{\frac{zW'}{W}\}]_{r=0} > 0$,

Case 4: $\beta_2 < 0$, $|\beta_2| < \frac{\beta_1 - \beta^*}{2}$, θ constant, $\frac{d}{dr}[\mathcal{I}\{\frac{zW'}{W}\}]_{r=0} < 0$.

For $\beta_2 = 0$ the result of Theorem A was established by Robertson [1] for $W = W_\alpha(z)$ and $W_C = W_{\alpha^*,C}(z)$ and by Brown [2] for $W = W_\beta(z)$, $W_C = W_{\beta^*,C}(z)$.

We will assume that $\frac{rW_C'(r)}{W_C(r)} > \beta^* - |\beta_2|$ for all $0 \leq r < R^*$ since if for some ρ, $0 < \rho < R^*$, we have $\frac{\rho W_C'(\rho)}{W_C(\rho)} = \beta^* - |\beta_2|$, we can restrict our attention to the interval $0 \leq r < \rho$ and then use Lemma 1 on the interval $\rho < r < R^*$.

PROOF OF CASE 1.

1. From Lemma 8 it follows that there exists a ρ^*, $0 < \rho^* < \rho_1$, such that for all θ $S(r) > S(0)$, $T(r) > T(0)$, $M(r) > M(0)$ and $N(r) > N(0)$ for all $0 < r \leq \rho^*$.
2. Now fix θ and apply Lemma 6 to the interval $\rho^* \leq r \leq \rho$ to obtain $T(r) > T(0)$ on $\rho^* \leq r \leq \rho_1$.
3. From definitions (3.10) and (3.11), the definition of the ρ_i and the monotonicity of $\frac{rW_C'(r)}{W_C(r)}$ on $0 < r < R^*$, it follows that $M(\rho_1) > M(0)$ and $N(\rho_1) > N(0)$.

4. Then by Lemma 5A applied to the interval $\rho^* \leq r \leq \rho_1$ we have $S(\rho_1) > S(0)$.

5. By Lemma 7A it then follows that $T(r) > T(0)$ on the interval $\rho_1 \leq r \leq \rho_2$.

6. By Lemma 6, $T(r) > T(0)$ on the interval $\rho_2 \leq r \leq \rho_3$.

7. By 4 above and Lemma 2 we have $S(\rho_2) > S(0)$.

8. As in 3 above we obtain $M(\rho_2) > M(0)$, $N(\rho_2) > N(0)$ and $N(\rho_3) > N(0)$.

9. Then by Lemma 5A we have $S(\rho_3) > S(0)$.

10. From 8, 9 and Lemma 7A it follows that $T(r) > T(0)$ on the interval $\rho_3 \leq r \leq \rho_4$.

By successive iterations of steps 6 through 10 it follows that if $T(r) > T(0)$ on $\rho_i \leq r \leq \rho_{i+1}$ then $T(r) > T(0)$ on $\rho_{i+1} \leq r \leq \rho_{i+2}$. Moreover, the proof actually demonstrates that $T(r) > T(0)$ on any interval of the ray θ = constant, $0 < r < R^*$, on which either $\mathcal{I}\{\frac{zW'}{W}\} - \beta_2 \geq 0$ or $\mathcal{I}\{\frac{zW'}{W}\} - \beta_2 \leq 0$. Thus it follows that $T(r) > T(0) = \beta_1 - \beta^* \geq 0$ for all $0 < r < R^*$ on any ray θ = constant.

PROOF OF CASE 2.

1. As in step 1 of Case 1 we have that there exists a ρ^*, $0 < \rho^* < \rho_1$, such that for all θ, $S(r) > S(0)$, $T(r) > T(0)$, $M(r) > M(0)$ and $N(r) > N(0)$ for all $0 \leq r \leq \rho^*$.

2. By Lemma 7A it then follows that $T(r) > T(0)$ on the interval $\rho^* \leq r \leq \rho_1$.

3. By Lemma 6 we have $T(r) > T(0)$ on $\rho_1 \leq r \leq \rho_2$.

4. By definition (3.10) and the monotonicity of $\frac{rW_C'(r)}{W_C(r)}$ on $0 < r < R^*$ we have $M(\rho_1) > M(0)$.

5. By 1 above and Lemma 2 we have $S(\rho_1) > S(0)$.

6. Then by Lemma 5A it follows that $S(\rho_2) > S(0)$.

7. By definition (3.11) and the monotonicity of $\frac{rW_C'(r)}{W_C(r)}$ on $0 < r < R^*$ we have $N(\rho_2) > N(0)$.

8. Then by 3, 6, 7, and Lemma 7A we have $T(r) > T(0)$ on the interval $\rho_2 \leq r \leq \rho_3$.

By successive iteration of steps 3 through 8 (omitting the reference to step 1 in step 5) it follows that if $T(r) > T(0)$ on $\rho_i \leq r \leq \rho_{i+1}$ then $T(r) > T(0)$ on $\rho_{i+1} \leq r \leq \rho_{i+2}$. Then as in Case 1 we obtain $T(r) > T(0) = \beta_1 - \beta^* \geq 0$ for all

$0 < r < R^*$ on any ray θ = constant.

PROOFS OF CASE 3 AND CASE 4. The proofs of Case 3 and Case 4 are identical to those of Case 2 and Case 1 respectively except that all A lemmas are replaced by corresponding B lemmas.

COROLLARY A. Theorem A remains true if $W(z)$ and $W_C(z)$ are replaced by either $W_\alpha(z)$ and $W_{\alpha^*,C}(z)$ or by $W_\alpha(z)$ and $W_{\beta^*,C}(z)$.

PROOF. The result follows from the fact that all of the lemmas used in the proof of Theorem A remain valid under the indicated substitutions.

Our Main Theorem will be derived from Theorem A precisely as the Main Theorem of [1] was derived from Theorem A of [1]. We will not reproduce Robertson's proofs but simply mention that his methods apply equally well to $W_C(z) = W_{\alpha^*,C}(z)$ and $W_C(z) = W_{\beta^*,C}(z)$, and then summarize the needed results in the following lemma.

LEMMA 4.1. Let $z^2p^*(z)$ satisfy the conditions of Theorem 2.1 and let R be fixed, $0 < R < 1$. Then there exists a $C \equiv C(R) > 0$ such that when $p_C^*(z) \equiv p_{C(R)}^*(z)$ we have $W'_{C(R)}(R) = 0$ and $W'_{C(R)}(r) > 0$ for all $0 < r < R$. Moreover, for fixed $z^2p^*(z)$ we have $\lim_{R\to 1^-} C(R) \equiv A(p^*) \equiv A$ is finite and $W'_A(r) > 0$ for all $0 < r < 1$. The value A, called the universal constant corresponding to $z^2p^*(z)$, is largest in the sense that for any $\epsilon > 0$ there exists an $r(\epsilon)$, $0 < r(\epsilon) < 1$, such that $W'_A(r(\epsilon)) = 0$ and $W'_{A+\epsilon}(r(\epsilon)) \leq 0$.

THE MAIN THEOREM. Let

$$z^2p^*(z) = p_0^* + p_1^*z + \cdots + p_n^*z^n + \cdots$$

be nonconstant and holomorphic in $|z| < 1$ and real on the real axis with $p_0^* \leq 1/4$. Let

$$\Re\{z^2p^*(z)\} \leqq |z|^2p^*(|z|) \text{ for } |z| < 1. \tag{4.1}$$

Let $z^2p_A^*(z) = A(z^2p^*(z) - p_0^*) + p_0^*$ where $A = A(p^*)$ is the universal constant corresponding to $z^2p^*(z)$. Let

$$W_A(z) \equiv W_{\beta^*,A}(z) = z^{\beta^*}[1 + \sum_{n=1}^{\infty} b_n^*(C)],\ |z| < 1,\ \beta^* > 0,$$

be the unique solution of

$$W''(z) + p_A^*(z)W(z) = 0$$

corresponding to the smaller root of the indicial equation. Then the function

$$F_A(z) \equiv [W_A(z)]^{1/\beta^*} = z + \cdots$$

is a holomorphic function, univalent and starlike in $|z| < 1$, and is not both holomorphic and univalent in any larger circle whenever $A > 0$.

Let $z^2p(z)$ be holomorphic in $|z| < 1$ with

$$|z^2p(z) - p_0| \le |z|^2p_A^*(|z|) - p_0^* \tag{4.2}$$

for all $|z| < 1$. Let

$$W(z) \equiv W_\beta(z) = z^\beta[1 + \sum_{n=1}^{\infty} b_n z^n],\ |z| < 1,$$

$\beta = \beta_1 + i\beta_2$, $\beta_1 > 0$, be the unique solution of

$$W''(z) + p(z)W(z) = 0$$

corresponding to the root β, with smaller real part, of the indicial equation.

Then if $|\beta_2| \le \frac{\beta_1 - \beta^*}{2}$ the function

$$F_\beta(z) = [W_\beta(z)]^{1/\beta} = z + \cdots$$

is a holomorphic function, univalent and spirallike in $|z| < 1$. The constant $A = A(p^*)$ is the largest possible one.

PROOF. From Theorem A we have

$$\Re\{\frac{\beta z F_\beta'(z)}{F_\beta(z)}\} \equiv \Re\{\frac{zW_\beta'(z)}{W_\beta(z)}\} \ge \frac{|z|W_A'(|z|)}{W_A(|z|)} > 0,\ |z| < 1. \tag{4.3}$$

Now if we choose $z^2p(z) \equiv z^2p_C^*(z)$ then $\beta_2 = 0$, $W_\beta(z) \equiv W_{\beta^*,C}(z) \equiv W_C(z)$ and from Theorem 3.23 of [2] we have

$$\Re\{\frac{zW_C'(z)}{W_C(z)}\} \ge \frac{|z|W_C'(|z|)}{W_C(|z|)},\ |z| < R(C). \tag{4.4}$$

Thus from (4.4) and the definition of A we have

$$\mathcal{R}\{\frac{zF_A'(z)}{F_A(z)}\} \geq \frac{1}{\beta^*}\ \frac{|z|W_A'(|z|)}{W_A(|z|)} > 0,\ |z| < 1. \tag{4.5}$$

From (4.3) it follows that $F_\beta(z)$ is univalent and spirallike in $|z| < 1$, and from (4.5) it follows that $F_A(z)$ is univalent and starlike in $|z| < 1$. Moreover, since equality holds in (4.4) when z is real and positive, and since $W'_{A+\epsilon}(R) = 0$ for some R, $0 < R < 1$, for arbitrarily small positive values of ϵ, it is clear that $F_{A+\epsilon}(z)$ is not univalent in $|z| < 1$ no matter how small a positive ϵ we take. Thus the constant A is the largest possible. The proof that the radius of univalence of $F_A(z)$ is precisely 1 is contained in [1] page 265 and will not be reproduced here.

COROLLARY B. The Main Theorem is true when $W = W_\alpha(z)$ and $W_C(z) = W_{\alpha^*,C}(z)$ whenever $|\alpha_2| \leq \frac{\alpha_1 - \alpha^*}{2}$ and also when $W = W_\alpha(z)$ and $W_C(z) = W_{\beta^*,C}(z)$ whenever $|\alpha_2| \leq \frac{\alpha_1 - \beta^*}{2}$.

The proof of this corollary is immediate since all of our previous results remain valid under the indicated substitutions for $W(z)$ and $W_C(z)$.

5. REMARKS.

We will now indicate how the results of our Main Theorem and Corollary B compare to or extend those of Robertson's Main Theorem in [1] for $\gamma = 0$.

In [1] the condition

$$\mathcal{R}\{z^2p(z)\} \leq |z|^2 p_A^*(|z|) \tag{4.6}$$

replaces our condition (4.2), there is no explicit bound on $|\alpha_2|$, and the results refer to the case $W = W_\alpha(z)$ and $W_C = W_{\alpha^*,C}(z)$.

Our condition (4.2) is independent of condition (4.6). When $\alpha^* \geq 1/2$ our Corollary B requires that

$$|\alpha_2| \leq \frac{\alpha_1}{2} - \frac{1}{4} \tag{4.7}$$

while (4.6) implies that

$$\alpha_1 - \alpha_1^2 + \alpha_2^2 \leq \alpha^* - \alpha^{*2}$$

so that

$$|\alpha_2| \leq \alpha_1 - \tfrac{1}{2}. \tag{4.8}$$

We refer now to the "exponent plane" of Figure 1 in which α_1 and β_1 are measured horizontally and α_2 and β_2 are measured vertically. Our conclusions are the following:

1. In region I only our Main Theorem applies.
2. In regions II only Robertson's Main Theorem applies.
3. In regions III $\cup$ IV Robertson's Main Theorem applies and our Corollary B applies with $W = W_\alpha(z)$ and $W_C = W_{\beta^*,C}(z)$.
4. In region IV Robertson's Main Theorem applies and our Corollary B applies with $W = W_\alpha(z)$ and $W_C = W_{\alpha^*,C}(z)$.

Thus it is in region I that we have an extension of the results of [1] and [2].

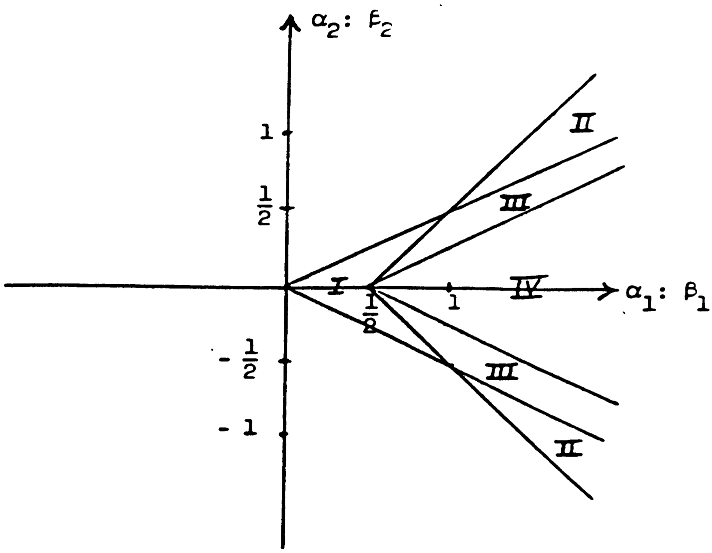

Figure 1

We note that we can obtain Robertson's Theorem A with $\gamma = 0$ from (3.18) with $W = W_\alpha(z)$ and $W_C = W_{\alpha^*,C}(z)$. This follows by noting first that the hypotheses of Theorem A of [1] imply that $\alpha_1 \geq \alpha^*$ and that $\tau(r) \geq 0$ for all $0 \leq r < 1$. Also, if $\alpha_1 > \alpha^*$ then $T(0) > 0$, while if $\alpha_1 = \alpha^*$ then $\alpha_2 = 0$ and as in the proof of Lemma 8 it follows that $T'(0) > 0$. Now if we let ρ be the smallest zero of $T(r)$ on $0 < r < 1$ we obtain from (3.18)

$$\frac{\rho\, dT(\rho)}{dr} = \tau(\rho) + \mathcal{I}^2\left\{\frac{\rho e^{i\theta} W'_\beta(\rho e^{i\theta})}{W_\beta(\rho e^{i\theta})}\right\}. \tag{4.9}$$

Thus either $\frac{dT(\rho)}{d\rho} > 0$ or both terms in the right member of (4.9) vanish. The former conclusion yields an immediate contradiction to the definition of ρ. If we assume the latter conclusion then an examination of the successive derivatives of (3.18) shows that the first non-vanishing derivative of $T(r)$ at ρ is positive and of even order. Thus $T(r) \geq 0$ for all $0 < r < R^*$.

Finally we point out that for $\gamma \neq 0$, $|\gamma| < \pi/2$, Robertson's Theorem A can be obtained by applying the same reasoning as above to the following analog of (3.18) where $W = W_\alpha(z)$ and $W_C = W_{\alpha^*, C}(z)$.

$$\frac{rdT}{dr} = \tau(r) + T(r)\left(1 - \frac{2rW'_C(r)}{W_C(r)}\right) - \sec \gamma T^2(r) + \sec \gamma\, \mathcal{I}^2\left\{\frac{zW'}{W}\right\}.$$

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