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STRICT TOPOLOGIES IN NON-ARCHIMEDEAN FUNCTION SPACES

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ABSTRACT. Let F be a non-trivial complete non-Archimedean valued field. Some locally F--convex topologies, on the space $C_b(X,E)$ of all bounded continuous functions from a zero--dimensional topological space X to a non-Archimedean locally F-convex space E, are studied. The corresponding dual spaces are also investigated.

KEY WORDS AND PHRASES: non-Archimedean spaces, spherically complete, zero-dimensional, Banaschewski compactification, strict topologies.
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1. INTRODUCTION.

Several authors have studied various topologies on spaces of continuous functions with values into either a valued field or a non-Archimedean locally convex space. Some of the papers on the subject are the [1]-[9]. The strict topology was introduced for the first time by Buck [10] in the space $C_b(X,E)$ of all bounded continuous functions from a locally compact space X to a locally convex space E. In recent years several other authors have extended Buck's results by generalizing the space X and taking E to be either the scalar field or a locally convex space or an arbitrary topological vector space. In [9] Prolla defined the strict topology in $C_b(X,E)$ assuming that X is locally compact Hausdorff zero-dimensional and E a non-Archimedean normed space over a locally compact non-Archimedean valued field F. In [7] the author studied the strict topology β_0 on $C_b(X,E)$ assuming that X is an arbitrary topological space and E a non-Archimedean locally F-convex space over a non-Archimedean valued field F.

In this paper we introduce and study the locally F-convex topologies β, β', β_1 and β_1' on $C_b(X,E)$ where X is zero-dimensional and E a non-Archimedean locally F-convex space. These topologies are defined by means of the Banaschewski compactification $\beta_0 X$ of X and yield as corresponding dual spaces certain spaces of E'-valued measures.

2. PRELIMINARIES.

Throughout this paper, X will denote a Hausdorff zero-dimensioanl (= ultraregular) topological space and $\beta_0 X$ its Banaschewski compactification (see [1]). For a continuous function f from X to an ultraregular topological space Y for which f(X) is relatively compact in Y, we will denote by $\hat{f}$ the unique continuous extension of f to all of $\beta_0 X$. For various notions on non-Archimedean spaces we will refer to [11]-[13].

Let F be a non-trivial complete non-Archimedean valued field and let E be a Hausdorff non-Archimedean locally F-convex space over F. Let $C_b(X,E)$ denote the space of all bounded continuous E-valued functions on X and let $C_{rc}(X,E)$ be the subspace of those f for which f(X) is relatively compact in E. For a subset A of X, we will denote by χ_A the F-characteristic function of A. Also, if f is a function from X to E and p a seminorm on E, we will define $\|f\|_{A,p}$ and $\|f\|_p$ by

$$\|f\|_{A,p} = \sup\{p(f(x)) : x \in A\}, \quad \|f\|_p = \|f\|_{X,p}.$$

For an F-valued function g on X, we define

$$\|g\|_A = \sup\{|g(x)| : x \in A\}, \quad \|g\| = \|g\|_X.$$

Let Γ be an upwards directed family of continuous non-Archimedean seminorms on E generating its topology. The uniform topology τ_u on a subspace of $C_b(X,E)$ is the locally F-convex topology generated by the family of non-Archimedean seminorms $f \mapsto \|f\|_p$, $p \in \Gamma$. The topology β_0, which was defined in [7], is the locally F-convex topology generated by the seminorms $p_\varphi(f) = \|\varphi f\|_p$ where $p \in \Gamma$ and φ is a bounded function from X to F which vanishes at infinity.

Let S(X) be the algebra of all clopen subsets of X. We will denote by M(X,E') (see [6]) the space of all finitely-additive E'-valued measures m on S(X) for which the set m(S(X)) is an equicontinuous subset of the dual space E' of E. For each $m \in M(X,E')$ there exists $p \in \Gamma$ such that $m_p(X) < \infty$ where, for $A \in S(X)$.

$$m_p(A) = \sup\{|m(B)s| : B \subset A,\ B \in S(X), p(s) \leq 1\}.$$

As it is shown in [6], we have $m_p(A \cup B) = \max\{m_p(A), m_p(B)\}$. We will denote by M(X,F) the space of all bounded finitely-additive F-valued measures on S(X). If $m \in M(X,E')$, then, for each $s \in E$, the set function $ms : S(X) \to E$, $(ms)(A) = m(A)s$, is in M(X,F).

For a decreasing sequence (A_n) (resp. net (A_α)), of clopen subsets of X, we will write $A_n \downarrow \emptyset$ (resp. $A_\alpha \downarrow \emptyset$) if $\bigcap A_n = \emptyset$ (resp. $\bigcap A_\alpha = \emptyset$). An element μ of M(X,F) is called σ-additive (resp. τ-additive) if for each sequence $G_n \downarrow \emptyset$ (resp. net $G_\alpha \downarrow \emptyset$) of clopen subsets of X we have $\lim \mu(G_n) = 0$ (resp. $\lim \mu(G_\alpha) = 0$). A member m of M(X,E') is called σ-additive (resp. τ-additive) if each ms, $s \in E$, is σ-additive (resp. τ-additive). We will denote by $M_\sigma(X,E')$ and $M_\tau(X,E')$ the spaces of all σ-additive and all τ-additive members of M(X,E'), respectively. For an $m \in M(X,F)$, we define $|m|$ on S(X) by

$$|m|(A) = \sup\{|m(B)| : B \in S(X),\ B \subset A\}.$$

Let now $m \in M(X,E')$ and $A \in S(X)$, $A \neq \emptyset$. Consider the family Ω_A of all $a = \{A_1, \dots, A_n; x_1, \dots, x_n\}$, where $A_1, \dots, A_n$ is a clopen partition of A and $x_i \in A_i$. The set Ω_A becomes directed by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the partition in α_2. If f is an E-valued function on X and $\alpha = \{A_1, \dots, A_n; x_1, \dots, x_n\} \in \Omega_A$, we define $\omega_\alpha(f,m) = \sum_{i=1}^{n} m(A_i) f(x_i)$. If the $\lim_{\alpha \in \Omega_A} \omega_\alpha(f,m)$ exists, then we say that f is m-integrable over A and we denote this limit by $\int_A f\,dm$. For $A=\emptyset$, we define $\int_A f\,dm=0$. We will write simply $\int f\,dm$ for $\int_X f\,dm$. It is shown in [6] that every $f\in C_{cr}(X,E)$ is m-integrable over each $A\in S(X)$. The function $T_m : C_{rc}(X,E)\to F$, $T_m(f)=\int f\,dm$, is linear and τ_u-continuous. Moreover, the mapping $T : M(X,E')\to (C_{rc}(X,E),\tau_u)'$, $T(m)=T_m$, is linear one-to-one and onto. Hence, we may identify $M(X,E')$ with the dual space of $(C_{rc}(X,E),\tau_u)$.

Finally, we recall that a subset A of a vector space over F is called F-absolutely convex (or simply absolutely convex) if $\gamma A+\delta A\subset A$ for all γ, $\delta\in F$ with $|\gamma|$, $|\delta|\leq 1$.

3. THE STRICT TOPOLOGIES $\beta,\beta',\beta_1,\beta_1'$.

Before defining the topologies $\beta,\beta'\beta_1,\beta_1'$, we prove the following

LEMMA 3.1. If f_1,f_2 are continuous F-valued functions on X, then there exists a continuous F-valued function f in X with $|f(x)|=\max\{|f_1(x)|,|f_2(x)|\}$ for all $x\in X$.

PROOF. For each positive real number r, the set $\{s\in F : |s|=r\}$ is open in F. Hence, the set

$$A_1=\{x\in X : |f_1(x)|=|f_2(x)|\neq 0\}$$

is open in X. Also open is the set

$$A_2=\{x\in X : |f_1(x)|\neq |f_2(x)|\}.$$

Define f on X by

$$\begin{aligned} f(x) &= f_1(x) \text{ if } x\in A_1\\ &= f_1(x)+f_2(x) \text{ if } x\notin A_1. \end{aligned}$$

It is easy to see that $|f(x)|=\max\{|f_1(x)|,|f_2(x)|\}$ for all $x\in X$. Also, f is continuous. In fact f is clearly continuous at each point of the open set $A_1\cup A_2$. Suppose now that $f_1(x)=f_2(x)=0$. Given $\varepsilon>0$, there exists a neighborhood V of x such that $|f_i(y)|<\varepsilon$ for all $y\in V$, $i=1,2$. If $y\in V$, then $|f(y)-f(x)|=|f(y)|<\varepsilon$ which proves that f is continuous at x. Thus f satisfies the requirements.

Let now Ω_1 (respectively Ω) denote the family of all F-zero (resp. compact) subsets of β_0X which are disjoint from X. For $A\in\Omega$, let C_A denote the family of all $h\in C_{rc}(X,F)$ such that $\hat{h}|A=0$. For each $p\in\Gamma$, let $\beta_{A,p}$ denote the locally F-convex topology on $C_b(X,E)$ generated by the family of non-Archimedean seminorms $\{p_\varphi : \varphi\in C_A\}$, where $p_\varphi(f)=\|\varphi f\|_p$. The locally F-convex topology β_A is defined by the family of seminorms $\{p_\varphi : p\in\Gamma,\varphi\in C_A\}$. The topology β_p (resp. $\beta_{1,p}$) is the locally F-convex inductive limit of the topologies $\beta_{A,p}$ $A\in\Omega$ (resp. $A\in\Omega_1$). The locally F-convex projective limit of the topologies β_p (resp. $\beta_{1,p}$), $p\in\Gamma$, is denoted by β' (resp. β_1'). Since $p\geq q$ implies $\beta_p\geqq\beta_q$, we have $\beta'=\bigcup_{p\in\Gamma}\beta_p$. Analoqously we have $\beta_1'=\bigcup_{p\in\Gamma}\beta_{1,p}$.

We define β to be the locally F-convex inductive limit of the topologies β_A, $A\in\Omega$. Thus β has as a base at zero the family of all F-absolutely convex subsets of $C_b(X,E)$ which are β_A-neighborhoods of zero for each $A\in\Omega$. Analogously, β_1 is the locally F-convex inductive limit of the topologies β_A, $A\in\Omega_1$.

We have the following easily established

LEMMA 3.2. 1) The topologies β,β',β_1, and β_1' are Hausdorff.

2) $\beta' \leqq \beta_1' \leqq \beta_1 \leqq \tau_u$ and $\beta' \leqq \beta \leqq \beta_1$.

LEMMA 3.3. Let $H \in \Omega$ and $p \in \Gamma$. If (A_n) is a sequence of clopen subsets of X such that the closure $\overline{A}_n$ in $\beta_0 X$, of each A_n, is disjoint from H and if $0 < \alpha_n \to \infty$, then the set

$$W_p(A_n,\alpha_n) = \bigcap_{n=1}^{\infty} \{f \in C_b(X,E) : \|f\|_{A_n,p} \leqq \alpha_n\}$$

is a $\beta_{H,p}$-neighborhood of zero.

PROOF. We may assume that $A_n \subset A_{n+1}$ for each n. Set $\gamma_n = \inf_{k \geqq n} \alpha_k$. Then, $0 < \gamma_n \to \infty$ and $\gamma_n \leqq \gamma_{n+1}$. Clearly $W_p(A_n,\gamma_n) \subset W_p(A_n,\alpha_n)$. Let $\lambda \in F$, $|\lambda| > 1$. For each positive integer n, there exists an integer m such that $|\lambda|^m \leqq \gamma_n < |\lambda|^{m+1}$. Take $\lambda_n = \lambda^m$ (if $m = 0$, we take $\lambda^0 = 1$). Since $|\lambda_n| > \gamma_n \cdot |\lambda|^{-1}$, we have $|\lambda_n| \to \infty$. Also, $|\lambda_n| \leq |\lambda_{n+1}|$ and $|\lambda_n| \leq \gamma_n$. Moreover

$$W_p(A_n,|\lambda_n|) \subset W_p(A_n,\gamma_n).$$

Each $\overline{A}_n$ is clopen in $\beta_0 X$. Define h on $\beta_0 X$ by

$$\begin{aligned} h(x) &= \lambda_1^{-1} \text{ if } x \in \overline{A}_1 \\ &= \lambda_n^{-1} \text{ if } x \in \overline{A}_n - \overline{A}_{n-1},\ n \geqq 2 \\ &= 0 \quad \text{if } x \notin \bigcup_{n=1}^{\infty} \overline{A}_n. \end{aligned}$$

Then h is continuous. In fact, h is clearly continuous on the open set $B = \bigcup_{n=1}^{\infty} \overline{A}_n$. Let $x_0 \notin B$ and $\varepsilon > 0$. Choose n such that $|\lambda_n| > 1/\varepsilon$. The set $V_n = \beta_0 X - \overline{A}_n$ is a neighborhood of x_0. Moreover, for $x \in V_n$, we have $|h(x) - h(x_0)| = |h(x)| \leqq |\lambda_n|^{-1} < \varepsilon$. So h is continuous at x_0. Also, $h = 0$ on H. Set $\varphi = h|X$. We will show that

$$W_1 = \{f \in C_b(X,E) : \|\varphi f\|_p \leqq 1\} \subset W_p(A_n,|\lambda_n|).$$

In fact, let $f \in W_1$ and $x \in A_n$. If $x \in A_1$, then $\varphi(x) = \lambda_1^{-1}$ and so $p(f(x)) \leqq |\lambda_1| \leqq |\lambda_n|$. If $x \notin A_1$, then $x \in A_k - A_{k-1}$ for some $k \leqq n$ and so $\varphi(x) = \lambda_k^{-1}$ which implies that $p(f(x)) \leqq |\lambda_k| \leqq |\lambda_n|$. Thus $\|f\|_{A_n,p} \leqq |\lambda_n|$ for all n and this completes the proof.

THEOREM 3.4. $\beta_{H,p}$ has a base at zero the family of all sets of the form $W_p(A_n,|\lambda_n|)$ where (A_n) is an increasing sequence of clopen subsets of X, such that the closure $\overline{A}_n$ in $\beta_0 X$ is disjoint from H, $\lambda_n \in F$ with $|\lambda_n| \leqq |\lambda_{n+1}|$ and $0 < |\lambda_n| \to \infty$.

PROOF. Using Lemma 3.1, we get that $\beta_{H,p}$ has as a base at zero the family of all sets of the form $W_{p,h} = \{f \in C_b(X,E) : \|hf\|_p \leqq 1\}$ where $h \in C_H$. Let now $h \in C_H$ and $W = W_{p,h}$. Let $\lambda \in F$ with $|\lambda| > 1$, $\|h\|$. Set

$$A_n = \{x \in X : |h(x)| \geqq |\lambda|^{-n}\}.$$

Clearly A_n is clopen, $A_n \subset A_{n+1}$ and $\overline{A}_n \subset \{x \in \beta_0 X : |\hat{h}(x)| \geqq |\lambda|^{-n}\}$ and so A_n is disjoint from H. Set $W_1 = W_p(A_n,|\lambda_n|)$ where $\lambda_1 = \lambda^{-1}$ and $\lambda_n = \lambda^{n-1}$ if $n \geqq 2$. We will show that $W_1 \subset W$. In fact, let $f \in W_1$. We need to show that

$$(*) \qquad p(h(x)f(x)) \leqq 1$$

for all $x \in X$. Clearly (*) holds if $x \notin \bigcup_{n=1}^{\infty} A_n$. If $x \in A_1$, then $p(f(x)) \leqq |\lambda|^{-1}$ and so (*)

holds since $|h(x)| < |\lambda|$. Finally, if $x \in A_m - A_{m-1}$, $m > 1$, then $|h(x)| < |\lambda|^{-m+1}$ and $p(f(x)) \leq |\lambda|^{m-1}$ which implies that $p(h(x)f(x)) \leq 1$. This proves that $W_1 \subset W$. This and the preceding Lemma complete the proof.

THEOREM 3.5. (i) $\beta_0 \leq \beta' \leq \beta$.

(ii) The topologies $\beta, \beta', \beta_1, \beta_1'$ and τ_u have the same bounded sets.

(iii) If X is locally compact, then $\beta_0 = \beta' = \beta$.

PROOF. Let W be an absolutely convex β_0-neighborhood of zero. There exist $p \in \Gamma$ and a bounded F-valued function h on X vanishing at infinity such that $W_1 = \{f \in C_b(X,E) : \|hf\|_p \leq 1\} \subset W$, By [7, Proposition 2.6], there exist a sequence (K_n) of compact subsets of X and $0 < \alpha_n \to \infty$ such that $W_p(K_n, \alpha_n) \subset W_1$. Let $H \in \Omega$. For each n, there exists a clopen subset B_n of $\beta_0 X$ containing K_n and disjoint from H. If $A_n = B_n \cap X$, then $W_p(A_n, \alpha_n)$ is a $\beta_{H,p}$-neighborhood of zero contained in W_1. Thus W is a $\beta_{H,p}$-neighborhood of zero for every $H \in \Omega$ and hence W is a β'-neighborhood of zero.

(ii) It follows from (i) and from Lemma 2.2, since β_0 and τ_u have the same bounded sets (see [7, Proposition 2.11]).

(iii) If X is locally compact, then the set $H = \beta_0 X - X$ is closed in $\beta_0 X$ (see [14, XI, 8.3]). If V is a β-neighborhood of zero, then V is a β_H-neighborhood of zero and hence a β_0-neighborhood of zero since every member of C_H vanishes at infinity. Thus $\beta \leq \beta_0$ and so $\beta = \beta_0$.

4. THE STRICT TOPOLOGIES ON $C_{rc}(X,E)$.

Throughout the rest of the paper, we will consider the strict topologies on the subspace $C_{rc}(X,E)$ of $C_b(X,E)$.

THEOREM 4.1. Let $p \in \Gamma$, $H \in \Omega$ and V an absolutely convex subset of $C_{rc}(X,E)$. Then V is a $\beta_{H,p}$-neighborhood of zero iff the following condition is satisfied: For each $d > 0$ there exist a clopen subset A of X, whose closure $\overline{A}$ in $\beta_0 X$ is disjoint from H, and $\delta > 0$ such that

$$\{f \in C_{rc}(X,E) : \|f\|_p \leq d, \ \|f\|_{A,p} \leq \delta\} \subset V.$$

PROOF. Suppose that V is a $\beta_{H,p}$-neighborhood of zero. By Theorem 3.4, there exist an increasing sequence (A_n) of clopen subsets of X, with $\overline{A}_n \cap H = \emptyset$, and $\lambda_n \in F$ with $0 < |\lambda_n| \to \infty$, $|\lambda_n| \leq |\lambda_{n+1}|$, such that $W_p(A_n, |\lambda_n|) \subset V$. Let now $d > 0$ and choose n such that $|\lambda_k| > d$ if $k > n$. Take $\delta = |\lambda_1|$ and $A = A_n$. It is clear that if $f \in C_{rc}(X,E)$ is such that $\|f\|_p \leq d$ and $\|f\|_{A,p} \leq \delta$, then $f \in W_p(A_n, |\lambda_n|) \subset V$.

Conversely, assume that the condition is satisfied. Let $\lambda \in F$, $|\lambda| > 1$. Choose an increasing sequence (A_n) of clopen sets, with $\overline{A}_n \cap H = \emptyset$, and a decreasing sequence (δ_n) of positive numbers such that $U_n \cap \lambda^n U \subset V$, where

$$U_n = \{f \in C_{rc}(X,E) : \|f\|_{A_n,p} \leq \delta_n\}, \ U = \{f : \|f\|_p \leq 1\}.$$

Set

$$V_1 = U_1 \cap \left[\bigcap_{n=1}^{\infty} (U_{n+1} + \lambda^n U)\right].$$

We will show that $V_1 \subset V$. In fact, let $f \in V_1$. Then $f \in U_1$ and, for each n, $f = g_n + h_n$ with $g_n \in \lambda^n U$, $h_n \in U_{n+1}$. Let N be such that $f \in \lambda^N U$. Set $f_1 = g_1$ and $f_k = g_k - g_{k-1}$ if $k > 1$. We have $f = f_1 + f_2 + \dots + f_N + h_N$. Since $f_1 = g_1 \in U$ and $f_1 = f - h_1 \in U_1 + U_2 \subset U_1 + U_1 \subset U_1$, we have $f_1 \in U_1 \cap \lambda U \subset V$. For $n > 1$, we have $f_n = g_n - g_{n-1} \in \lambda^n U + \lambda^{n-1} U$ and so $f_n \in \lambda^n U$. Also, $f_n = h_{n-1} - h_n \in U_n + U_{n+1} \subset U_n$ and hence $f_n \in U_n \cap \lambda^n U \subset V$. Finally, $h_N = f - g_N \in \lambda^N U + \lambda^N U \subset \lambda^N U$ and $h_N \in U_{N+1} \subset U_N$ and so again $h_N \in V$. It follows that $f \in V$ since V is absolutely convex. Thus $V_1 \subset V$. Let now $\lambda_1 \in F$ with $0 < |\lambda_1| < 1$, δ_1 and $\lambda_n = \lambda^{n-1}$ if $n > 1$. We will finish the proof by showing that

$$V_2 = \bigcap_{n=1}^{\infty} \{f \in C_{rc}(X,E) : \|f\|_{A_n,p} \leq |\lambda_n|\} \subset V_1.$$

So, let $f \in V_2$. Since $\|f\|_{A_1,p} \leq |\lambda_1| < \delta_1$, we have $f \in U_1$. Let m be any positive integer. Since $f(A_{m+1})$ is relatively compact, there are $x_1, \dots, x_n$ in A_{m+1} such that

$$f(A_{m+1}) \subset \bigcup_{i=1}^{n} \{s : p(s - f(x_i)) \leq 1\}$$

and so

$$A_{m+1} \subset \bigcup_{i=1}^{n} G_i = G$$

where $G_i = \{x \in X : p(f(x) - f(x_i)) \leq 1\}$. Clearly G is clopen. Moreover, if $x \in G_i$, then $p(f(x)) \leq \max\{1, p(f(x_i))\} \leq |\lambda|^m$. Let $g = \chi_G \cdot f$, $h = f - g$. Then $h = 0$ on A_{m+1} and so $h \in U_{m+1}$. Also, $\|g\|_p \leq |\lambda|^m$ and so $g \in \lambda^m U$. This proves that $f \in V_1$ and so the result follows by Lemma 3.3.

For $p \in \Gamma$, let u_p denote the locally F-convex topology generated by the non-Archimedean seminorms $f \mapsto \|f\|_p$.

THEOREM 4.2. (i) For $H \in \Omega$, $\beta_{H,p}$ is the finest locally F-convex topology on $C_{rc}(X,E)$ which agrees with $\beta_{H,p}$ on u_p-bounded sets.

(ii) β_p (resp. $\beta_{1,p}$) is the finest locally F-convex topology on $C_{rc}(X,E)$ which agrees with β_p (resp. $\beta_{1,p}$) on u_p-bounded sets.

PROOF. (i) Let τ be a locally F-convex topology on $C_{rc}(X,E)$ which agrees with $\beta_{H,p}$ on u_p-bounded sets and let V be an absolutely convex τ-neighborhood of zero. Given $d > 0$ there exists a $\beta_{H,p}$-neighborhood V_1 of zero such that

$$V_1 \cap \{f \in C_{rc}(X,E) : \|f\|_p \leq d\} \subset V.$$

By Theorem 4.1, there exist a clopen set A in X, whose closure in $\beta_0 X$ is disjoint from H, and $\delta > 0$ such that

$$\{f \in C_{rc}(X,E) : \|f\|_p \leq d,\ \|f\|_{A,p} \leq \delta\} \subset V_1.$$

Thus

$$\{f \in C_{rc}(X,E) : \|f\|_p \leq d,\ \|f\|_{A,p} \leq \delta\} \subset V.$$

This, by the preceding Theorem, implies that V is a $\beta_{H,p}$-neighborhood of zero. Thus $\tau \leq \beta_{H,p}$.

(ii) It follows easily from (i).

5. DUAL SPACES FOR THE STRICT TOPOLOGIES.

Since each of the topologies $\beta,\beta',\beta_1,\beta_1'$ is coarser than τ_u and since (by [6]) $(C_{rc}(X,E),\tau_u)' = M(X,E')$, it follows that the dual space of $C_{rc}(X,E)$ under any one of the topologies $\beta,\beta',\beta_1,\beta_1'$ is a subspace of $M(X,E')$.

THEOREM 5.1. (i) $(C_{rc}(X,E),\beta)' \subset M_\tau(X,E')$.

(ii) $(C_{rc}(X:E),\beta_1)' \subset M_\sigma(X,E')$.

PROOF. (i) Let $m \in M(X,E')$ be in the dual space of $(C_{rc}(X,E),\beta)$ and let $s \in E$. Given $\varepsilon > 0$, the set

$$W = \left\{ f \in C_{rc}(X,E) : \left|\int f\,dm\right| \leqq \varepsilon \right\}$$

is a β-neighborhood of zero. Let now (A_α) be a net of clopen subsets of X with $A_\alpha \downarrow \emptyset$. The closure B_α of A_α in $\beta_0 X$ is clopen and $B_\alpha \downarrow Q \in \Omega$. Since W is a β_Q-neighborhood of zero, there exist $h \in C_Q$ and $p \in \Gamma$ such that

$$W_1 = \{f \in C_{rc}(X,E) : \|hf\|_p \leqq 1\} \subset W.$$

Choose $\delta > 0$ such that $\delta \cdot p(s) \leqq 1$ and set

$$B = \{x \in \beta_0 X : |\hat{h}(x)| \leqq \delta\}.$$

Since $\beta_0 X - B$ is compact,there exists α_0 with $B_{\alpha_0} \subset B$. Let now $\alpha \geqq \alpha_0$. If $f = \chi_{A_\alpha} s$, then $f \in W_1$ and so

$$|m(A\ )s| = \left|\int f\,dm\right| \leqq \varepsilon.$$

This proves that $\lim m(A_\alpha)s = 0$ for every $s \in E$ and so $m \in M_\tau(X,E')$.

(ii) The proof is analogous to that of (i).

THEOREM 5.2. $(C_{rc}(X,F),\beta_1)' = M_\sigma(X,F)$.

PROOF. By the preceding Theorem, it suffices to show that if $m \in M_\sigma(X,F)$, then the mapping $f \mapsto \int f\,dm$ is β_1-continuous on $C_{rc}(X,F)$. So, let $m \in M_\sigma(X,F)$ and set

$$W = \left\{ f \in C_{rc}(X,F) : \left|\int f\,dm\right| \leqq 1 \right\}.$$

Let $Q \in \Omega_1$. There exists a decreasing sequence (B_n) of clopen sets in $\beta_0 X$ with $Q = \bigcap B_n$. Let $A_n = B_n \cap X$. Since $A_n \downarrow \emptyset$, we have $|m|(A_n) \to 0$ (see [6, Theorem 3.2]). Let now $d > 0$ and choose λ,μ in F with $|\lambda| \geqq d$, $|\mu|\cdot|m|(X) \leqq 1$, $\mu \neq 0$. Choose n such that $|m|(A_n) < |\lambda|^{-1}$ and take $A = X - A_n$. Clearly A is clopen and its closure in $\beta_0 X$ is contained in $\beta_0 X - B_n$ and so it is disjoint from Q. Let now $f \in C_{rc}(X,F)$ with $\|f\| \leqq d$ and $\|f\|_A \leqq |\mu|$. Then

$$\left|\int_{A_n} f\,dm\right| \leqq |\lambda|\cdot|m|(A_n) \leqq 1 \quad \text{and} \quad \left|\int_A f\,dm\right| \leqq |\mu|\cdot|m|(A) \leqq 1.$$

Hence

$$\left|\int f\,dm\right| \leqq \max\left\{ \left|\int_{A_n} f\,dm\right|, \left|\int_A f\,dm\right| \right\} \leqq 1$$

which proves that $f \in W$. By Theorem 4.1. W is a β_Q-neighborhood of zero. Since this is true for all $Q \in \Omega_1$ and since W is absolutely convex, it follows that W is a β_1-neighborhood of zero and so $m \in (C_{rc}(X,F),\beta_1)'$.

DEFINITION 5.3. Let $H \subset M(X,E)$. Then, H is called:

(i) uniformly σ-additive iff the following condition is satisfied: If (A_n) is a sequence of clopen sets with $A_n \downarrow \emptyset$, then $m(A_n) \to 0$ uniformly for $m \in H$.

(ii) uniformly τ-additive iff the following condition is satisfied: If (A_α) is a net of clopen subsets of X with $A_\alpha \downarrow \emptyset$, then $m(A_\alpha) \to 0$ uniformly for $m \in H$.

THEOREM 5.4. Let $H \subset M(X,F)$. Then:

(i) H is uniformly τ-additive iff $|m|(A_\alpha) \to 0$ uniformly for $m \in H$ whenever $A_\alpha \downarrow \emptyset$.

(ii) H is uniformly σ-additive iff for each sequence (A_n) of clopen subsets of X with $A_n \downarrow \emptyset$, we have $|m|(A_n) \to 0$ uniformly for $m \in H$.

PROOF. (i) The condition is clearly sufficient. Conversely, assume that H is uniformly τ-additive and let $A_\alpha \downarrow \emptyset$. Suppose, by way of contradiction, that there exists $\varepsilon > 0$ such that $\sup_{m \in H} |m|(A_\alpha) > \varepsilon$ for all α. Let α_0 be fixed and choose $m \in H$ with $|m|(A_{\alpha_0}) > \varepsilon$. There exists $B_0 \in S(X)$ contained in A_{α_0} such that $|m(B_0)| > \varepsilon$. Since $A_\alpha \cap (X-B_0) \downarrow \emptyset$, there exists $\alpha_1 \geqq \alpha_0$ such that $|m(A_{\alpha_1} \cap (X-B_0))| < \varepsilon$. Let $B_1 = B_0 \cup A_{\alpha_1}$. Then, $A_{\alpha_1} \subset B_1 \subset A_{\alpha_0}$. Moreover, since $|m(B_0)| > \varepsilon$ and $|m(A_{\alpha_1} \cap (X-B_0))| < \varepsilon$ and since $m(B_1) = $ $= m(B_0) + m(A_{\alpha_1} \cap (X-B_0))$, we have $|m(B_1)| = |m(B_0)| > \varepsilon$. Thus, for each α there exist $\alpha_1 \geqq \alpha$, $m \in H$ and clopen set B with $A_{\alpha_1} \subset B \subset A_\alpha$ and $|m(B)| > \varepsilon$. Let D denote the set of all $B \in S(X)$ with the following property: There are α_1, α_2, $\alpha_1 \geqq \alpha_2$, and $m \in H$ such that $A_{\alpha_1} \subset B \subset A_{\alpha_2}$ and $|m(B)| > \varepsilon$. For each α there exists (by the first part of the proof) $B \in D$ with $B \subset A_\alpha$. Thus $\bigcap_{B \in D} B = \emptyset$. Also, let $B_1, B_2 \in D$. There are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $A_{\alpha_1} \subset B_1 \cap A_{\alpha_2}$ and $A_{\alpha_3} \subset B_2 \subset A_{\alpha_4}$. Let $\alpha \geqq \alpha_i$, $i = 1,2,3,4$ and choose $B \in D$ with $B \subset A_\alpha$. Then $B \subset B_1 \cap B_2$. Thus D is directed downwards to the empty set. Since, for each $B \in D$ there exists $m \in H$ with $|m(B)| > \varepsilon$, it follows that H is not uniformly τ-additive and this contradiction completes the proof of (i).

(ii) Suppose that there exists a sequence (A_n) of clopen subsets of X and $\varepsilon > 0$ such that $\sup_{m \in H} |m|(A_n) > \varepsilon$ for each n. We will show that for each n there exist $k > n$, $m \in H$ and $A_k \subset B \subset A_n$ with $|m(B)| > \varepsilon$. In fact, there exist $m \in H$ and $B_0 \subset A_n$ with $|m(B_0)| > \varepsilon$. Since $A_k \cap (X-B_0) \downarrow \emptyset$, there exists $k > n$ with $|m(A_n \cap (X-B_0))| < \varepsilon$. Now it suffices to take $B = A_k \cup B_0$. We get now inductively a sequence of indices $1 = n_1 < n_2 < \ldots$, a sequence (B_i) of clopen sets and a sequence (m_i) in H such that $A_{n_{i+1}} \subset B_i \subset A_{n_i}$ and $|m_i(B_i)| > \varepsilon$. Since $B_i \downarrow \emptyset$, H is not uniformly σ-additive. It is clear now that the result of (ii) follows.

COROLLARY 5.5. Let $m \in M(X,F)$. Then m is τ-additive iff $|m|(A_\alpha) \to 0$ whenever $A_\alpha \downarrow \emptyset$.

THEOREM 5.6. Let $H \subset M(X,F)$. Then:

(i) H is an equicontinuous subset of the dual space of $(C_{rc}(X,F),\beta)$ iff H is norm--bounded (i.e. $\sup_{m \in H} |m|(X) < \infty$) and uniformly τ-additive.

(ii) H is an equicontinuous subset of the dual space of $(C_{rc}(X,F),\beta_1)$ iff H is norm bounded and uniformly σ-additive.

PROOF. (i) Suppose that H β-equicontinuous. Then the polar H^0 of H in $C_{rc}(X,F)$ is a β-neighborhood of zero and hence a τ_u-neighborhood of zero. Thus there exists $\lambda \neq 0$ in F such that

$$W = \{f \in C_{rc}(X,F) : \|f\| \leq |\lambda|\} \subset H^0.$$

If now $A \in S(X)$, then $\lambda \chi_A \in W$ and so $|m(A)| \leq |\lambda|^{-1}$ for all $m \in H$. It follows that $|m|(X) \leq |\lambda|^{-1}$ for all $m \in H$ and so H is norm-bounded. Let now $A_\alpha \downarrow \emptyset$. If $B_\alpha = \overline{A}_\alpha$ is the closure of A_α in $\beta_0 X$, then $B_\alpha \downarrow Q \in \Omega$. Since H^0 is a β-neighborhood of zero, there exists $h \in C_Q$ such that

$$W_1 = \{f \in C_{rc}(X,F) : \|hf\| \leq 1\} \subset H^0.$$

Let $\varepsilon > 0$ and choose $\mu \neq 0$ in F with $|\mu| \leq \varepsilon$. The set

$$G = \{x \in \beta_0 X : |\hat{h}(x)| \leq |\mu|\}$$

is clopen and contains Q. Since $B_\alpha \downarrow Q$, there exists α_0 such that $B_{\alpha_0} \subset G$. If $\alpha \geq \alpha_0$, then $\mu^{-1} \cdot \chi_{A_\alpha} \in W_1$ and so $|\mu|^{-1} \cdot |m(A_\alpha)| \leq 1$ for all $m \in H$. Thus $|m(A_\alpha)| \leq |\mu| \leq \varepsilon$ for all $m \in H$ and all $\alpha \geq \alpha_0$. This proves that H is uniformly τ-additive.

Conversely, suppose that H is norm-bounded and uniformly τ-additive. Let $d > 0$. Choose $\lambda \in F$ with $|\lambda| \geq d$ and a non-zero $\gamma \in F$ such that $|\gamma| \cdot |m|(X) \leq 1$ for all $m \in H$. Let $Q \in \Omega$. There exists a decreasing net (B_α) of clopen sets in $\beta_0 X$ with $\bigcap B_\alpha = Q$. If $A_\alpha = B_\alpha \cap X$, then $A_\alpha \downarrow \emptyset$. By hypothesis and by Theorem 5.4 there exists α such that $|m|(A_\alpha) < |\lambda|^{-1}$ for all $m \in H$. Let $D = X - A_\alpha$. Then D is clopen and its closure in $\beta_0 X$ is disjoint from Q. If now $f \in C_{rc}(X,F)$ is such that $\|f\| \leq d$ and $\|f\|_D \leq |\mu|$, then, for all $m \in H$, we have

$$\left|\int_{A_\alpha} f\,dm\right| \leq |\lambda| \cdot |m|(A_\alpha) \leq 1, \quad \left|\int_D f\,dm\right| \leq |\mu| \cdot |m|(D) \leq 1$$

and so $\left|\int f\,dm\right| \leq 1$. It follows that

$$\{f \in C_{rc}(X,F) : \|f\| \leq d, \ \|f\|_D \leq |\mu|\} \subset H^0.$$

By Theorem 4.1, H^0 is a β_Q-neighborhood of zero for all $Q \in \Omega$ and so H^0 is a β-neighborhood of zero which implies that H is β-equicontinuous.

(ii) The proof is analogous to that of (i).

Using the preceding Theorem and Theorem 5.1. we get the following

THEOREM 5.7. $(C_{rc}(X,F),\beta)' = M_\tau(X,F)$.

THEOREM 5.8. Let $H \subset M(X,E')$ and $p \in \Gamma$. The following are equivalent:

(i) H is an equicontinuous subset of the dual space of $(C_{rc}(X,E),\beta_p)$.

(ii) a) $\sup_{m \in H} m_p(X) < \infty$.

b) If $A_\alpha \downarrow \emptyset$, then $m_p(A_\alpha) \to 0$ uniformly for $m \in H$.

(iii) The set $H_p = \{ms : m \in H, p(s) \geq 1\}$ is norm bounded and uniformly τ-additive.

(iv) H_p is an equicontinuous subset of the dual space of $(C_{rc}(X,F),\beta)$.

PROOF. (i $\Longrightarrow$ ii) Since $\beta_p \leq u_p$, H is u_p-equicontinuous and from this follows that $\sup_{m \in H} m_p(X) < \infty$. Let now $A_\alpha \downarrow \emptyset$ and let $B_\alpha = \overline{A}_\alpha$ be the closure of A_α in $\beta_0 X$. Then $B_\alpha \downarrow Q \in \Omega$.

There exists $h \in C_Q$ such that

$$W_1 = \{f \in C_{rc}(X,E) : \|hf\|_p \leqq 1\} \subset H^0.$$

Let $\varepsilon > 0$ and choose $\lambda \neq 0$ in F with $|\lambda| \leqq \varepsilon$. There exists α_0 such that

$$A_{\alpha_0} \subset \{x \in X : |h(x)| \leqq |\lambda|\}.$$

Let now $\alpha \geqq \alpha_0$ and $B \in S(X)$ with $B \subset A_\alpha$. If $p(s) \leqq 1$, then $\lambda^{-1} \cdot \chi_A s$ is in W_1 and so $|m(A)s| \leqq |\lambda| \leqq \varepsilon$ for all $m \in H$. It follows that $m_p(A_\alpha) \leqq \varepsilon$ for all $m \in H$ and all $\alpha \geqq \alpha_0$.

(ii $\Longrightarrow$ i) It follows by an argument analogous to the one used in the proof of Theorem 5.8.

By Theorem 5.8, (iii) is equivalent to (iv). Finally, it is easy to see that (ii) is equivalent to (iii).

The proof of the following Theorem is analogous to the one of the preceding Theorem.

THEOREM 5.9. Let $H \subset M(X,E')$ and $p \in \Gamma$. The following are equivalent:

(i) H is an equicontinuous subset of the dual space of $(C_{rc}(X,E), \beta_{1,p})$.

(ii) a) $\sup_{m \in H} m_p(X) < \infty$.

b) If (A_n) is a sequence in $S(X)$ with $A_n \downarrow \emptyset$, then $m_p(A_n) \to 0$ uniformly for $m \in H$.

(iii) The set $H_p = \{ms : m \in H, p(S) \leqq 1\}$ is norm-bounded and uniformly σ-additive.

(iv) H_p is an equicontinuous subset of the dual space of $(C_{rc}(X,F), \beta_1)$.

For $p \in \Gamma$, let $M'_{\sigma,p}(X,E')$ (resp. $M'_{\tau,p}(X,E')$) be the set of those $m \in M_p(X,E') = \{\mu \in M(X,E') : \mu_p(X) < \infty\}$ for which for each sequence (A_n) (resp. net (A_α)) of clopen sets with $A_n \downarrow \emptyset$ (resp. $A_\alpha \downarrow \emptyset$) we have $m_p(A_n) \to 0$ (resp. $m_p(A_\alpha) \to 0$). Let

$$M'_\sigma(X,E') = \bigcup_{p \in \Gamma} M'_{\sigma,p}(X,E'), \quad M'_\tau(X,E') = \bigcup_{p \in \Gamma} M'_{\tau,p}(X,E').$$

By Theorem 5.8 and 5.9, we have the following

THEOREM 5.10. $(C_{rc}(X,E), \beta')' = M'_\tau(X,E')$ and $(C_{rc}(X,E), \beta'_1)' = M'_\sigma(X,E')$.

THEOREM 5.11. Suppose that F is spherically complete and that E is a non-Archimedean normed space over F. Then:

(i) If (f_n) is a sequence in $C_{rc}(X,E)$ such that $\|f_n(x)\| \downarrow 0$ for all $x \in X$, then $f_n \xrightarrow{\beta_1} 0$.

(ii) If (f_α) is a net in $C_{rc}(X,E)$ such that $\|f_\alpha(x)\| \downarrow 0$ for all $x \in X$, then $f_\alpha \xrightarrow{\beta} 0$.

PROOF. (i) Let $p = \|\cdot\|$ be the non-Archimedean norm of E and let W be a β_1-closed absolutely convex β_1-neighborhood of zero. The polar $H = W^0$ of W, in the dual space of $(C_{rc}(X,E), \beta_1)$, is β_1-equicontinuous. Let $\alpha \in F$, $|\alpha| > 1$. By [13, Theorems 4.14 and 4.15], we have $H^0 \subset \alpha \cdot W$. Choose $\gamma, \delta \neq 0$ in F such that $|\gamma| \geqq \|f_1\|$, $|\gamma\delta| \leqq |\alpha|^{-1}$ and $|\delta| \cdot m_p(X) \leqq |\alpha|^{-1}$ for all $m \in H$. Let

$$A_n = \{x \in X : \|f_n(x)\| \geqq |\delta|\}.$$

Then, $A_n \downarrow \emptyset$ and so, by Theorem 5.9, there exists n_0 such that $m_p(A_n) < |\delta|$ for all $m \in H$ and all $n \geqq n_0$. Let now $n \geqq n_0$. For all $m \in H$, we have

$$\left|\int_{A_n} f_n dm\right| \leq |\gamma|\cdot m_p(A_n) \leq |\alpha|^{-1}$$

and

$$\left|\int_{X-A_n} f_n dm\right| \leq |\delta|\cdot m_p(X) \leq |\alpha|^{-1}.$$

Thus, $\left|\int \alpha f_n dm\right| \leq 1$ for all $m \in H$ which implies that $\alpha f_n \in H^0 \subset \alpha W$ and so $f_n \in W$.

(ii) The proof is analogous to that of (i).

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