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ON RANK 5 PROJECTIVE PLANES

OTTO BACHMANN

Département de mathématiques
Ecole polytechnique fédérale
CH-1015 Lausanne, Suisse

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ABSTRACT. In this paper we continue the study of projective planes which admit collineation groups of low rank (Kallaher [1] and Bachmann [2,3]). A rank 5 collineation group of a projective plane $\mathbb{P}$ of order $n \neq 3$ is proved to be flag-transitive. As in the rank 3 and rank 4 case this implies that $\mathbb{P}$ is not desarguesian and that n is (a prime power) of the form m^4 if m is odd and $n = m^2$ with $m \equiv 0 \bmod 4$ if n is even. Our proof relies on the classification of all doubly transitive groups of finite degree (which follows from the classification of all finite simple groups).

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1. INTRODUCTION.

All known finite projective planes with a transitive collineation group G are desarguesian. It has been conjectured that all such planes are desarguesian. Under additonal assumptions this has been proved: If G is 2-transitive, i.e. if G has rank 2, then the plane is desarguesian (Theorem of Ostrom and Wagner). If G has rank 3 then (Kallaher [1] and Bachmann [2]) the order of the plane is either 2 or an odd fourth power; moreover, if $n > 2$, the plane is non-desarguesian and G is non-solvable and flag-transitive. If G has rank 4 then (Bachmann [3]) the same conclusions hold for G; the plane is always non-desarguesian and its order is either an odd fourth power or an even square divisible by 16.

Probably the only rank 3 plane is the plane of order 2 and there is no rank 4 plane.

In this paper we will investigate rank 5 planes. The main difficulty consists in showing that, with one exception, G is flag-transitive (see &3).

THEOREM 1. Let $\mathbb{P}$ be a projective plane of finite order n with a rank 5 collineation group G. If $n \neq 3$, then G is flag-transitive.

The desarguesian plane $\mathbb{P}(3) = (P, L)$ of order 3 has a rank 5 collineation group G which is not flag-transitive:

Let $P = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$,

$L = \{\{0,1,3,9\},\{0,4,7,5\},\{0,8,12,2\}\} \cup \{\{0,6,10,11\}\} \cup \{\{1,2,4,10\},\{4,9,8,11\},\{8,5,1,6\}\} \cup \{\{1,7,11,12\},\{4,12,6,3\},\{8,3,10,7\}\} \cup \{\{2,3,5,11\},\{9,7,2,6\},\{5,12,9,10\}\}$,

$G = \langle\alpha,\beta\rangle$ where $\alpha = (0\ 1\ 2\ \dots\ 12)$, $\beta = (1\ 4\ 8)(2\ 9\ 5)(3\ 7\ 12)(6\ 10\ 11)$.

Then $|G| = 39$, $\langle\alpha\rangle \triangleleft G$, $G_0 = \langle\beta\rangle$; G is solvable and not flag-transitive and acts as a Frobenius group on P.

Obviously, $\mathbb{P}(3)$ admits no rank 5 collineation group which is flag-transitive.

As in the rank 3 and rank 4 case one deduces from Theorem 1 the following theorem (see &4).

THEOREM 2. Let $\mathbb{P}$ be a projective plane of finite order $n \neq 3$ with a rank 5 collineation group G. Then

a) G is non-solvable,

b) $\mathbb{P}$ is not desarguesian,

c) n is a power of a prime, $n = m^4$ if n is odd and $n = m^2$ with $m \equiv 0 \bmod 4$ if n is even.

Our proof of Theorem 1 strongly relies on the fact (following from the classification of all finite simple groups) that the doubly transitive groups of finite degree are of known type (Cameron [4], p. 8 and 9). We also make use of the classification of all subgroups of GL(n,p) which are transitive on $V(n,p)\setminus\{0\}$ (Hering [5]; Huppert and Blackburn [6], p. 386).

2. DEFINITIONS AND PRELIMINARY RESULTS.

We shall in general use standard notation. A point (resp. line) will be identified with the set of lines (points) on it. We shall frequently use the following results (Dembowski [7]):

A collineation group of a projective plane has equally many point orbits and line orbits. The point and line ranks of a transitive collineation group of a projective plane are equal. If a transitive collineation group G of a projective plane $\mathbb{P}$ contains a nontrivial central collineation, then $\mathbb{P}$ is desarguesian and G contains all elations of $\mathbb{P}$ and is 2-transitive on the points (and lines) of $\mathbb{P}$. A 2-transitive group has a unique minimal normal subgroup, which is elementary abelian or simple (Burnside [8], p. 202).

The following lemmas will be useful.

LEMMA 1. Let $\mathbb{P} = (P, L)$ be a finite projective plane with a transitive collineation group G and let $P_o \in P$, $l_o \in L$. Then the following holds:

a) $|l_o^{G_{P_o}}| = |P_o^{G_{l_o}}|$

b) If G_{P_o} (G_{l_o}) induces on P_o (l_o) line (point) orbits of length $a_1,\dots,a_r$ ($b_1,\dots,b_s$), then $r = s$ and $a_1,\dots,a_r$ and $b_1,\dots,b_s$ coincide up to order.

PROOF. a) By counting the set $(P_o,l_o)^G$ in two ways we obtain $|P||l_o^{G_{P_o}}| = |(P_o,l_o)^G| = |L||P_o^{G_{l_o}}|$ whence $|l_o^{G_{P_o}}| = |P_o^{G_{l_o}}|$.

b) Let $P_o = l_1^{G_{P_o}} \cup \ldots \cup l_r^{G_{P_o}}$ with $|l_i^{G_{P_o}}| = a_i$ and $l_o = P_1^{G_{l_o}} \cup \ldots \cup P_s^{G_{l_o}}$ with $|P_j^{G_{l_o}}| = b_j$. Then $a_i|P| = |(P_o,l_i)^G|$, $b_j|L| = |(P_j,l_o)^G|$ and b) follows from the fact that, by the counting principle, $\{(P_o,l_i)^G: i = 1,2,\ldots,r\} = \{(P_j,l_o)^G: j = 1,2,\ldots,s\}$.

LEMMA 2. Let $\mathbb{P} = (P,L)$ be a projective plane of finite order n with a rank 5 collineation group G. Then $n \neq 2,4$.

PROOF. Let $P_o \in P$. Assume $n = 2$. Then, for any $P \in P\setminus\{P_o\}$, $G_{P_o,P} = 1$, for otherwise G would contain central collineations and then would be 2-transitive. $|G| = |G_{P_o,P}||P^{G_{P_o}}||P| = 7\ |P^{G_{P_o}}|$ implies that all point orbits of G_{P_o} have length 1, which is impossible.

Assume now $n = 4$. G is not flag-transitive for otherwise G would contain all elations (Higman and Mc Laughlin [9]) and thus would be 2-transitive. It follows that G_{P_o} induces on P_o line orbits of length 1 and 4 or 2 and 3.

Assume at first that $P_o = l_o^{G_{P_o}} \cup l_1^{G_{P_o}}$ where $|l_o^{G_{P_o}}| = 1$ and $|l_1^{G_{P_o}}| = 4$. By Lemma 1, G_{l_o} induces the orbits $\{P_o\}$, $l_o\setminus\{P_o\}$ on l_o. It follows that G_{P_o} induces 3 orbits on $P\setminus l_o$. Hence G_{l_1} leaves invariant two points P_2 and P_3 on $l_1\setminus\{P_o\}$. This implies that $|G_{P_o,l_1}| = 2$ whence $|G_{P_o}| = 8$. Thus G_{P_o} is either a dihedral or a quaternion group. In any case, the fact that G_{P_o} contains a (planar) involution in the center leads immediately to a contradiction.

Now assume that $P_o = l_o^{G_{P_o}} \cup l_2^{G_{P_o}}$, where $l_o^{G_{P_o}} = \{l_o,l_1\}$ and $l_2^{G_{P_o}} = \{l_2,l_3,l_4\}$. By Lemma 1, G_{l_i} induces orbits of length 2 and 3 on l_i $(i = 0,1,2,3,4)$ such that the point P_o lies in the orbits of length 2 (resp. 3) if $i = 0,1$ $(i = 2,3,4)$. Thus the lengths of the point orbits of G_{P_o} are 1,2,6,6,6. This is impossible, since G_{P_o} fixes the line joining the two points in the orbit of length 2.

3. PROOF OF THEOREM 1

Let $\mathbb{P} = (P,L)$ be a projective plane of finite order n with a rank 5 collineation group G and let $P_o \in P$. Assume that G is not flag-transitive. By Lemma 2 and since $\mathbb{P}(3)$ admits no flag-transitive rank 5 collineation group, we have $n \geq 5$. By the result at the beginning of the preceding section about transitive collineation groups with central collineations we may assume throughout that G contains no central collineation.

G_{P_o} defines five point orbits $\mathcal{P}_i = P_i^{G_{P_o}}$ and five line orbits $\mathcal{L}_i = l_i^{G_{P_o}}$ $(i = 0,1,2,3,4)$. G_{P_o} induces on P_o two, three or four line orbits. Thus we are lead to the following cases:

Case I : $P_o = \mathcal{L}_o \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$

Case II : $P_o = \mathcal{L}_o \cup \mathcal{L}_1 \cup \mathcal{L}_2$

Case III: $P_o = \mathcal{L}_o \cup \mathcal{L}_1$.

Theorem 1 will be proved if we can show that none of these cases can occur.

Case I. Since G_{P_o} has four point orbits on $\mathcal{P}\setminus\{P_o\}$, G_{l_o,P_o} is transitive on $l_o\setminus\{P_o\}$. It follows that G_{l_o} has the point orbits $\{P_o\}$ and $l_o\setminus\{P_o\}$ on l_o. This contradicts Lemma 1.

REMARK. In case I the group G is transitive on non-incident point-line pairs. Thus the impossibility of case I also follows from Ostrom [10], where such collineation groups are shown to be 2-transitive.

Case II. As G_{P_o} has four point orbits on $\mathcal{P}\setminus\{P_o\}$, we may assume that it is transitive on $l_o\setminus\{P_o\}$. Therefore G_{l_o} induces the orbits $l_o\setminus\{P_o\}$ and $\{P_o\}$ on l_o, which contradicts Lemma 1.

The main difficulty lies in the proof that case III is impossible.

Case III. It suffices to discuss the following two subcases:

Case III1: $P_1,P_2,P_3 \in l_o$; $P_4 \in l_1$

Case III2: $P_1,P_2 \in l_o$; $P_3,P_4 \in l_1$.

In the following two subsections we will show that the cases III1 and III2 cannot occur.

3.1. CASE III1.

By Lemma 1, G_l induces two point orbits on l, for every line l. G_{l_1} induces the two point orbits $\{P_o\}$ and $l_1\setminus\{P_o\}$ on l_1 whence $G_{l_1} = G_{P_o}$. It follows that $G_{l_o} = G_P$ for some point $P \in l_o$. Clearly $P \neq P_o$. We may assume that $P = P_3$. Then $G_{l_o} = G_{P_3}$ and G_{l_o} acts transitively on $l_o\setminus\{P_3\}$.

Put $s_i = |P_i^{G_{P_o,l_o}}|$ $(i = 1,2)$ and assume that $s_1 \geq s_2$. We have $s_1 + s_2 + 1 = n$.

For $R \in \mathcal{P}$ let l_R denote the (uniquely determined) line for which G_{l_R} fixes the point $R \in l_R$. Put $\bar{\mathcal{L}} = \{l_R\colon R \in l_1\setminus\{P_o\}\}$. Since G_{P_o} is transitive on $\mathcal{P}_i$ and fixes l_1, the symbol $(\bar{\mathcal{L}},\mathcal{P}_i)$, i.e. the number of lines of $\bar{\mathcal{L}}$ through each point of $\mathcal{P}_i$, is well-defined.

LEMMA 3. $(\bar{\mathcal{L}},\mathcal{P}_1) \leq 1$.

PROOF. Suppose that $(\bar{\mathcal{L}},\mathcal{P}_1) \geq 2$. It follows that $\binom{n}{2} = \binom{|\bar{\mathcal{L}}|}{2} \geq |\mathcal{P}_1| = s_1|l_o^{G_P}| = s_1 n$, whence $s_1 = s_2 = (n-1)/2$ and $\binom{n}{2} = s_1 n$.

Thus every point of P_1 is incident with exactly two lines of $\bar{L}$ and any two lines of $\bar{L}$ intersect in a point of P_1. This implies that the action of G_{P_o} on $l_1\setminus\{P_o\}$ is 2-homogeneous. Since this action is also faithful, it follows (Kantor [11]) that G_{P_o} has odd order. So G has odd order and is solvable.
Now we show that G is primitive on the points (see Higman and Mc Laughlin [9], p. 386). Assume that G is imprimitive and denote the number of imprimitive classes by ν. If C is an imprimitive class and $P \in C$, then $l_P \cap C = \{P\}$, since G_P is transitive on $l_P\setminus\{P\}$. Each point of $C\setminus\{P\}$ is on exactly one line of $P\setminus\{l_P\}$ and as G_P is transitive on $P\setminus\{l_P\}$, each line of $P\setminus\{l_P\}$ meets C in $t > 1$ points, where t is a fixed number. So $|C| = n(t-1)+1$ and thus $n^2+n+1 = |P| = \nu|C| = \nu(n(t-1)+1)$. This implies that $n(n+1-\nu(t-1)) = \nu - 1 \geq 1$ whence $n+1-\nu(t-1) \geq 1$ and $n \leq \nu - 1$. This leads to the contradiction $n \leq \nu - 1 < \nu \leq \nu(t-1) \leq n$.
So G is solvable and primitive on the points; it follows (Dembowski [7], p. 212) that n^2+n+1 is a prime and hence that G is a Frobenius group. This implies that $1 = G_{P_o,P_3} = G_{P_o,l_o}$ whence the contradiction $n = 3$.

LEMMA 4. $(\bar{L},P_1) \neq 1$.

PROOF. Suppose that $(\bar{L},P_1) = 1$. Put $\alpha = (\bar{L},P_2)$. Then $|\bar{L}\setminus P_3| = s_1 + \alpha s_2$ whence $s_1 + \alpha s_2 \leq n = s_1 + s_2 + 1$ and thus $\alpha \in \{0,1,2\}$.
If $\alpha = 1$, then each point of $P_1 \cup P_2 \cup P_3$ is contained in exactly one line of $\bar{L}$, which contradicts the fact that the lines of $\bar{L}$ intersect in points of $P_1 \cup P_2 \cup P_3$.
If $\alpha = 2$, then $s_2 = 1$ and $s_1 = n-2$. Counting the set $\{(P,l)\colon P \in P_2, l \in \bar{L}, P \in l\}$ in two ways leads to $(P_2,\bar{L}) = 2$, i.e. each line of $\bar{L}$ contains exactly two points of P_2. Fix now some line $l_S \in \bar{L}$. Each line of $\bar{L}\setminus\{l_S\}$ intersects l_S in a point of P_2. Thus $n - 1 = 2$ which is impossible.
If finally $\alpha = 0$, then $(\bar{L},P_3) = s_2 + 1$. Counting the set $\{(P,l)\colon P \in P_3, l \in \bar{L}, P \in l\}$ in two ways leads to $(P_3,\bar{L}) = s_2 + 1$. Fix some line $l_S \in \bar{L}$. Through each point of $P_3 \cap l_S$ go s_2 lines of $\bar{L}\setminus\{l_S\}$ and this gives all lines of $\bar{L}\setminus\{l_S\}$; hence

$$n - 1 = s_2(s_2+1) \quad \text{and} \quad s_1 = s_2^2 \qquad (*)$$

On the other hand G_{l_o} acts as a rank 3 permutation group on $l_o\setminus\{P_3\}$. From Higman [12] we deduce that $\mu s_1 = s_2(s_2 - \lambda - 1)$ for integers λ and μ. As $\mu = 0$, by (*), G_{l_o} is imprimitive on $l_o\setminus\{P_3\}$. Hence $s_2 + 1 \mid n$, which contradicts (*).

LEMMA 5. $(\bar{L},P_1) \neq 0$.

PROOF. Suppose that $(\bar{L},P_1) = 0$. Then $(P_2,\bar{L}) + (P_3,\bar{L}) = n$. Counting the set

$$\left\{\begin{array}{l}\{(P,l)\colon P \in P_2, l \in \bar{L}, P \in l\}\\ \{(P,l)\colon P \in P_3, l \in \bar{L}, P \in l\}\end{array}\right\} \text{ in two ways gives } \left\{\begin{array}{l}(P_2,\bar{L}) = (\bar{L},P_2)s_2\\ (P_3,\bar{L}) = (\bar{L},P_3)\end{array}\right\}.$$

Fix some line $l_S \in \bar{L}$ and count the set $\{l\colon l \in \bar{L}\setminus\{l_S\}, l \cap l_S \neq S\}$ in two ways:

$(P_2,\bar{L})((\bar{L},P_2) - 1) + (P_3,\bar{L})((\bar{L},P_3) - 1) = n - 1$, whence $(n - (P_3,\bar{L}))((\bar{L},P_2) - 1) = n - (P_3,\bar{L})((P_3,\bar{L}) - 1) - 1$. This implies that either $(P_3,\bar{L}) = 1$, $(\bar{L},P_2) = 2$ or $(\bar{L},P_2) = 1 < (P_3,\bar{L})$, $(P_3,\bar{L})((P_3,\bar{L}) - 1) = n - 1$. In the first case we obtain $(P_2,\bar{L}) = n - 1$ and then $n - 1 = 2s_2$, i.e. $s_1 = s_2 = (n - 1)/2$. Hence we may interchange the roles of P_1 and P_2; we then have $(\bar{L},P_1) = 2$, contrary to Lemma 3. In the second case we obtain $(P_2,\bar{L}) = s_2$, i.e. $(P_3,\bar{L}) = n - s_2 = s_1 + 1$, and then $(s_1 + 1)s_1 = n - 1$. This contradicts $2s_1 \geq n - 1$.

The Lemmas 3, 4 and 5 prove that the case III1 cannot occur.

3.2. CASE III2.

By Lemma 1, G_{l_o} (G_{l_1}) induces two point orbits Γ and Δ (Γ' and Δ') on l_o (l_1). We may assume that $\Gamma = \{P_o\} \cup P_1^{G_{P_o,l_o}}$, $\Delta = P_2^{G_{P_o,l_o}}$, $\Gamma' = \{P_o\} \cup P_3^{G_{P_o,l_1}}$, $\Delta' = P_4^{G_{P_o,l_1}}$. Clearly G_{l_o} (G_{l_1}) is 2-transitive on Γ (Γ'). Let $\gamma \in G$ take l_1 into l_o. If $\Gamma = \Gamma'^{\gamma}$, then there would exist some collineation in G taking the flag (P_o,l_1) into (P_o,l_o). This is impossible; hence $\Delta = \Gamma'^{\gamma}$ and $\Gamma = \Delta'^{\gamma}$. It follows that $|\Gamma| \leq |\Delta|$ or $|\Gamma'| \leq |\Delta'|$. By interchanging the roles of l_o and l_1, if necessary, we may assume for the following that $|\Gamma| \leq |\Delta|$. It also follows that G_{l_o} is 2-transitive on Δ. Moreover we see that $G_{l_o,X}$ is transitive on Δ (Γ) for any $X \in \Gamma$ (Δ).

We may summarize the situation obtained up to now by the following lemma.

LEMMA 6. Let $\mathbb{P}$ be a finite projective plane with a rank 5 collineation group G which is not flag-transitive. Then, for any line l, G_l induces two orbits Γ and Δ on l and is 2-transitive on Γ and Δ such that, for any $X \in \Gamma$ (Δ), $G_{l,X}$ is transitive on Δ (Γ).

REMARK. If $|\Gamma| < |\Delta|$, then the fact that, for any $X \in \Gamma$, $G_{l,X}$ is transitive on Δ also follows from Hilfssatz 1 of Itô [13].

Clearly, the dual of the situation described in the lemma also holds.

We will prove in Lemma 10 that G_{l_o} acts faithfully on Δ. Thus G_{l_o} has a unique minimal normal subgroup which is elementary abelian or simple (Burnside [8], p. 202). If the socle is simple (and not abelian) then it is 2-transitive on Δ with one exception (the group PSL(2,8) of degree 28) (Cameron [4], p. 8 and 9). In the Lemmas 12, 13 and 15 we will exclude the elementary abelian, the 2-transitive and the exceptional case, whereby the case III2 will be shown to be impossible.

LEMMA 7. $|\Gamma| \geq 3$.

PROOF. Clearly $|\Gamma| \geq 2$. Assume that $|\Gamma| = 2$: $\Gamma = \{P_o,P_1\}$. Then, by Lemma 1, $|P_1^{G_{P_o}}| = 2$, i.e. $P_1^{G_{P_o}} = \{P_1,P\}$ for some point $P \notin l_o$. This implies that G_{P_o} fixes the line P_1P. Hence G_{P_o,l_1} fixes the point $l_1 \cap P_1P$. As $|\Delta'| \geq 2$, we then obtain $l_1 \cap P_1P = \{P_3\}$. So $|\Gamma'| = 2$ and $n = 3$, which is impossible.

Hence we may assume in the following that $(|\Delta| \geq) |\Gamma| \geq 3$.

As an immediate consequence of Lemma 7 we have the next lemma.

LEMMA 8. For any point P (line 1), G_P (G_1) fixes no line (point).

LEMMA 9. Let $|\Delta| = p^d$, where p is a prime. Then the following holds:

a) If d is even, then no involution in G fixes Γ pointwise.

b) If $p \mid n$, then ZG_{1_o,P_2} contains no involution.

PROOF. a) Suppose that $\sigma \in G$ is a (planar) involution which fixes Γ pointwise. Then $|\Gamma| \leq \sqrt{n} + 1$ and therefore $n + 1 = |\Gamma| + p^d \leq \sqrt{n} + 1 + p^d$. This implies that $\sqrt{n}(\sqrt{n} - 1) \leq p^d$, whence $\sqrt{n} \leq p^{d/2}$ as d is even. But then $n \leq p^d$, which is impossible.
b) Suppose that $p \mid n$ and that $\sigma \in ZG_{1_o,P_2}$ is a (planar) involution. If σ fixes some point of $\Delta\setminus\{P_2\}$, then σ fixes every point of Δ and no point of Γ, since G_{1_o,P_2} is transitive on $\Delta\setminus\{P_2\}$ and Γ. But then $p^d = \sqrt{n} + 1$, which is impossible. It follows that σ fixes every point of Γ and no point of $\Delta\setminus\{P_2\}$. Hence $\sqrt{n} = |\Gamma| = n + 1 - p^d$, which again is a contradiction.

Let A (resp. B) denote the kernel of the permutation representation induced by G_{1_o} on Γ (Δ). Dually let $\bar{A}$ ($\bar{B}$) denote the kernel of the representation induced by G_{P_o} on $\bar{\Gamma} = 1_o^{G_{P_o}}$ ($\bar{\Delta} = 1_1^{G_{P_o}}$). By Lemma 1 we have $|\Gamma| = |\bar{\Gamma}|$, $|\Delta| = |\bar{\Delta}|$.

LEMMA 10. G_{1_o} acts faithfully on Δ, i.e. $B = 1$.

PROOF. Suppose that $B \neq 1$. Clearly $A \cap B = 1$. If B contains a (planar) involution, then we obtain the contradiction $|\Delta| \geq (n + 1)/2 > \sqrt{n} + 1$. Hence B is of odd order ≥ 3. G_{1_o}/A is (faithful and) 2-transitive on Γ and so has a unique minimal normal subgroup M/A with $A \triangleleft M \trianglelefteq G_{1_o}$. Since AB/A is a normal subgroup of G_{1_o}/A of odd order ≥ 3, it follows that M/A is a solvable normal subgroup of the primitive group G_{1_o}/A and therefore regular, elementary abelian and of odd prime power order p^r.
$1 \triangleleft (M\cap B)A/A \trianglelefteq G_{1_o}/A$ implies that $M\cap B$ is transitive on Γ and we deduce from $(M\cap B)A/A \leq M/A$ and $(M\cap B)\cap A = 1$ that $M\cap B$ is elementary abelian of order p^s with $s \leq r$. It follows that $M\cap B$ is regular on Γ.
Now let $\alpha \in (M\cap B)\setminus 1$. α fixes no element of Γ. Therefore, if the structure $\mathbb{F}(\alpha)$ of elements which are fixed by α is a subplane of $\mathbb{P}$, then its order is $|\Delta| - 1 \geq (n - 1)/2 > \sqrt{n}$, which is impossible. If all the lines of $\mathbb{F}(\alpha)$ go through a point of Δ, then we get a contradiction to the fact that A commutes elementwise with α and is transitive on Δ, as $1 \triangleleft AB/B \trianglelefteq G_{1_o}/B$ and G_{1_o}/B is 2-transitive on Δ. It remains the possibility that $\mathbb{F}(\alpha)$ is not a subplane but contains a point $R \notin 1_o$. Then A leaves R fixed. Moreover $|R^{G_{1_o}}| \neq 1$, by Lemma 8. It follows that A fixes elementwise a subplane $\mathbb{P}' = (P', L')$ of

$\mathbb{P}$ of order $|\Gamma| - 1 = p^r - 1$. G_{1_o} acts as a collineation group on $\mathbb{P}'$. $M \cap B$ is regular on Γ and thus fixes at most one point of $P' \setminus \Gamma$. As $p \nmid (|\Gamma| - 1)^2 = |P' \setminus \Gamma|$, $M \cap B$ fixes exactly one point of $P' \setminus \Gamma$. This point is also left fixed by G_{1_o}, contrary to Lemma 8.

By Lemma 10 G_{1_o} has a unique minimal normal subgroup. Let us denote this subgroup by M.

LEMMA 11. G_{1_o} doesn't act faithfully on Γ, i.e. $A \neq 1$.

PROOF. Suppose that $A = 1$. By Lemma 10 we also have $B = 1$. If the socle M is elementary abelian of order p^r, then M fixes a point $R \notin 1_o$, since $p \nmid n = |\Gamma| + |\Delta| - 1 = 2p^r - 1$. As M doesn't fix any point on 1_o, R is the only point not on 1_o which is fixed by M. Thus R is also left fixed by G_{1_o}, contrary to Lemma 8.

Hence M is not elementary abelian. Then M is simple and (Cameron [4], p. 8 and 9) either 2-transitive on Γ and Δ or isomorphic to PSL(2,8) with $|\Gamma|$ or $|\Delta|$ equal to 28. In the following we will show that actually M cannot be isomorphic to any one of the (non-abelian) simple groups which can occur as socles of 2-transitive groups (see Cameron [4], p. 8 and 9). This will give the contradiction proving Lemma 11.

Assume at first that $|\Gamma| = |\Delta|$. Since G contains involutions but no central collineations $n = 2|\Gamma| - 1$ is a square. This immediately excludes the following possibilities for M:

PSL(2,11) of degree 11, PSL(2,8) of degree 28, A_7 of degree 15, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, HS, Co_3.

Now put $n = (2c + 1)^2$. Then $2c(c + 1) = |\Gamma| - 1$. We conclude that $|\Gamma|$ is odd and $|\Gamma| - 1$ is not a prime power > 4. It follows that M is distinct from PSp(2d,2), PSL(2,q) of degree $q + 1$ $(q > 4)$, $PSU(3,q^2)$ $(q > 2)$, $Sz(q)$ $(q > 2)$, ${}^2G_2(q)$ $(q > 3)$. If $M \cong PSL(3,q)$, $|\Gamma| = (q^3 - 1)/(q - 1) = q^2 + q + 1$, then $2c(c + 1) = q(q + 1)$. This is easily seen to be impossible if $q \neq 3$.

By considering the number of points on 1_o which are fixed by appropriate involutions one can handle the remaining cases A_k of degree $k \geq 5$, PSL(3,3) of degree 13 and PSL(d,q) of degree $(q^d - 1)/(q - 1)$ $(d \geq 4)$:

Suppose that $M \cong A_k$, $|\Gamma| = k$ $(k \geq 5)$. Then M has involutions fixing $k - 4$ points in Γ. Since $k - 4 > \sqrt{2k - 1} + 1 = \sqrt{n} + 1$ if $k \geq 10$, we have $5 \leq k \leq 10$. The fact that $n = 2k - 1$ is a square then implies $k = 5$ and $n = 9$. Since any involution in A_5 (acting on a set of five elements) fixes exactly one element, any involution in M fixes two points in $\Gamma \cup \Delta$. This is impossible.

Now suppose that M is similar to PSL(3,3) in its action on the point or line set of the projective plane $\mathbb{P}(3)$. Then $|\Gamma| = 13$. Since every involution of PSL(3,3) fixes five points and five lines in $\mathbb{P}(3)$, the involutions in M fix $2.5 > 6 = \sqrt{2.13 - 1} + 1 = \sqrt{n} + 1$ elements in $\Gamma \cup \Delta$, which is impossible.

Finally suppose that M is similar to PSL(d,q) ($d \geq 4$), where PSL(d,q) is considered as acting on the set of points or hyperplanes in the projective space $\mathbb{P}(d-1,q)$. There are involutions in PSL(d,q) fixing $(q^{d-1}-1)/(q-1)+1$ (if q is odd) or $(q^{d-1}-1)/(q-1)$ (if q is even) points resp. hyperplanes in $\mathbb{P}(d-1,q)$. Since (for $d \geq 4$) $(q^{d-1}-1)/(q-1) > \sqrt{2(q^d-1)/(q-1)-1}+1 = \sqrt{n}+1$, we get again a contradiction.

Assume now that $|\Gamma| < |\Delta|$. To exclude this case we show that M cannot be isomorphic to a group that appears as the socle of a 2-transitive group which admits 2-transitive permutation representations of different degrees. The fact that $n = |\Gamma| + |\Delta| - 1$ is a square implies that M is not isomorphic to PSL(2,4) (of degree 5 and 6), PSL(2,7) (7,8), PSL(2,9) (6,10), PSL(4,2) (8,15), PSL(2,11) (11,12), A_7 (7,15), M_{11} (11,12), PSp(2d,2) $(2^{d-1}(2^d+1), 2^{d-1}(2^d-1))$ (since $n+1 \not\equiv 2 \bmod 4$ and $|\Gamma| + |\Delta| = 2^{2d} \equiv 0 \bmod 4$). If M is isomorphic to PSL(2,8) of degree 9 and 28 then $n = 36$. Hence any involution of M would fix a subplane of order 6, which is impossible.

This completes the proof of Lemma 11.

LEMMA 12. M is not elementary abelian.

PROOF. Assume that M is elementary abelian of order p^d. As $M \trianglelefteq A \neq 1 = B$, M is regular on Δ, $|\Delta| = p^d$ and M fixes each point of Γ.

Assume at first that $p \nmid n$. Then M fixes equally many points and lines. The lines fixed by M are not concurrent, since $M \trianglelefteq G_{1_o}$ and G_{1_o} is transitive on Γ. Suppose that the lines distinct from 1_o which M leaves fixed all go through a point $R \notin 1_o$. Then G_{1_o} fixes R, contrary to Lemma 8. It follows that the fixed structure $\mathbb{F}(M) = (P', L')$ of M is a subplane of order $|\Gamma| - 1$ and hence $|\Gamma| - 1 = \sqrt{n}$ or $(|\Gamma| - 1)|\Gamma| \leq n - 2$.

If $|\Gamma| - 1 = \sqrt{n}$, then $n = p^d + \sqrt{n}$, a contradiction.

Assume now that $(|\Gamma| - 1)|\Gamma| \leq n - 2$. Then G_{1_o} is transitive on $L' \setminus \{1_o\}$, as it has five line orbits on L. Let's consider the line orbits induced on $P_2 \setminus \{1_o\}$ by G_{1_o,P_2}. The lengths of these orbits are $|\Delta| - 1$ and $|\Gamma|$. On the other hand one of these orbits consists of the lines of P_2 which contain one point of $P' \setminus 1_o$ and hence has length $(|\Gamma| - 1)^2$. Therefore $|\Delta| - 1 = (|\Gamma| - 1)^2$, whence the contradiction $(|\Gamma| - 1)|\Gamma| = |\Gamma| + |\Delta| - 2 = n - 1$.

Now suppose that $p \mid n$. We may assume for the following that $p \neq 2$ for otherwise the involutions of M would fix $|\Gamma| = \sqrt{n} + 1$ points on 1_o, whence the contradiction $n = 2^d + \sqrt{n}$.

$\mathbb{F}(M) = (\Gamma, \{1_o\})$ constitutes the only possibility for $\mathbb{F}(M)$ not excluded by the proof above. To cover this case we use the fact that the action of G_{1_o} on Δ is similar to the action of a subgroup of the affine group A(d,p) on the vector space $\mathbb{V}(d,p)$ (Huppert [14], p. 162). We identify Δ with the set $\mathbb{V}(d,p)$. Then $H \doteq G_{1_o,P_2}$ is a subgroup of

$GL(d,p)$ which is transitive on $\mathbb{V}(d,p)\setminus\{0\}$. Put $A = MW$, where $W \trianglelefteq H$. We have $H/W \cong MH/MW = G_{1_o}/A$. So H/W has a faithful 2-transitive representation on Γ.

Hering [5] has classified all the subgroups of $GL(d,p)$ which are transitive on $\mathbb{V}(d,p)\setminus\{0\}$. We shall show that none of these can occur here (see the list given in Huppert and Blackburn [6], p. 386). For this reason let L be a subfield of $Hom(\mathbb{V},\mathbb{V})$ containing the identity map and maximal with respect to the condition that L is normalized by H and put $|L| = p^e$. Then $\mathbb{V}(d,p)$ can be considered as a vector space $\mathbb{V}(d/e,p^e)$ of dimension d/e over L and we have $H \leq \Gamma L(d/e,p^e)$.

The cases (3),(6),(7) and (9) of the list cannot occur, since $p \neq 2$.

Case (1): $SL(d/e,p^e) \leq H \leq \Gamma L(d/e,p^e)$.

Assume d/e is even. Then there is an involution $\sigma \in SL(d/e,p^e) \cap Z\Gamma L(d/e,p^e)$. Hence $\sigma \in ZH$. This is in conflict with Lemma 9b).

Assume now d/e is odd and $d/e \geq 3$. As $ZSL(d/e,p^e) \trianglelefteq H$, we have $ZSL(d/e,p^e)W/W \trianglelefteq H/W$. Suppose that $ZSL(d/e,p^e)W/W \neq 1$. Then H/W has a cyclic minimal normal subgroup $\langle\alpha W\rangle$, $\alpha \in ZSL(d/e,p^e)$, of prime order $|\Gamma| = q \geq 3$. Every involution in $SL(d/e,p^e)$ fixes elements of Γ, since the number of fixed points in Δ is a power of p and so is inferior to $\sqrt{n} + 1$. It follows that every involution in $SL(d/e,p^e)$ fixes all points of Γ. But then all involutions of $SL(d/e,p^e)$ fix the same number of points in Δ. This implies that $d/e = 3$. But the involutions of $SL(3,p^e)$ leave p^e points fixed. Thus $q + p^{3e} = n + 1$ and $q + p^e = \sqrt{n} + 1$, whence $p^e(p^{2e} - 1) = \sqrt{n}(\sqrt{n} - 1)$. So $\sqrt{n} = p^e n^*$, where $p \nmid n^*$, and then $p^e(n^{*2} - p^e) = n^* - 1$. This leads to $n^* > p^e$, whence $\sqrt{n} > p^{2e}$, which is impossible. This contradiction implies that $ZSL(d/e,p^e)W/W = 1$, i.e. $ZSL(d/e,p^e) \leq W$. Since $PSL(d/e,p^e)$ is simple, we have either $SL(d/e,p^e) \cap W = ZSL(d/e,p^e)$ or $SL(d/e,p^e) \cap W = SL(d/e,p^e)$. In the second case every involution of $SL(d/e,p^e)$ leaves Γ elementwise fixed, whence a contradiction as before. In the first case we deduce from $PSL(d/e,p^e) = SL(d/e,p^e)/(SL(d/e,p^e) \cap W) \cong SL(d/e,p^e)W/W \trianglelefteq H/W$ and Bannai [15] (Theorem 1) that the action of the subgroup $SL(d/e,p^e)W/W$ of H/W on Γ is similar to the natural action of $PSL(d/e,p^e)$ on the set of points or hyperplanes of the projective space $\mathbb{P}((d/e) - 1,p^e)$. Hence $|\Gamma| = (p^d - 1)/(p^e - 1)$ and so $n = |\Gamma| + |\Delta| - 1 = ((p^d - 1)/(p^e - 1)) + p^d - 1 = p^e(p^d - 1)/(p^e - 1)$. Since $SL(d/e,p^e)$ has an involution fixing p^{d-2e} points of Δ, we must have $p^{2(d-2e)} \leq p^e(p^d - 1)/(p^e - 1)$. It follows that $p^e < p^{5e-d} + 1$, i.e. $d/e = 3$. Now consider an involution $\sigma \in SL(3,p^e)$. σ fixes p^e elements in Δ and leaves either $p^e + 2$ or $p^e + p^{e/2} + 1$ elements in Γ invariant, since these are the numbers of points or lines in the projective plane $\mathbb{P}(p^e)$ which are left invariant by any involution in $PSL(3,p^e)$. Thus σ fixes either $2(p^e + 1)$ or $2p^e + p^{e/2} + 1$ elements on 1_o. But this is impossible.

Assume finally that $d/e = 1$. We have $1 \leq H \leq \Gamma L(1,p^e)$ and $H'W/W \trianglelefteq H/W$. $H'W/W \neq 1$, since H/W is not abelian, and H' is cyclic, since $H' \leq \Gamma L'(1,p^e)$. So H/W is solvable and has a cyclic minimal normal subgroup of (odd) prime order q. By Huppert [16] H/W is similar to a subgroup of the semilinear group $\Gamma(q)$ acting on GF(q). In particular, H/W is a Frobenius group. Now consider an involution $\sigma \in H$. By Lemma 9b) we have $\sigma \notin ZH$. This implies that e (and d) are even. Thus $\sigma \notin W$, by Lemma 9a). So σ leaves exactly one point in Γ fixed. But any involution in $\Gamma L(1,p^e)$ fixes at most $p^{e/2}$ elements in $GF(p^e)$. Hence $\sqrt{q + p^e - 1} + 1 = \sqrt{n} + 1 \leq p^{e/2} + 1$, which is absurd.

In the remaining four cases (2),(4),(5) and (8) ZH is easily seen to contain an involution. Thus these cases are excluded by Lemma 9b). This completes the proof of Lemma 12.

LEMMA 13. M is not similar to PSL(2,8) of degree 28.

PROOF. If $M \cong PSL(2,8)$ and $|\Delta| = 28$, then $|\Gamma| \geq 9$, since $n = |\Gamma| + |\Delta| - 1 = |\Gamma| + 27$ must be a square. Moreover, the involutions in M fix all points of Γ, as $M \leq A$. This gives the contradiction $9 \leq |\Gamma| \leq \sqrt{n} + 1 = \sqrt{|\Gamma| + 27} + 1$.

LEMMA 14. $\bar{A} \neq 1 = \bar{B}$.

PROOF. By Lemma 10 and 11 and their duals, either $\bar{A} \neq 1 = \bar{B}$ or $\bar{A} = 1 \neq \bar{B}$. Suppose that $\bar{A} = 1 \neq \bar{B}$. The socle M (resp. $\bar{M}$) of G_{1_o} (G_{P_o}) contains involutions, by Lemma 12 and its dual. As $M \trianglelefteq A$ and $\bar{M} \trianglelefteq \bar{B}$, it follows that $|\Gamma| \leq \sqrt{n} + 1$ and $|\Delta| = |\bar{\Delta}| \leq \sqrt{n} + 1$, whence the contradiction $n = |\Gamma| + |\Delta| - 1 \leq 2\sqrt{n} + 1$.

LEMMA 15. M is not 2-transitive on Δ.

PROOF. Suppose that M is 2-transitive on Δ. $A \neq 1 = B$ and $\bar{A} \neq 1 = \bar{B}$, by Lemma 10,11 and 14. The socles M and $\bar{M}$ of G_{1_o} and G_{P_o} are simple, by Lemma 12 and 13 and their duals, and $M \trianglelefteq G_{1_o,P_o}$ and $\bar{M} \trianglelefteq G_{P_o,1_o}$. So M and $\bar{M}$ are minimal normal subgroups of $G_{P_o,1_o}$. But $G_{P_o,1_o}$ is 2-transitive on Δ, by Lemma 6, and hence has a unique minimal normal subgroup. Therefore $M = \bar{M}$ and M fixes each line of $\bar{\Gamma}$. If we apply the same arguments to any point of Γ, we see that M fixes lines through each point of Γ. It follows that M fixes elementwise a subplane $\mathbb{F}(M) = (P', L')$ of order $|\Gamma| - 1$. Hence $|\Gamma| - 1 = \sqrt{n}$ or $(|\Gamma| - 1)|\Gamma| \leq n - 2$.

Assume that $|\Gamma| - 1 = \sqrt{n}$. Then $\mathbb{F}(M)$ is a Baer subplane and every line of $\mathbb{P}$ contains points of $\mathbb{F}(M)$. This implies that M_{P_2} fixes all lines through P_2; hence $M_{P_2} = 1$ and thus $|\Delta| \leq 2$, contrary to Lemma 7.

Now assume that $(|\Gamma| - 1)|\Gamma| \leq n - 2$. This case can be excluded as in the proof of Lemma 12.

In view of Lemma 10,12,13,15 and the results in Cameron [4], p. 8 and 9, the case III2 cannot occur. This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2.

To prove Theorem 2 we essentially proceed as in the rank 3 case (Kallaher [1]).

Let $\mathbb{P} = (P, L)$ be a projective plane of finite order $n \neq 3$ with a rank 5 collineation group G. G is flag-transitive, by Theorem 1, and $n > 4$, by Lemma 2. By Ott [17] and [18] n is a prime power.

If $\mathbb{P}$ is desarguesian, then (Higman and Mc Laughlin [9]) G contains all elations and so is 2-transitive. This contradiction proves b).

Assume that G is solvable. Since G is primitive on P, $n^2 + n + 1$ must be a prime. Hence G acts as a Frobenius group on P. Fix some flag (P_o, l_o) and let $P_i^{G_{P_o}}$, where $P_i \in l_o$ and $i = 0,1,2,3,4$, denote the point orbits of G_{P_o}. Then $|P_i^{G_{P_o},l_o}| = n/4$, whence $|G| = (n^2 + n + 1)(n + 1)n/4$. Since G acts as a Frobenius group on P, it contains no involutions. So $|G|$ is odd and thus $n = 4$. Hence we have a contradiction. This proves a).

To complete the proof of Theorem 2 assume first that n is odd. Then (Higman and Mc Laughlin [9], Proposition 10) n is a fourth power. Now assume n is even. Then a), b) and the lemma in Keiser [19] imply that $n = m^2$ with $m \equiv 0 \bmod 4$.

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