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THE EXTENDING TOPOLOGIES

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ABSTRACT. Let H be a topological group and a subgroup of G. A topology on G is called an extending topology if and only if it makes G a topological group and it induces the given topology on H. The set of all such topologies is studied.

KEY WORDS AND PHRASES. Topological group, extending topologies, normal subgroup.
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1. INTRODUCTION.

Comfort and Ross demonstrated in [2] that a number of topologies exist on an infinite abelian group that make the group continuous. Sharma in [3] extended this result by showing that nontrivial Hausdorff topologies can be placed on any group that has an infinite center. Implicit in Sharma's work is the fact that every topology which makes the center of a group into a topological group can be used to create a topology which makes the whole group a topological group and which extends the topology on the center.

Let (H,t) be a topological group and suppose that H is a subgroup of G. The purpose of this paper is to consider the collection of topologies on G which make G into a topological group and yet agree on the fixed subgroup H.

2. RESULTS.

DEFINITION. T is called an extending topology for t if and only if (G,T) is a topological group and $\{U \cap H \mid U \in T\} = t$. Let ξ be the set of all such extending topologies.

As an example of this situation consider $\mathbb{R}^2$ with the usual vector addition. Let H be the y-axis and t the usual topology on H. The collection of sets of the form $\{(x,y) \mid y = \alpha x + c,\ r_1 < c < r_2\}$ with α fixed forms a basis for a topology

T_α on $\mathbb{R}^2$. In each case T_α is an extending topology for t. We note that the topology $T_\alpha \cap T_\beta$ fails to extend the topology t whenever $\alpha \neq \beta$, but the topology $T_\alpha \vee T_\beta$ using $T_\alpha \cup T_\beta$ as a subbasis is the usual topology for $\mathbb{R}^2$ whenever $\alpha \neq \beta$. This is not an accident.

THEOREM 1. If $\mathfrak{E}$ is not empty, then $(\mathfrak{E}, \vee)$ is a complete subsemilattice of the complete lattice of group topologies on G.

The sets of the form $\{gU \mid U \in t, g \in G\}$ form a basis for a topology T^* on T. Following the terminology of [1], we will call T^* the translation topology whenever (G, T^*) is a topological group.

THEOREM 2. If $T \in \mathfrak{E}$, then $T \subset T^*$.

PROOF. Let $x \in U$ with $U \in T$. Then $H \cap x^{-1}U = V \in t$. So $xV \in T^*$ and $xV \subset U$.

By [1] we know that for normal H, $\mathfrak{E} \neq \emptyset$ iff $T^* \in \mathfrak{E}$. In this case T^* is the supream of $\mathfrak{E}$.

Returning to our example of $\mathbb{R}^2$ with extending topology T_α, we note that the natural projection from $(\mathbb{R}^2, T_\alpha)$ to $\mathbb{R}^2/H$ yields the indiscrete topology on $\mathbb{R}^2/H$. This happens for every $\alpha \in \mathbb{R}^1$. Yet no two different topologies of the form T_α and T_β are comparable. This also is no accident.

THEOREM 3. Suppose T and T' are in $\mathfrak{E}$ and induce the same topology on G/H. If $T \subset T'$, then $T = T'$.

PROOF. Let $V \in T'$ and $a \in V$. We can find sets $V_1, V_2 \in T'$ with $e \in V_1$, $a \in V_2$ and $V_1V_2 \subset V$. There exists a set $U_1 \in T$ such that $U_1 \cap H = V_1 \cap H$ and a set $U_2 \in T$ such that $a \in U_2$ and $U_2U_2^{-1} \subset U_1$. Since $U_2 \in T'$ and the natural projection $p' : (G, T') \to G/H$ is an open map we have that $p'(U_2 \cap V_2)$ is open in G/H. Let $U = p^{-1}(p'(U_2 \cap V_2) \cap U_2$ where p is the natural projection from (G, T) to G/H. Certainly $a \in U$ and $U \in T$.

Let $x \in U$. Since $p(x) \in p'(U_2 \cap V_2)$ we have that $p(x) = p(y)$ for some $y \in U_2 \cap V_2$. Also $xy^{-1} \in U_1$ since both x and y are in U_2 and $xy^{-1} \in H$ since $xH = yH$. Thus since $y \in V_2$, $(xy^{-1})y \in V$. Hence $T = T'$.

In [1] it was shown that (G, T^*) was homeomorphic to $H \times G/H$ where H is given the topology t and G/H is given the discrete topology. This occurs even when H fails to be an algebraic factor of G. The homeomorphism was given by picking a fixed representation of the form g_*H for each coset in G/H and defining $F : G \to H \times G/H$ by $F(g) = (h, gH)$ where $g_*h = g$ and $g_*H = gH$. Without loss of generality we can choose eH to be the fixed representation of H in G/H. One might hope to find other extensions of t by placing other topologies on G/H which make G/H a topological group and by varying the choice of g_*H representations. In particular we might look for minimal elements in $\mathfrak{E}$ by placing the indiscrete topology on G/H. There are limitations to this approach.

THEOREM 4. Let (H,t) be Hausdorff and $F : (G,T) \to (H \times G/H, t \times \tau)$ be a homeomorphism. Suppose also that either τ is the indiscrete topology or H is totally disconnected and $(G/H, \tau)$ is connected. Then if $T \in \xi$, H is a direct factor of G.

PROOF. Define $f : G \to H$ by $f(g) = g_*^{-1}g$ where $gH = g_*H$. Let $g_1, g_2 \in G$ and let $h_1 = f(g_1)$ and $h_2 = f(g_2)$. Suppose τ is the indiscrete topology. If $h_1h_2 \neq f(g_1g_2) = h$, then we can find a $V \in t$ such that $h \in V$ and $h_1h_2 \notin V$. $U = F^{-1}(V \times G/H)$ is an open set in G with $g_1g_2 \in U$. We can find open sets $U_1, U_2 \in T$ with $g_1 \in U_1$, $g_2 \in U_2$, and $U_1U_2 \subset U$. But $h_1 \in U_1$ and $h_2 \in U_2$ and hence $h_1h_2 \in U$. Since this is a contradiction we must conclude that f is a homomorphism. If $i : H \to G$ is the inclusion map, we have that $id_H = f \circ i$ and hence H is a direct factor of G. In the other case $h_1 \times G/H$, and $h_2 \times G/H$ are connected and so their product must be in the component $f(g_1g_2) \times G/H$. So once again we have that h_1h_2 must be $f(g_1g_2)$.

If H is normal in G and $T \in \xi$, we may use topologies on G/H, finer than the one induced by T, to find other topologies in ξ.

THEOREM 5. Let H be normal in G. Let $T \in \xi$ induce τ on G/H. If τ' is a group topology on G/H which is finer than τ, then there is a unique $T' \in \xi$ which induces τ' and is finer than T.

PROOF. We define the injective group homomorphism $f : G \to (G/H \times G, \tau' \times T)$ by $f(g) = (gH, g)$. Let T' be the topology making f an embedding. Hence (G,T) may be considered as a subtopological group of $(G/H \times G, \tau' \times T)$. Since $f^{-1}(G/H \times U) = U$, $T' \supset T$. Also if $H \cap f^{-1}(V \times U) \neq \emptyset$, $H \cap f^{-1}(V \times U) = H \cap U$ so that T and T' agree on H. Let $p : (G,T) \to (G/H, \tau)$ and $p' : (G,T') \to (G/H, \tau')$ be the natural projections. Then $p'(f^{-1}(V \times U)) = V \cap p(U) \in \tau'$. Hence τ' is induced by T'.

If T'' also induces τ' on G/H then $f : (G,T'') \to (G/H \times G, \tau' \times T)$ is continuous. Thus $T' \subset T''$. If $T'' \in \xi$, then $T' = T''$ by Theorem 3.

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