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ALMOST-PERIODICITY IN LINEAR TOPOLOGICAL SPACES AND APPLICATIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

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ABSTRACT. Let E be a complete locally convex space (l.c.s.) and $f : R \to E$ a continuous function; then f is said to be almost-periodic (a.p.) if, for every neighbourhood (of the origin in E) U, there exists $\ell = \ell(U)>0$ such that every interval $[a,a+\ell]$ of the real line contains at least one point τ such that $f(t+\tau) - f(t) \in U$ for every $t \in R$. We prove in this paper many useful properties of a.p. functions in l.c.s. and give Bochner's criteria in Fréchet spaces.

KEY WORDS AND PHRASES. *Almost-periodic functions, Bochner's criteria, weakly almost-periodic functions, abstract differential equations, perfect Fréchet spaces, infinitesimal generator of equi-continous C_0-group.*
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1. INTRODUCTION

The notion of almost-periodic functions has been introduced by Bohl and Esclangon at the beginning of the century and widely studied by Bochner [1], [2] and many other mathematicians. The reader can see [3], [4], [5], [6], [7],... for what is written on the subject.

A definition of almost-periodic functions on a group and with values in a linear topological space is contained in the important 1935 paper of Bochner and Von Neumann [2]; we consider here the one suggested in [6] which is very easy to handle (see definition 1 below). Most of the results of Part I of this paper are known in Banach spaces. We give their extensions to linear topological spaces.

In Section 5 of our paper, we study almost-periodicity of solutions of some abstract differential equations of the form : $x'(t) = Ax(t) + f(t)$, $-\infty<t<\infty$, in Fréchet spaces.

We suppose the reader is acquainted with elementary properties of linear topological spaces (see for example [8]).

We consider a locally convex space $E = E(\tau)$ over the field $\phi(\phi = R$ or $C)$; its topology τ is generated by a family of continous semi-norms $Q = \{p,q,...\}$.

We assume E is a Hausdorff space. A basis of neighbourhoods (of the origin in E) contains sets of the form $U = U(\varepsilon; p_i, 1 \le i \le n) = \{x \in E;\ p_i(x)<\varepsilon, i=1,..,n\}$ $p_i \in Q$. E is called a Fréchet space if τ is induced by an invariant and complete metric. If

E is a Fréchet space, we may take $Q = \{p_i\}_{i=1}^{\infty}$. A subset $D \subseteq E$ is dense in E if every $x \in E$ is the limit of a generalized sequence of elements of D. A linear operator $A : D(A) \to E$ with domain D(A) dense in E is closed if its graph G(A) is a closed subset of the product space E x E.

THEOREM. (See [9]). Let E be a complete locally convex space. Then the linear operator $A : D(A) \to E$ is closed iff for every generalized sequence (x_μ) in D(A) such that $\lim_\mu x_\mu = x$ and $\lim_\mu Ax_\mu = y$ we have $x \in D(A)$ and $Ax = y$.

COROLLARY. Every continous linear operator defined on all E is closed.

In a locally convex space E, a subset X is called totally bounded if, for every neighbourhood (of the origin) U, there corresponds a finite set Y such that $X \subset Y+U$.

2. ALMOST PERIODIC FUNCTIONS WITH VALUES IN A LOCALLY CONVEX SPACE.

DEFINITION 1. Let E be a complete locally convex space (l.c.s.). A continous function $f : R \to E$ is called almost-periodic (a.p.) if for each neighbourhood (of the origin) U, there exists a real number $\ell = \ell(U)>0$ such that every interval $[a,a+\ell]$ contains at least a point τ such that $f(t+\tau) - f(t) \in U$ for every $t \in R$.

Obviously $\tau = \tau_U$ and we call it a U-translation number of the function f. The following two theorems are known (see [6]). We give here a proof of the second one.

THEOREM 1. (a) If $f : R \to E$ is a.p. then f is uniformly continuous on R.
(b) If $(f_n)_{n=1}^{\infty}$ is a sequence of a.p. functions which converge uniformly on R to a function f, then f is also a.p..

THEOREM 2. If f is a.p., then $\{f(t);\ t \in R\}$ is totally bounded in E.

PROOF. Let U be a given neighbourhood, and V a symmetric neighbourhood such that $V + V \subset U$; let $\ell = \ell(V)$ as in definition 1. By continuity of f, the set $\{f(t);\ t \in [0,\ell]\}$ is compact in E (see [8] proposition 7, p. 53). But in a l.c.s., every compact set is totally bounded (see [8] theorem 5, p. 60) : therefore there exists $x_1,\dots,x_\nu$ such that for every $t \in [0,\ell]$, we have $f(t) \in \bigcup_{j=1}^{\nu} (x_j+V)$.

Take an arbitrary $t \in R$ and consider $\tau \in [-t, -t+\ell]$ a V - translation number of f. Then we have:

$$f(t+\tau) - f(t) \in V. \qquad (2.1)$$

Choose x_k between $x_1,\dots,x_\nu$ such that

$$f(t+\tau) \in x_k + V. \qquad (2.2)$$

Let us write $f(t) - x_k = [f(t) - f(t+\tau)] + [f(t+\tau) - x_k]$. Then by (2.1) and (2.2) we get $f(t) - x_k \in U$ and therefore $f(t) \in x_k + U$; as t is arbitrary we conclude:

$$\{f(t);\ t \in R\} \subset \bigcup_{j=1}^{\nu} (x_j+U).$$

The theorem is proved.

REMARK 1. If E is a Fréchet space, then $\{f(t);\ t \in R\}$ is relatively compact in E if f is a.p.. For in every complete metric space, relative compacity and totally boundedness are equivalent ([13], p.13). We then conclude every sequence $(f(t_n))_{n=1}^{\infty}$ contains a convergent subsequence.

THEOREM 3. Let E be a complete l.c.s. If $f : R \to E$ is a.p. then the functions $\lambda f\ (\lambda \in \Phi)$ and $\bar{f}$ defined by $\bar{f}(t) \equiv f(-t)$ are also a.p..

PROOF. λf is obviously a.p.. Let us consider $\bar{f}$; by almost-periodicity of f, if U is a given neighbourhood, there exists $\ell = \ell(U)$ such that every interval $[a,a+\ell]$

contains τ such that $f(t+\tau) - f(t) \in U$ for every $t \in R$. Put $s = -t$; we get :

$$\bar{f}(s-\tau) - \bar{f}(s) = f(-s+\tau) - f(-s) = f(t+\tau) - f(t).$$

Therefore $\bar{f}(s-\tau) - \bar{f}(s) \in U$ for every $s \in R$, which shows f is a.p. with $-\tau$ as a U-translation number.

3. BOCHNER'S CRITERIA AND OTHER PROPERTIES.

We first give theorem 4 we prove as theorem 6.6 in [6].

THEOREM 4. Let E be a Fréchet space and $f : R \to E$ a.p.; then for every real sequence $(s'_n)_{n=1}^{\infty}$, there exists a subsequence $(s_n)_{n=1}^{\infty}$ such that $(f(t+s_n))_{n=1}^{\infty}$ is uniformly convergent in $t \in R$.

PROOF. Consider the sequence of functions $(f_{s_n})_{n=1}^{\infty}$ corresponding to $(s_n)_{n=1}^{\infty}$ and let $S = (\eta_n)_{n=1}^{\infty}$ be a dense sequence in R. By remark 1, we can extract from $(f(\eta_1+s_n))_{n=1}^{\infty}$ a convergent subsequence, for $\{f(t);\ t \in R\}$ is relatively compact in E.

Let $(f_{s_{1,n}})_{n=1}^{\infty}$ be the subsequence of $(f_{s_n})_{n=1}^{\infty}$ which converges at η_1. We apply the same argument as above to the sequence $(f_{s_{1,n}})_{n=1}^{\infty}$ to choose a subsequence $(f_{s_{2,n}})_{n=1}^{\infty}$ which converges at η_2. We continue the process and consider the diagonal sequence $(f_{s_{n,n}})_{n=1}^{\infty}$ which converges for each η_n in S. Call this last sequence by $(f_{r_n})_{n=1}^{\infty}$. Now we are going to show it is uniformly convergent in R, i.e. for every neighbourhood U, there exists $N = N_U$ such that $f(t+r_n) - f(t+r_m) \in U$ for every $t \in R$ if $n, m > N$.

Consider an arbitrary neighbourhood U and a symmetric neighbourhood V such that $V+V+V+V+V \subset U$. Let $\ell = \ell(V)$ as in definition 1. By uniform continuity of f over R (theorem 1), there exists $\delta = \delta_V > 0$ such that

$$f(t) - f(t') \in V \tag{3.1}$$

for every $t,t' \in R$ with $|t-t'| < \delta$.

We divide the interval $[0,\ell]$ into ν subintervals of length smaller than δ. Then, in each interval, we choose a point of S and get $S_0 = \{\xi_1,\ldots,\xi_\nu\}$. As S_0 is finite, $(f_{r_n})_{n=1}^{\infty}$ is uniformly convergent over S_0; therefore there exists $N = N_V$ such that

$$f(\xi_i+r_n) - f(\xi_i+r_m) \in V \tag{3.2}$$

for every $i = 1,\ldots,\nu$ and if $n, m > N$.

Let $t \in R$ be arbitrary and $\tau \in [-t,-t+\ell]$ such that $f(t+\tau) - f(t) \in V$. Choose ξ_i such that $|t+\tau-\xi_i| < \delta$; then $f(t+\tau+r_n) - f(\xi_i+r_m) \in V$, for every n. Therefore, if $n, m > N$, we get:

$$f(t+r_n) - f(t+r_m) \in U, \tag{3.3}$$

which proves uniform convergence of $(f(t+r_n))_{n=1}^{\infty}$.

To see (3.3) we write :

$$\begin{aligned} f(t+r_n) - f(t+r_m) &= [f(t+r_n) - f(t+r_n+\tau] \\ &+ [f(t+r_n+\tau) - f(\xi_i+r_n)] + [f(\xi_i+r_n) - f(\xi_i+r_m)] \\ &+ [f(\xi_i+r_m) - f(t+r_m+\tau)] \\ &+ [f(t+r_m+\tau) - f(t+r_m)], \end{aligned}$$

and we apply (3.2) or (3.3) to each term in square brackets. The theorem is proved.

We now state and prove Bochner's criteria :

THEOREM 5. Let E be a Fréchet space. Then $f : R \to E$ is a.p. iff for every real sequence $(s'_n)_{n=1}^{\infty}$ there exists a subsequence $(s_n)_{n=1}^{\infty}$ such that $(f(t+s_n))_{n=1}^{\infty}$ converges uniformly in $t \in R$.

PROOF. The condition is obviously necessary by theorem 4; let us show it is sufficient; suppose f is not a.p.; then there exists a neighbourhood U such that for every $\ell > 0$, there exists an interval of length ℓ which contains no U-translation number of f, or :

there exists an interval $[-a,-a+\ell]$ such that for every

$\tau \in [-a,-a+\ell]$ there exists $t = t_\tau$ such that $f(t+\tau) - f(t) \notin U$.

Let us consider $\tau_1 \in R$ and an interval (a_1-b_1) with $b_1-a_1 > 2|\tau_1|$ which contains no U-translation number of f. Now let $\tau_2 = \frac{a_1-b_1}{2}$; then $\tau_2-\tau_1 \in (a_1,b_1)$ and therefore $\tau_2-\tau_1$ cannot be a U-translation number of f. Let us consider another interval (a_2,b_2) with $b_2-a_2 > 2\ (|\tau_1|+|\tau_2|)$, which contains no U-translation number of f. Let $\tau_3 = \frac{a_2-b_2}{2}$; then $\tau_3-\tau_1$, $\tau_3-\tau_2 \in (a_2,b_2)$ and therefore $\tau_3-\tau_1$ and $\tau_3-\tau_2$ cannot be U-translation number of f. We proceed and get a sequence $(\tau_n)_{n=1}^{\infty}$ such that no

$\tau_m-\tau_n$ is a U-translation number of f;

$$f(t+\tau_m-\tau_n) - f(t) \notin U. \tag{3.4}$$

Put $\sigma = \sigma_{mn} = t-\tau_n$; then (3.4) becomes:

$$f(\sigma+\tau_m) - f(\sigma+\tau_n) \notin U. \tag{3.5}$$

Suppose there exists a subsequence $(\tau'_n)_{n=1}^{\infty}$ of $(\tau_n)_{n=1}^{\infty}$ such that $(f(t+\tau'_n))_{n=1}^{\infty}$ converges uniformly in $t \in R$; then for every neighbourhood V, there exists $N = N_V$ such that if m, n > N (we may take m > n), then we have:

$$f(t+\tau'_m) - f(t+\tau'_n) \in V \tag{3.6}$$

for every $t \in R$.

But this contradicts (3.5); it suffices to take U = V and $\sigma_{mn} = t_{mn}$ in (3). Therefore $(f(t+\tau_n))_{n=1}^{\infty}$ does not contain any subsequence which converges uniformly in t. The theorem is proved.

REMARK 2. Here we do not use metrizability of E in the proof of the sufficiency of the condition.

THEOREM 6. Let E be a Fréchet space and consider the functions f, g, f_1, f_2: $R \to E$; then we have:

a) f + g is a.p. in E if f and g are a.p. in E

b) $F = (f_1, f_2)$ is a.p. in the product space E x E if f_1 and f_2 are a.p. in E.

PROOF. It is very easy to prove a) and b) by using Bochner's criteria; we omit it. The reader can see [9].

COROLLARY 1. If f_1 and f_2 are a.p. in the Fréchet space E, then for every neighbourhood U, f_1 and f_2 have common U-translation numbers.

PROOF. Let U be a given neighbourhood in E; by theorem 6, the function f(t) = $(f_1(t), f_2(t))$ is a.p.. Consider now τ a U x U-translation number of f; then $f(t+\tau) - f(t) \in U \times U$, for every $t \in R$ and therefore $f_i(t+\tau) - f_i(t) \in U$, i = 1,2, for every $t \in R$; τ is then a U-translation number of f_1 and f_2.

REMARK 3. Theorem 6, b) and corollary 1 are true even for n functions, $n \geq 2$.

4. WEAKLY A.P. FUNCTIONS; INTEGRATION OF A.P. FUNCTIONS.

Let E be a complete locally convex space.

DEFINITION. A function $f: R \to E$ is called weakly a.p. (we write W.a.p.) in E if the numerical function $(x^*f)(t)$ is a.p. for every $x^* \varepsilon E$ where E^* is the dual space of E.

Obviously every a.p. function is w.a.p.; and if f is w.a.p. then it is weakly continous and weakly bounded.

THEOREM 7. Let E be a complete l.c.s. and f a w.a.p. and continous function; assume $\{F(t);\ t \varepsilon R\}$ is weakly bounded, where $F(t) = \int_0^t f(\sigma)d\sigma$; then F(t) is w.a.p..

PROOF. We first note existence of the integral because of continuity of f over R. Take any $x^* \varepsilon E^*$; then $(x^*f)(t)$ is a.p.. By continuity of x^*, we have $(x^*F)(t) = \int_0^t (x^*f)(\sigma)d\sigma$, which is bounded by our assumption. Now $(x^*F)(t)$ is a.p. (see [6], theorem 6.20). The theorem is proved.

THEOREM 8. Let E be a Fréchet space and $f: R \to E$ a given function; then f is a.p. if f is w.a.p. and $\{f(t):\ t \varepsilon R\}$ is relatively compact in E.

PROOF. The condition is obviously necessary. Let us show it is sufficient by contradiction. Suppose there exists t_o such that f is discontinuous at t_o. Then we can find a neighbourhood U and two sequences $(s'_{n_1})_{n=1}^{\infty}$ and $(s'_{n_2})_{n=1}^{\infty}$ such that $\lim_{n\to\infty} s'_{n_1} = 0 = \lim_{n\to\infty} s'_{n_2}$ and

$$f(t_0+s'_{n_1}) - f(t_0+s'_{n_2}) \notin U \tag{4.1}$$

for every $n \varepsilon N$. By relative compacity of $\{f(t);\ t \varepsilon R\}$ we can extract $(s_{n_1})_{n=1}^{\infty}$ and $(s_{n_2})_{n=1}^{\infty}$ from the respective first two sequences such that $\lim_{n\to\infty} f(t_0+s_{n_1}) = a_1 \varepsilon E$ and $\lim_{n\to\infty} f(t_0+s_{n_2}) = a_2 \varepsilon E$. Consequently, using (4.1), we get $a_1 - a_2 \varepsilon E$ and therefore by the Hahn-Banach theorem ([13], corollary 1, p. 108), there exists $x^* \varepsilon E^*$ such that $x^*(a_1-a_2) \neq 0$; hence

$$x^*(a_1) \neq x^*(a_2). \tag{4.2}$$

By continuity of x^*, we have:

$$x^*(a_1) = \lim_{n\to\infty} x^*f(t_0+s_{n_1}) = \lim_{n\to\infty} x^*f(t_0+s_{n_2}) = x^*(a_2)$$

which contradicts (4.2); f is therefore continuous over R.

We are now going to show almost-periodicity of f; but first of all, we state and prove:

LEMMA 1. Let E be a Fréchet space and $\Phi: R \to E$ be a.p.. Let $(s_n)_{n=1}^{\infty}$ be a real sequence such that $\lim_{n\to\infty} \Phi(s_n+\eta_k)$ exists for each $k = 1,2,\ldots$, where $(\eta_k)_{k=1}^{\infty}$ is dense in R. Then $(\Phi(t+s_n))_{n=1}^{\infty}$ is uniformly convergent in $t \varepsilon R$.

PROOF. Suppose by contradiction $(\Phi(t+s_n))_{n=1}^{\infty}$ is not uniformly convergent in t; then there exists a neighbourhood U such that for every $N = 1,2,\ldots$, there exists n_N, m_N N and $t_N \varepsilon R$ such that:

$$\Phi(t_N+s_{n_N}) - \Phi(t_N+s_{m_N}) \notin U. \tag{4.3}$$

By Bochner's criteria we can extract two subsequences $(s'_{n_N}) \subset (s_{n_N})$ and $(s'_{m_N}) \subset (s_{m_N})$ such that

$$\lim_{N\to\infty} \Phi(t+s'_{n_N}) = g_1(t) \text{ uniformly in } t \in R,$$

$$\lim_{N\to\infty} (t+s'_{m_N}) = g_2(t) \text{ uniformly in } t \in R.$$

Let V be a symmetric neighbourhood with $V+V+V \subset U$. Then there exists $N_0 = N_{0_V}$ such that if $N > N_0$,

$$\Phi(t_N+s'_{n_N}) - g_1(t_N) \in V,$$

$$\Phi(t_N+s'_{m_N}) - g_2(t_N) \in V.$$

We conclude $g_1(t_N) - g_2(t_N) \notin V$. If not, we should get

$$\begin{aligned}\Phi(t_N+s'_{n_N}) - \Phi(t_N+s'_{m_N}) = \Phi(t_N+s'_{n_N}) &- g_1(t_N)\\ &+ g_1(t_N) - g_2(t_N)\\ &+ g_2(t_N) - \Phi(t_N+s'_{m_N})\end{aligned}$$

and therefore $\Phi(t_N+s'_{n_N}) - \Phi(t_N+s'_{m_N}) \in U$; this contradicts (1).

We have found V with the property that if N is large enough, there exists $t_N \in R$ such that

$$g_1(t_N) - g_2(t_N) \notin V.$$

But this is impossible; because if we take a subsequence $(\xi_k)_{k=1}^{\infty} \subset (\eta_k)_{k=1}^{\infty}$ and $\xi_k \to t_N$, then we have

$$\lim_{N\to\infty} \Phi(\xi_k+s'_{n_N}) = \lim_{N\to\infty} \Phi(\xi_k+s'_{m_N})$$

for every k, and therefore $g_1(\xi_k) = g_2(\xi_k)$ for every k; by continuity of g_1 and g_2, $g_1(t_N) = g_2(t_N)$, thus $g_1(t_N) - g_2(t_N)$ belongs to every neighbourhood of 0. The lemma is proved.

Let us now continue proving theorem 8. Consider arbitrary real sequences $(h_n)_{n=1}^{\infty}$ and $(\eta_r)_{r=1}^{\infty}$ the rational numbers.

By relative compacity of $\{f(t), t \in R\}$, we can extract a subsequence $(h_n)_{n=1}^{\infty}$ (we do not change notation) such that for each r,

$$\lim_{n\to\infty} f(\eta_r+h_n) = x_r \text{ exists in E.} \tag{4.4}$$

Now $(f(\eta_r+h_n))_{n=1}^{\infty}$ is uniformly convergent in r. Suppose it is not; then we find a neighbourhood U and three subsequences $(\xi_r)_{r=1}^{\infty} \subset (\eta_r)_{r=1}^{\infty}$, $(h'_r)_{r=1}^{\infty} \subset (h_r)_{r=1}^{\infty}$, $(h''_r)_{r=1}^{\infty} \subset (h_r)_{r=1}^{\infty}$ and

$$f(\xi_r+h'_r) - f(\xi_r+h''_r) \notin U. \tag{4.5}$$

By relative compacity of $\{f(t); t \in R\}$ we may say

$$\lim_{r\to\infty} f(\xi_r+h'_r) = b' \in E, \tag{4.6}$$

$$\lim_{r\to\infty} f(\xi_r+h''_r) = b'' \in E,$$

and using (4.5), we get

$$b' - b'' \notin U. \tag{4.7}$$

By the Hahn-Banach theorem, there exists $x^* \in E^*$ such that

$$x^*(b') \neq x^*(b''). \tag{4.8}$$

But $f(t)$ is w.a.p. hence $(x^*f)(t)$ is a.p. and consequently it is uniformly continuous over R.

Consider the sequence of functions $(\varphi_n)_{n=1}^{\infty}$ defined by:

$$\varphi_n(t) = (x^*f)(t+h_n),\ n = 1,2,\ldots$$

The equality $\varphi_n(t+\tau) - \varphi_n(t) = x^*f(t+\tau+h_n) - x^*f(t+h_n)$ shows almost-periodicity of each φ_n. Also $(\varphi_n)_{n=1}$ is equi-uniformly continuous over R because (x^*f) is uniformly continuous over R, as it is easy to see. Using (4.4), we can say

$$\lim_{n\to\infty} x^*f(\eta_r+h_n) = x^*(x_r)$$

for every r. Therefore, by lemma 1, $(x^*f(t+h_n))_{n=1}^{\infty}$ is uniformly convergent in t.

Consider now the sequences $(\xi_r+h'_r)_{r=1}^{\infty}$ and $(\xi_r+h''_r)_{r=1}^{\infty}$. By Bochner's criteria, we extract two subseqeunces (we use the same notations) such that $(x^*f(t+\xi_r+h'_r))_{r=1}^{\infty}$ and $(x^*f(t+\xi_r+h''_r))_{r=1}^{\infty}$ are uniformly convergent in $t \in R$.

Let us now prove

$$\lim_{r\to\infty} x^*f(t+\xi_r+h'_r) = \lim_{r\to\infty} x^*f(t+\xi_r+h''_r). \tag{4.9}$$

Consider the inequality:

$$\begin{aligned}|x^*f(t+\xi_r+h'_r) - x^*f(t+\xi_r+h''_r)| &\\ \leq |x^*f(t+\xi_r+h'_r) - x^*f(t+\xi_r+h_r)| &\\ + |x^*f(t+\xi_r+h_r) - x^*f(t+\xi_r+h''_r)| &\end{aligned} \tag{4.10}$$

$r = 1,2,\ldots$

Let $\varepsilon > 0$ be given; as $(x^*f(t+h_r))_{r=1}^{\infty}$ is uniformly convergent in t, we choose η_ε such that for r, s > η_ε, we have $|x^*f(t+h_s) - x^*f(t+h_r)| < \frac{\varepsilon}{2}$, for $t \in R$; then for r, s > η_ε, we get

$$|x^*f(t+\xi_r+h_s) - x^*f(t+\xi_r+h_r)| < \frac{\varepsilon}{2}. \tag{4.11}$$

Consequently, for $r > \eta_\varepsilon$, we get:

$$|x^*f(t+\xi_r+h'_r) - x^*f(t+\xi_r+h_r)| < \frac{\varepsilon}{2},$$

$$|x^*f(t+\xi_r+h''_r) - x^*f(t+\xi_r+h_r)| < \frac{\varepsilon}{2}$$

and the inequality (4.10) gives:

$$|x^*f(t+\xi_r+h'_r) - x^*f(t+\xi_r+h''_r)| < \varepsilon$$

for $t \in R$. (4.9) is then proved.

Now take $t = 0$; then using (4.6) we get:

$$x^*(b') = \lim_{r\to\infty} x^*f(\xi_r+h'_r) = \lim_{r\to\infty} x^*f(\xi_r+h''_r) = x^*(b'')$$

which contradicts (4.8) and uniform convergence in r for $(f(\eta_r+h_n))_{n=1}^{\infty}$.

If i, j > N, we have

$$f(\eta_r+h_i) - f(\eta_r+h_u) \in U, \text{ for every r.} \tag{4.11}$$

Therefore if $t \in R$, we take a subsequence of $(\eta_r)_{r=1}^{\infty}$ which converges to t and using continuity of f and the relation (4.11), we obtain, for i, j > N,

$$f(t+h_i) - f(t+h_j) \in U.$$

f is then a.p..

THEOREM 9. Let E be a Fréchet space. If $f: R \to E$ is a.p. and $\{F(t);\ t \in R\}$ where $F(t) = \int_0^t f(\sigma)d\sigma$ is relatively compact in E, then F is a.p..

PROOF. Immediate from theorems 7 and 8.

THEOREM 10. Let E be a complete l.c.s.. If f is a.p. and its derivative f' uniformly continuous over the real line, then f' is also a.p..

PROOF. Consider the sequence of a.p. functions $\{n(f(t+\frac{1}{n}) - f(t))\}_{n=1}^{\infty}$; it suffices to prove it converges uniformly over the real line to f'(t).

Let $U = U(\varepsilon; p_i, 1\leq i\leq n)$; by uniform continuity of f', we can choose $\delta = \delta_U > 0$ such that $f'(t_1) - f'(t_2) \in U$ for every t_1, t_2 with $|t_1-t_2| < \delta$.
We write

$$f'(t) - n(f(t+\frac{1}{n}) - f(t)) = n \int_0^{\frac{1}{n}} [f'(t) - f'(t+\sigma)]d\sigma .$$

Therefore, if we take $N = N_U > \frac{1}{\delta}$, then for n N, we have:

$$p_i[f'(t) - n(f(t+\frac{1}{n}) - f(t))] \leq n \int_0^{\frac{1}{n}} p_i[f'(t) - f'(t+\sigma)]d\sigma < \varepsilon$$

for every semi-norm p_i and every $t \in R$. The theorem is proved.

THEOREM 11. Let E be a Fréchet space; then the set of all a.p. $f:R \to E$ is a Banach space under the supremum norm.

PROOF. Obvious; use theorems 1, 2 and 6.

5. APPLICATIONS TO ABSTRACT DIFFERENTIAL EQUATIONS

A. A.P. SOLUTIONS OF $(\frac{d}{dx} - A)\, x = 0$

Consider in a complete l.c.s. E the differential equation

$$\frac{dx}{dt} = Ax(t), \quad -\infty<t<\infty, \tag{5.1}$$

where A is a continuous linear operator such that $\{A^k;\ k = 1,2,\ldots\}$ is equi-continuous. A solution of (5.1) is a continuously differentiable function which satisfies (5.1). It is easy to construct (as in [13] p. 244-246) a solution of the form:

$$e^{tA}x(0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x(0).$$

We can say more:

PROPOSITION 1. The function $e^{tA}x_0$: $R \to E$ is the unique solution of the Cauchy problem:

$$\frac{dx}{dt} = Ax(t); \quad -\infty<t<\infty, \tag{5.2}$$

$$x(0) = x_0.$$

PROOF. Suppose there exists another solution y(t) with $y(0) = x_0$; consider the function $v(\tau) = e^{(t-\tau)A}y(\tau)$, with fixed t and show it is constant over the real line; therefore $v(\tau) = v(0)$ for every $\tau \in R$, which means $v(t) = v(0)$, or $y(t) = e^{tA}x_0$, proving uniqueness of the solution (see[9] for a complete proof).

Now, define a perfect Fréchet space E as a Fréchet space with the following property: every function f: $R \to E$ with (i) $\{f(t);\ t \in R\}$ is bounded in E; (ii) f'(t) is a.p. in E; is necessarily a.p. in E.

We state and prove the two following theorems inspired from a result of PEROV (see [15] theorem 1.1) but they are not direct generalisations. In fact they are new results.

THEOREM 1. Let E be a perfect Fréchet space; assume (i) A is a compact linear operator; (ii) $\{A^k;\ k = 1,2,\ldots\}$ is equi-continuous; (iii) for every semi-norm p, there exists a semi-norm q such that $p[e^{tA}x] \leq q(x)$ for every $x \in E$ and every $t \in R$.

Then every solution $x(t)$ of (5.1) is a.p. in E.

PROOF. Because $x(t) = e^{tA}x(0)$, then $x(t)$ is bounded in E by (iii). E being a perfect Fréchet space, it suffices to prove $x'(t)$ is a.p..

$\{Ax(t);\ t \in R\}$ is also relatively compact in E for A is a compact operator; consequently $\{x(t);\ t \in R\}$ is also relatively compact. Let $(s'_n)_{n=1}^{\infty}$ be an arbitrary real sequence; we then can extract a subsequence $(s_n)_{n=1}^{\infty}$ such that $(x'(s_n))_{n=1}^{\infty}$ is a Cauchy sequence in E. But we have:

$$x'(t+s_n) = Ax(t+s_n) = Ae^{(t+s_n)A}x(0) = Ae^{tA}e^{s_nA}x(0)$$

$$= Ae^{tA}x(s_n) = e^{tA}Ax(s_n) = e^{tA}x'(s_n)$$

for every $n = 1,2,\ldots$, and every $t \in R$. If p is a given semi-norm, there exists a semi-norm q such that

$$p[x'(t+s_n) - x'(t+s_m)] = p[e^{tA}(x'(s_n) - x'(s_m))]$$

$$\leq q[x'(s_n) - x'(s_m)]$$

for every $t \in R$ and every $n, m \in N$. Therefore $(x'(t+s_n))_{n=1}^{\infty}$ is uniformly Cauchy in t; we then conclude almost-periodicity of $x'(t)$ by Bochner's criteria.

THEOREM 2. Let E be a Fréchet space; assume conditions (i) - (iii) in theorem 1 are satisfied and moreover the range R(A) of A is dense in E. Then every solution $x(t)$ of (5.1) is a.p. in E.

We remark the first part of the proof of theorem 1 tells us if $x(t)$ is a solution of (5.1) with $x(0) \in D(A) = E$, then $x'(t)$ is a.p.. Before proving Theorem 2 let us state and prove:

LEMMA 1. Every solution of (5.1) with initial data in R(A) is a.p..

PROOF. Let $a \in R(A)$ and consider the solution $y(t)$ with $y(0) = a$; there exists $x_0 \in D(A) = E$ such that $Ax_0 = a$. We have $y(t) = e^{tA}a = e^{tA}Ax_0 = ae^{tA}x_0 = Ax(t) = x'(t)$ where $x(t) = e^{tA}x_0$; therefore $x'(t)$ (and consequently $y(t)$) is a.p.. The lemma is proved.

PROOF OF THEOREM 2. Consider a solution $x(t)$ of (5.1) with $x(0) \in E$; as R(A) is dense in E, there exists a sequence $(a_n)_{n=1}^{\infty}$ in R(A) such that $a_n \to x(0)$. Consider a sequence of solutions $(y_n(t))_{n=1}^{\infty}$ with $y_n(0) = a_n$, $n = 1,2,\ldots$ To prove almost-periodicity of $x(t)$ it suffices to prove $y_n(t) \to x(t)$ uniformly in $t \in R$ for every $y_n(t)$ is a.p. by lemma 1. We have $x(t) = e^{tA}x(0)$, $y_n(t) = e^{tA}a_n$, $n = 1,2,\ldots$ Now given a semi-norm p there exists, by assumption (iii), a semi-norm q such that $p(y_n(t) - x(t)) \leq q(a_n - x(0))$, for every $t \in R$ and every $n \in N$. The conclusion is immediate.

B. A.P. SOLUTIONS OF $(\frac{d}{dx} - A)\ x = f$

We now consider the non-homogeneous differential equation

$$\frac{dx}{dt} = Ax(t) + f(t), \quad -\infty<t<\infty \tag{5.3}$$

where A is a closed linear operator with domain D(A) dense in a Fréchet space E; the function $f(t)$ is a.p. in E. Let us recall some useful definitions (see 13). A family of continuous linear operators $T(t)$, $t \in R$, is an equi-continuous C_0-group:

(i) $T(t_1+t_2)x = T(t_1)T(t_2)x$, $T(0)x = x$, for every $x \in E$ and every t_1, $t_2 \in R$;

(ii) for every semi-norm p, there exists a semi-norm q such that $p[T(t)x] \leq q(x)$ for every $x \in E$ and every $t \in R$.

(iii) $\lim_{t \to t_0} T(t_0)x$, for every $x \in E$ and every $t_0 \in R$.

Now consider an equi-continuous C_0-group $T(t)$. A is called the infinitesimal generator of $T(t)$ if $Ax = \lim_{\eta \to 0} \frac{T(\eta)x - x}{\eta}$, i.e., A is the linear operator with domain $D(A) = \{x \in E; \lim_{\eta \to 0} \frac{T(\eta)x - x}{\eta}$ exists in E$\}$ and for every $x \in D(A)$, $Ax = \lim_{\eta \to 0} \frac{T(\eta)x - x}{\eta}$.

It can be proved $\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax$ for every $x \in D(A)$ (see[13] for the case of a semi-group).

We are going to prove the following theorem 3 which is a generalization of theorem 3.2 [15] due to ZAIDMAN.

THEOREM 3. Let E be a Fréchet space. Suppose $x(t)$ is a solution of equation (5.3) with relatively compact trajectory; A is the infinitesimal generator of equi-continuous C_0-group $T(t)$ such that $T(t)x : R \to E$ is a.p. for every $x \in E$; $f(t)$ is a.p.. Then $x(t)$ is also a.p..

Before we prove theorem 3, let us mention some useful lemmas (see [9] for proofs):

LEMMA 2. Let E be a complete l.c.s.. If $f(\sigma)$ is continuous, then $T(t-\sigma)f(\sigma)$: $R \to E$ is also continuous for every $t \in R$.

LEMMA 3. In a complete l.c.s. E, every solution of (5.3) admits the integral representation:

$$x(t) = T(t)x(0) + \int_0^t T(t-\sigma)f(\sigma)d\ .$$

LEMMA 4. Let E be a Fréchet space. If $\{T(t)x;\ t \in R\}$ is relatively compact in E for every $x \in E$ and $\{f(t);\ t \in R\}$ is also relatively compact in E, then $\{T(t)f(t);\ t \in R\}$ is relatively compact in E.

PROOF. Let $(t''_n)_{n=1}^{\infty}$ be an arbitrary real sequence; by our assumption on $f(t)$, we can extract a subsequence $(t'_n)_{n=1}^{\infty} \subset (t''_n)_{n=1}$ such that $\lim_{n\to\infty} f(t'_n)$ exists in E; let x be this limit.

Take another subsequence $(t_n)_{n=1}^{\infty} \subset (t'_n)_{n=1}^{\infty}$ such that $(T(t_n)x)_{n=1}^{\infty}$ is a Cauchy sequence in E. Write:

$$\begin{aligned} T(t_n)f(t_n) - T(t_m)f(t_m) &= [T(t_n) - T(t_m)]\ [f(t_n - x] \\ &+ [(T(t_n) - T(t_m))x] \\ &+ T(t_m)[f(t_n) - f(t_m)]\ . \end{aligned}$$

Let p be any semi-norm; then we have

$$\begin{aligned} p[T(t_n)f(t_n) - T(t_m)f(t_m)] &< p[[T(t_n) - T(t_m)]\ [f(t_n) - x]] \\ &+ p[T(t_n) - T(t_m))x] \\ &+ p[T(t_m)[f(t_n) - f(t_m)]]. \end{aligned}$$

Using equi-continuity of $T(t)$, we can take a semi-norm q such that

$$p[T(t_n)[f(t_n) - f(t_m)]] \leq q[f(t_n) - f(t_m)]$$

and

$$p[[T(t_n) - T(t_m)]\ [f(t_n) - x]] \leq 2q[f(t_n) - x].$$

Now we choose n and m sufficiently large such that

$$q[f(t_n) - f(t_m)] < \frac{\varepsilon}{3},\ q[f(t_n) - x] < \frac{\varepsilon}{6},\ p[(T(t_n) - T(t_m))x] < \frac{\varepsilon}{3}$$

then we obtain:

$$p[T(t_n)f(t_n) - T(t_m)f(t_m)] < \varepsilon$$

which shows $(T(t_n)f(t_n))_{n=1}^{\infty}$ is a Cauchy sequence. The lemma is proved.

LEMMA 5. Let E be a Fréchet space and consider the equi-continuous C_0-group $T(t)$ such that $T(t)x : R \to E$ is a.p. for every $x \in E$. Suppose also $f(t)$ is a.p.. Then $T(t)f(t) : R \to E$ is a.p..

PROOF. Consider $U = U(\varepsilon;\ p_i,\ 1 \le i \le n)$ a given neighbourhood; because of equi-comtinuity of $T(t)$, there corresponds to each semi-norm p_i, a semi-norm q_i such that:

(i) $p_i(T(t)x) \le q_i(x),\ x \in E,\ t \in R.$

Consider also the symmetric neighbourhood

$$V = V(\tfrac{\varepsilon}{4};\ p_i,\ q_i,\ 1 \le i \le n);\ V + V + V + V \subset U.$$

As $\{f(t);\ t \in R\}$ is totally bounded, there exists $t_1,\dots,t_\nu$ such that for every $t \in R$ we have $f(t) \in \bigcup_{k=1}^{\nu} (f(t_k) + V)$. Consider now the following a.p. functions: $f(t)$, $T(t)f(t_k)$, $k = 1,\dots,\nu$. Then they have the same V-translation numbers; therefore we can say there exists $\ell=\ell(V) > 0$ such that any interval $[a,a+\ell]$ contains τ with

$$f(t+\tau) - f(t) \in V,\ t \in R \tag{5.4}$$

$$T(t+\tau)f(t_k) - T(t)f(t_k) \in V,\ k = 1,\dots,\ t \in R.$$

Take $t \in R$ arbitrary; then there exists k $(1 \le k \le \nu)$ such that

$$f(t) \in f(t_k) + V. \tag{5.5}$$

Write:

$$T(t+\tau)f(t+\tau) - T(t)f(t) = \{T(t+\tau)[f(t+\tau) - f(t)]\}$$
$$+ \{T(t+\tau)[f(t) - f(t_k)]\} + \{T(t+\tau)f(t_k) - T(t)f(t_k)\}$$
$$+ \{T(t)[f(t_k) - f(t)]\}.$$

For every semi-norm p_i; there exists a semi-norm q_i such that:

$$p_i[T(t+\tau)f(t+\tau) - T(t)f(t)] \le q_i[f(t+\tau) - f(t)]$$
$$+ q_i[f(t) - f(t_k)] + p_i[T(t+\tau)f(t_k) - T(t)f(t_k)]$$
$$+ q_i[f(t_k) - f(t)] < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \text{ , (using (5.3) - (5.5).}$$

Therefore $T(t+\tau)f(t+\tau) - T(t)f(t) \in U$ for every $t \in R$, which is almost-periodicity for $T(t)f(t)$.

PROOF OF THEOREM 3. By lemma 3 we have : $x(t) = T(t)x(0) + \int_0^t T(t-\sigma)f(\sigma)d\sigma$. But $T(t)x(0)$ is a.p.. It remains to prove the function $v(t) = \int_0^t T(t-\sigma)f(\sigma)d\sigma$ is also a.p..

As $\{x(t);\ t \in R\}$ and $\{T(t)x(0);\ t \in R\}$ are relatively compact, then $\{v(t);\ t \in R\}$ also is relatively compact. Let us write $v(t) = \int_0^t T(t)T(-\sigma)f(\sigma)d\sigma$ $= T(t) \int_0^t T(-\sigma)f(\sigma)d\sigma$. Then $T(-t)v(t) = \int_0^t T(-\sigma)f(\sigma)d\sigma$.

By theorem 3 of chapter 1, $T(-t)x$ is a.p. for every $x \in E$, therefore $\{T(-t)x;\ t \in R\}$ is relatively compact for every $x \in E$. By lemma 4, $\{T(-t)v(t);\ t \in R\}$ and consequently $\{\int_0^t T(-\sigma)f(\sigma)d\sigma\ ;\ t \in R\}$ is relatively compact. By lemma 5, $T(-t)f(t)$ is a.p., therefore $\int_0^t T(-\sigma)f(\sigma)d\sigma$ is a.p.. We apply again lemma 5 to conclude almost-periodicity of $\int_0^t T(t-\sigma)f(\sigma)d\sigma$. Theorem 3 is proved.

THEOREM 4. Let E be a Fréchet space. Solutions of the equation $x'(t) = Ax(t)$, $-\infty<t<\infty$, with relatively compact trajectory are precisely almost-periodic ones, if A is the infinitesimal generator of equi-continuous C_0-group $T(t)$.

PROOF. Let $x(t)$ be a solution of the given equation. It suffices to prove that if $x(t)$ has a relatively compact trajectory, then $x(t)$ is a.p.. Take an arbitrary real sequence $(s'_n)_{n=1}^{\infty}$; we can extract a subsequence $(s_n)_{n=1}^{\infty} \subset (s'_n)_{n=1}^{\infty}$ such that $(x(s_n))_{n=1}^{\infty}$ is a Cauchy sequence in E; but we have

$$x(t+s_n) = T(t+s_n)x(0) = T(t)T(s_n)x(0) = T(t)x(s_n), \quad n = 1,2,\ldots$$

Therefore

$$x(t+s_n) - x(t+s_m) = T(t)[x(s_n) - x(s_m)], \quad n, m \in N.$$

Let p be a given semi-norm; by equi-continuity of $T(t)$, there exists a semi-norm q such that:

$$p[x(t+s_n) - x(t+s_m)] \leq q[x(s_n) - x(s_m)], \quad t \in R.$$

Which shows $(x(t+s_n))_{n=1}^{\infty}$ is a Cauchy sequence, uniform in $t \in R$. We conclude using Bochner's criteria.

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